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Research article

Existence of ground state for coupled system of biharmonic Schrödinger equations

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Abstract: In this paper we consider the following system of coupled biharmonic Schrödinger equations

$$\begin{cases} \Delta^2 u + \lambda_1 u = u^3 + \beta u v^2, \\ \Delta^2 v + \lambda_2 v = v^3 + \beta u^2 v, \end{cases}$$

where $(u, v) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$, $1 \leq N \leq 7$, $\lambda_i > 0 (i = 1, 2)$ and β denotes a real coupling parameter. By Nehari manifold method and concentration compactness theorem, we prove the existence of ground state solution for the coupled system of Schrödinger equations. Previous results on ground state solutions are obtained in radially symmetric Sobolev space $H^2_r(\mathbb{R}^N) \times H^2_r(\mathbb{R}^N)$. When β satisfies some conditions, we prove the existence of ground state solution in the whole space $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$.

Keywords: system of Schrödinger equations; ground state solution; Nehari manifold; concentration-compactness principle
Mathematics Subject Classification: 35J35, 35J50, 35Q55, 47J35

1. Introduction

In this paper, we consider the existence of standing waves for the following coupled system of biharmonic Schrödinger equations

$$\begin{cases} i\partial_t E_1 - \triangle^2 E_1 + |E_1|^2 E_1 + \beta |E_2|^2 E_1 = 0, \\ i\partial_t E_2 - \triangle^2 E_2 + |E_2|^2 E_2 + \beta |E_1|^2 E_2 = 0, \end{cases} \tag{1.1}$$

where $E_1 = E_1(x,t) \in \mathbb{C}$, $E_2 = E_2(x,t) \in \mathbb{C}$ and β is a constant. This system describes the interaction of two short dispersive waves. By standing waves we mean solutions of type

$$(E_1(x,t), E_2(x,t)) = (e^{i\lambda_1 t}u(x), e^{i\lambda_2 t}v(x)), \tag{1.2}$$

where u, v are real functions. This leads us to study the following biharmonic Schrödinger system

$$\begin{cases} \Delta^2 u + \lambda_1 u = u^3 + \beta uv^2, \\ \Delta^2 v + \lambda_2 v = v^3 + \beta u^2 v, \end{cases} \tag{1.3}$$

where $(u,v) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$. In this paper we assume that $1 \le N \le 7, \lambda_i > 0 (i = 1,2)$ and β is a coupling parameter.

In order to describe wave propagation, some models with higher-order effects and variable coefficients, such as the third-, fourth- and fifth-order dispersions, self-steepening and symmetric perturbations, have been proposed in physical literatures (see e.g. [26]). Karpman investigated the stability of the soliton solutions for fourth-order nonlinear Schrödinger equations (see [13, 14]). To understand the differences between second and fourth order dispersive equations, one can refer to [11].

Physically, the interaction of the long and short waves can be described by a system of coupled nonlinear Schrödinger and Korteweg-de Vries equations. Recently, a fourth-order version of such system was considered by P. Alvarez-Caudevilla and E. Colorado [5]. Using the method of Nehari manifold, they proved the existence of ground state in radially symmetric space $H_r^2(\mathbb{R}^N) \times H_r^2(\mathbb{R}^N)$. In their proof, the compact embedding of radially symmetric function space is essential. A natural problem is whether there exists a ground state in the Sobolev space $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$.

On the other hand, the second order counterparts of (1.1) and (1.3) are respectively

$$\begin{cases} i\partial_t E_1 - \triangle E_1 + |E_1|^2 E_1 + \beta E_1 |E_2|^2 = 0, \\ i\partial_t E_2 - \triangle E_2 + |E_2|^2 E_2 + \beta |E_1|^2 E_2 = 0. \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta u + \lambda_1 u = u^3 + \beta uv^2, \\ \Delta v + \lambda_2 v = v^3 + \beta u^2 v. \end{cases} \tag{1.5}$$

Since pioneering works of [2–4, 18, 19, 22], system (1.5) and its extensions to more general second order elliptic systems have been extensively studied by many authors, e.g. [8, 9, 12, 21, 23]. For the similar problem for fractional order elliptic system, one can refer to [7, 10, 25].

Motivated by the above developments, using techniques of variation principle and concentration-compactness lemma, we consider the existence of ground state for system (1.3). By ground state, we mean a nontrivial least energy solution of the system.

We organize the paper as follows. In Section 2, we give some notations, elementary results and statements of our main theorems. In Section 3, we study some properties of Palais-Smale sequence. In Section 4, we give the proof of our main theorems.

2. Preliminaries and main theorems

In $H^2(\mathbb{R}^N)$, we define the following norm:

$$\langle u, v \rangle_i := \int_{\mathbb{R}^N} (\Delta u \cdot \Delta v + \lambda_i uv), \quad \|u\|_i^2 := \langle u, u \rangle_i, \quad i = 1, 2. \tag{2.1}$$

For $u \in L^p(\mathbb{R}^N)$, we set $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p\right)^{\frac{1}{p}}$ for $1 \le p < \infty$. Accordingly, the inner product and induced norm on

$$\mathbb{H} := H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N).$$

are given by

$$\begin{aligned} \langle (u,v),(\xi,\eta)\rangle &= \int_{\mathbb{R}^N} (\Delta u \cdot \Delta \xi + \Delta v \cdot \Delta \eta + \lambda_1 u\xi + \lambda_2 v\eta), \\ \|(u,v)\|^2 &= \|u\|_1^2 + \|v\|_2^2. \end{aligned} \tag{2.2}$$

The energy functional associated with system (1.3) is

$$\Phi(\mathbf{u}) = \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{4}\int_{\mathbb{R}^N}(u^4 + v^4) - \frac{1}{2}\beta\int_{\mathbb{R}^N} u^2v^2. \tag{2.3}$$

for $\mathbf{u} = (u,v) \in \mathbb{H}$.

Set

$$I_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{4}\int_{\mathbb{R}^N} u^4, \quad I_2(v) = \frac{1}{2}\|v\|_1^2 - \frac{1}{4}\int_{\mathbb{R}^N} v^4,$$

$$\Psi(\mathbf{u}) = \Phi'(\mathbf{u})[\mathbf{u}] = \|\mathbf{u}\|^2 - \int_{\mathbb{R}^N}(u^4 + v^4) - 2\beta\int_{\mathbb{R}^N} u^2v^2. \tag{2.4}$$

and the Nehari manifold

$$\mathcal{N} = \{\mathbf{u} = (u,v) \in \mathbb{H}\backslash\{(0,0)\} : \Psi(\mathbf{u}) = 0\}.$$

Remark 2.1. *(see [1, 5, 16])*

Let

$$2^* = \begin{cases} \dfrac{2N}{N-4}, & if \quad N > 4, \\ \infty, & if \quad 1 \le N \le 4. \end{cases}$$

Then we have the following Sobolev embedding:

$$H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad for \begin{cases} 2 \le p \le 2^*, & if\ N \ne 4, \\ 2 \le p < 2^*, & if\ N = 4. \end{cases}$$

Proposition 2.1. *Let $\Phi_{\mathcal{N}}$ be the restriction of Φ on $\mathcal{N}$. The following properties hold.*

i) $\mathcal{N}$ is a locally smooth manifold.

ii) $\mathcal{N}$ is a complete metric space.

iii) $\mathbf{u} \in \mathcal{N}$ is a critical point of Φ if and only if $\mathbf{u}$ is a critical point of $\Phi_{\mathcal{N}}$.

iv) Φ is bounded from below on $\mathcal{N}$.

Proof. i) Differentiating expression (2.4) yields

$$\Psi'(\mathbf{u})[\mathbf{u}] = 2\|\mathbf{u}\|^2 - 4\int_{\mathbb{R}^N}(u^4 + v^4) - 8\beta\int_{\mathbb{R}^N} u^2v^2. \tag{2.5}$$

By the definition of Nehari manifold, for $\mathbf{u} \in \mathcal{N}$, $\Psi(\mathbf{u}) = 0$ and hence

$$\Psi'(\mathbf{u})[\mathbf{u}] = \Psi'(\mathbf{u})[\mathbf{u}] - 3\Psi(\mathbf{u}) = -2\|\mathbf{u}\|^2 < 0. \tag{2.6}$$

It follows that $\mathcal{N}$ is a locally smooth manifold near any point $\mathbf{u} \neq \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$.

ii) Let $\{\mathbf{u}_n\} \subset \mathcal{N}$ be a sequence such that $\|\mathbf{u}_n - \mathbf{u}_0\| \to 0$ as $n \to +\infty$. By Gagliardo-Nirenberg-Sobolev inequality and interpolation formula for L^p space, we have $|u_n - u_0|_p \to 0$ and $|v_n - v_0|_p \to 0$ for $2 \leq p < 2^*$. It easily follows that $\Phi'(\mathbf{u}_n)[\mathbf{u}_n] - \Phi'(\mathbf{u}_0)[\mathbf{u}_0] \to 0$. Since $\Phi'(\mathbf{u}_n)[\mathbf{u}_n] = 0$, we have $\Phi'(\mathbf{u}_0)[\mathbf{u}_0] = 0$.

Claim: There exists $\rho > 0$ such that for all $\mathbf{u} \in \mathcal{N}$, $\|\mathbf{u}\| > \rho$.

Since $\mathbf{u}_n \in \mathcal{N}$ for all n and $\|\mathbf{u}_n - \mathbf{u}_0\| \to 0$, we get $\mathbf{u}_0 \neq (0, 0)$. Hence $\mathbf{u}_n \in \mathcal{N}$and $\mathcal{N}$ is a complete metric space.

Proof of the claim: Taking the derivative of the functional Φ in the direction $\mathbf{h} = (h_1, h_2)$, it follows that

$$\Phi'(\mathbf{u})[\mathbf{h}] = \int_{\mathbb{R}^N} (\Delta u h_1 + \lambda_1 u h_1 + \Delta v h_2 + \lambda_2 v h_2) - \int_{\mathbb{R}^N} (u^3 h_1 + v^3 h_2) - \beta \int_{\mathbb{R}^N} (uv^2 h_1 + u^2 v h_2).$$

Taking the derivative of $\Phi'(\mathbf{u})[\mathbf{h}]$ in the direction $\mathbf{h}$ again, it follows that

$$\Phi''(\mathbf{u})[\mathbf{h}]^2 = \|\mathbf{h}\|^2 - 3 \int_{\mathbb{R}^N} (u^2 h_1^2 + v^2 h_2^2) - \beta \int_{\mathbb{R}^N} (u^2 h_2^2 + v^2 h_1^2 + 4uv h_1 h_2).$$

Note that $[\mathbf{h}]^2$ means $[\mathbf{h}, \mathbf{h}]$ and $\mathbf{h} = (h_1, h_2)$. Let $\mathbf{u} = \mathbf{0}$, we obtain $\Phi''(\mathbf{0})[\mathbf{h}]^2 = \|\mathbf{h}\|^2$, which implies that $\mathbf{0}$ is a strict minimum critical point of Φ. In a word, we can deduce that $\mathcal{N}$ is a smooth complete manifold and there exists a constant $\rho > 0$ such that

$$\|\mathbf{u}\|^2 > \rho \quad \text{for all } \mathbf{u} \in \mathcal{N}. \tag{2.7}$$

iii) Assume that $(u_0, v_0) \in \mathcal{N}$ is a critical point of $\Phi_{\mathcal{N}}$. Then there is a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that

$$\Phi'(u_0, v_0) = \Lambda \Psi'(u_0, v_0). \tag{2.8}$$

Hence

$$0 = (\Phi'(u_0, v_0), (u_0, v_0)) = \Lambda(\Psi'(u_0, v_0), (u_0, v_0)). \tag{2.9}$$

From (2.6) and (2.9) , we get $\Lambda = 0$. Now (2.10) shows that $\Phi'(u_0, v_0) = 0$, i.e. (u_0, v_0) is a critical point of Φ.

iiii) By (2.3), (2.4) and (2.7), we have

$$\Phi_{\mathcal{N}}(\mathbf{u}) = \frac{1}{4}\|\mathbf{u}\|^2, \tag{2.10}$$

and

$$\Phi(\mathbf{u}) \geq \frac{1}{4}\rho \quad \text{for all } \mathbf{u} \in \mathcal{N}. \tag{2.11}$$

Then Φ is bounded from below on $\mathcal{N}$. □

Lemma 2.1. *For every* $\mathbf{u} = (u, v) \in \mathbb{H}\backslash\{(0, 0)\}$, *there is a unique number* $t > 0$ *such that* $t\mathbf{u} \in \mathcal{N}$.

Proof. For $(u, v) \in \mathbb{H}\backslash\{(0,0)\}$ and $t > 0$, define

$$\omega(t) := \Phi(tu, tv) = \frac{1}{2}t^2\|\mathbf{u}\|^2 - \frac{1}{4}t^4\int_{\mathbb{R}^N}(u^4 + v^4) - \frac{1}{2}\beta t^4\int_{\mathbb{R}^N}u^2v^2.$$

For fixed $(u, v) \neq (0, 0)$, we have $\omega(0) = 0$ and $\omega(t) \geq C^{'}t^2$ for small t. On the other hand, we have $\omega(t) \to -\infty$ as $t \to \infty$. This implies that there is a maximum point $t_m > 0$ of $\omega(t)$ such that $\omega^{'}(t_m) = \Phi^{'}(t_m\mathbf{u})\mathbf{u} = 0$ and hence $t_m\mathbf{u} \in \mathcal{N}$. Actually, since Φ has special structure, by direct computation we can also get the unique t_m. □

Lemma 2.2. *([20, page 125])*

Let $u \in L^q(\mathbb{R}^N)$ and $D^m u \in L^r(\mathbb{R}^N)$ for $1 \leq r,\ q \leq \infty$. For $0 \leq j < m$, there exists a constant $C > 0$ such that the following inequalities hold:

$$\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha},$$

where

$$\frac{1}{p} = \frac{j}{N} + (\frac{1}{r} - \frac{m}{N})\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha \leq 1.$$

and $C = C(n, m, j, q, r, \alpha)$.

The main results of the present paper are as follows:

Theorem 2.1. *There exist two positive numbers Λ^- and Λ^+, $\Lambda^- \leq \Lambda^+$, such that*

(i) If $\beta > \Lambda^+$, the infimum of Φ on $\mathcal{N}$ is attained at some $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ with $\Phi(\tilde{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}$ and both $\tilde{u}$ and $\tilde{v}$ are non-zero.

(ii) If $0 < \beta < \Lambda^-$, then Φ constrained on $\mathcal{N}$ has a mountain pass critical point $\mathbf{u}^*$ with $\Phi(\mathbf{u}^*) >$ $\max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}$.

The definitions of $\Lambda^+, \Lambda^-, \mathbf{u}_1$ and $\mathbf{v}_2$ will be given in section 4.

3. Palais-Smale sequence

Let

$$c = \inf_{\mathcal{N}} \Phi(\mathbf{u}).$$

Lemma 3.1. *There exists a bounded sequence $\mathbf{u}_n = (u_n, v_n) \subset \mathcal{N}$ such that $\Phi(\mathbf{u}_n) \to c$ and $\Phi^{'}(\mathbf{u}_n) \to 0$ as $n \to +\infty$.*

Proof. From Proposition 2.1, Φ is bounded from below on $\mathcal{N}$. By Ekeland's variational principle [24], we obtain a sequence $\mathbf{u}_n \subset \mathcal{N}$ satisfying

$$\begin{aligned}&\Phi(\mathbf{u}_n) \leq \inf_{\mathcal{N}} \Phi(\mathbf{u}) + \frac{1}{n},\\ &\Phi(\mathbf{u}) \geq \Phi(\mathbf{u}_n) - \frac{1}{n}\|\mathbf{u}_n - \mathbf{u}\| \quad \text{for any } \mathbf{u} \in \mathcal{N}.\end{aligned} \tag{3.1}$$

Since

$$c + \frac{1}{n} \geq \Phi(\mathbf{u}_n) = \frac{1}{4}\|\mathbf{u}_n\|^2, \tag{3.2}$$

there exists $C > 0$ such that

$$\|\mathbf{u}_n\|^2 \leq C. \tag{3.3}$$

For any $(y, z) \in \mathbb{H}$ with $\|(y, z)\| \leq 1$, denote

$$F_n(s, t) = \Phi'(u_n + sy + tu_n, v_n + sz + tv_n)(u_n + sy + tu_n, v_n + sz + tv_n). \tag{3.4}$$

Obviously, $F_n(0, 0) = \Phi'(u_n, v_n)(u_n, v_n) = 0$ and

$$\frac{\partial F_n}{\partial t}(0, 0) = (\Psi'(u_n, v_n), (u_n, v_n)) = -2\|\mathbf{u}_n\|^2 < 0. \tag{3.5}$$

Using the implicit function theorem, we get a C^1 function $t_n(s) : (-\delta_n, \delta_n) \to \mathbb{R}$ such that $t_n(0) = 0$ and

$$F_n(s, t_n(s)) = 0, \quad s \in (-\delta_n, \delta_n). \tag{3.6}$$

Differentiating $F_n(s, t_n(s))$ in s at $s = 0$, we have

$$\frac{\partial F_n}{\partial s}(0, 0) + \frac{\partial F_n}{\partial t}(0, 0)t_n'(0) = 0. \tag{3.7}$$

From (2.4) and (2.7), it follows that

$$\left|\frac{\partial F_n}{\partial t}(0, 0)\right| = |(\Psi'(u_n, v_n), (u_n, v_n))| = 2\|\mathbf{u}_n\|^2 > 2\rho. \tag{3.8}$$

By Hölder's inequality and Sobolev type embedding theorem, it yields

$$\begin{aligned} \left|\frac{\partial F_n}{\partial s}(0, 0)\right| &= |(\Psi'(u_n, v_n), (y, z))| \\ &\leq |2((u_n, v_n), (y, z))| + |4\int_{\mathbb{R}^N}(u_n^3 y + v_n^3 z)| + |4\beta\int_{\mathbb{R}^N}(u_n v_n^2 y + u_n^2 v_n z)| \\ &\leq C_1. \end{aligned} \tag{3.9}$$

From (3.7)–(3.9), we obtain

$$|t_n'(0)| \leq C_2. \tag{3.10}$$

Let

$$(\overline{y}, \overline{z})_{n,s} = s(y, z) + t_n(s)(u_n, v_n), \quad (y, z)_{n,s} = (u_n, v_n) + (\overline{y}, \overline{z})_{n,s}. \tag{3.11}$$

In view of (3.1), we have

$$|\Phi(y, z)_{n,s} - \Phi(u_n, v_n)| \leq \frac{1}{n}\|(\overline{y}, \overline{z})_{n,s}\|. \tag{3.12}$$

Applying a Taylor expansion on the left side of (3.12), we deduce that

$$\begin{aligned} \Phi(y, z)_{n,s} - \Phi(u_n, v_n) &= (\Phi'(u_n, v_n), (\overline{y}, \overline{z})_{n,s}) + r(n, s) \\ &= (\Phi'(u_n, v_n), s(y, z)) + (\Phi'(u_n, v_n), t_n(s)(u_n, v_n)) + r(n, s) \\ &= s(\Phi'(u_n, v_n), (y, z)) + r(n, s), \end{aligned} \tag{3.13}$$

where $r(n,s) = o\|(\overline{y},\overline{z})_{n,s}\|$ as $s \to 0$.

From (3.3), (3.10), (3.11) and $t_n(0) = 0$, we have

$$\limsup_{|s|\to 0} \frac{\|(\overline{y},\overline{z})_{n,s}\|}{|s|} \le C_3, \tag{3.14}$$

where C_3 is independent of n for small s. Actually, it follows from (3.10), (3.11) that $r(n,s) = O(s)$ for small s.

From (3.3), (3.12)–(3.14), we have

$$|(\Phi'(u_n,v_n),(y,z))| \le \frac{C_3}{n}. \tag{3.15}$$

Hence $\Phi'(u_n,v_n) \to 0$ as $n \to \infty$. We complete the proof of the lemma. □

From the above lemma, we have a bounded PS sequence such that $\Phi'(u_n,v_n) \to 0$ and $\Phi(u_n,v_n) \to c$. Then, there exists $(u_0,v_0) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ such that $(u_n,v_n) \rightharpoonup (u_0,v_0)$.

Lemma 3.2. *Assume that* $(u_n,v_n) \rightharpoonup (u_0,v_0)$ *and* $\Phi'(u_n,v_n) \to 0$ *as* $n \to \infty$. *Then* $\Phi'(u_0,v_0) = 0$.

Proof. For any $v = (\varphi,\psi), \varphi,\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\Phi'(u_n,v_n)v = \langle (u_n,v_n),(\varphi,\psi)\rangle - \int_{\mathbb{R}^N} (u_n^3\varphi + v_n^3\psi) - \beta \int_{\mathbb{R}^N} (u_n v_n^2\varphi - u_n^2 v_n\psi). \tag{3.16}$$

The weak convergence $\{\mathbf{u}_n\}$ implies that $\langle (u_n,v_n),(\varphi,\psi)\rangle \to \langle (u_0,v_0),(\varphi,\psi)\rangle$. Let $K \subset \mathbb{R}^N$ be a compact set containing supports of φ,ψ, then it follows that

$$\begin{aligned} &(u_n,v_n) \to (u_0,v_0) \quad \text{in } L^p(K)\times L^p(K) \text{ for } 2 \le p < 2^*,\\ &(u_n,v_n) \to (u_0,v_0) \quad \text{for a.e. } x \in \mathbb{R}^N. \end{aligned}$$

From [6], there exist a_K and $b_K \in L^4(K)$ such that

$$|u_n(x)| \le a_K(x) \quad \text{and} \quad |v_n(x)| \le b_K(x) \quad \text{for a.e. } x \in K.$$

Define $c_K(x) := a_K(x) + b_K(x)$ for $x \in K$. Then $c_K \in L^4(K)$ and

$$|u_n(x)|, |v_n(x)| \le |u_n(x)| + |v_n(x)| \le a_K(x) + b_K(x) = c_K(x) \quad \text{for a.e. } x \in K.$$

It follows that, for a.e. $x \in K$,

$$\begin{aligned} u_n v_n^2\varphi &\le c_K^3|\varphi|,\\ u_n^2 v_n\psi &\le c_K^3|\psi|, \end{aligned}$$

and hence

$$\begin{aligned} \int_K c_K^3|\varphi|dx &\le |c_K\chi_K|_4^3|\varphi\chi_K|_4,\\ \int_K c_K^3|\psi|dx &\le |c_K\chi_K|_4^3|\psi\chi_K|_4. \end{aligned}$$

By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \int_K u_n v_n^2 \varphi dx &\to \int_K u_0 v_0^2 \varphi dx, \\ \int_K u_n^2 v_n \psi dx &\to \int_K v_0 u_0^2 \psi dx. \end{aligned} \tag{3.17}$$

Similarly, there exists $d_K(x) \in L^4(K)$ such that $|u_n| \leq d_K(x)$ for a.e. $x \in K$ and

$$u_n^3 \varphi \leq |u_n|^3 |\varphi| \leq d_K(x)^3 |\varphi| \quad for\ a.e.\ x \in K.$$

By Lebesgue's dominated convergence theorem, it yields

$$\int_K u_n^3 \varphi dx \to \int_K u_0^3 \varphi dx. \tag{3.18}$$

By (3.16)–(3.18), we obtain

$$\Phi^{'}(u_n, v_n)(\varphi, \psi) \to \Phi^{'}(u_0, v_0)(\varphi, \psi) \tag{3.19}$$

and $\Phi^{'}(u_0, v_0) = 0$. Thus (u_0, v_0) is a critical point of Φ. □

Lemma 3.3. *([24, Lemma 1.21]) If u_n is bounded in $H^2(\mathbb{R}^N)$ and*

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \to 0 \ \ as \ \ n \to \infty, \tag{3.20}$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$.

Lemma 3.4. *Assume that $\{\mathbf{u}_n\}$ is a PS sequence constrained on $\mathcal{N}$ and*

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |\mathbf{u}_n|^2 dx = \sup_{z \in \mathbb{R}^N} (\int_{B(z,1)} |u_n|^2 dx + \int_{B(z,1)} |v_n|^2 dx) \to 0. \tag{3.21}$$

Then $\|\mathbf{u}_n\| \to 0$.

Proof. Since $\{\mathbf{u}_n\} \in \mathcal{N}$ and thus

$$\|\mathbf{u}_n\| = \int_{\mathbb{R}^N} (u_n^4 + v_n^4) + 2\beta \int_{\mathbb{R}^N} u_n^2 v_n^2.$$

From Lemma 3.3, we have that $u_n \to 0, v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. By Hölder's inequality, it follows that

$$\int_{\mathbb{R}^N} (u_n^4 + v_n^4) + 2\beta \int_{\mathbb{R}^N} u_n^2 v_n^2 \to 0,$$

and hence $\|\mathbf{u}_n\| \to 0$. □

4. Proof of main results

System (1.3) has two kinds of semi-trivial solutions of the form $(u,0)$ and $(0,v)$. So we take $\mathbf{u}_1 = (U_1,0)$ and $\mathbf{v}_2 = (0,V_2)$, where U_1 and V_2 are respectively ground state solutions of the equations

$$\triangle^2 f + \lambda_i f = f^3, \quad i = 1,2$$

in $H^2(\mathbb{R}^N)$ which are radially symmetric(see [15]). Moreover, if we denote w a ground state solution of (4.1)

$$\triangle^2 w + w = w^3, \tag{4.1}$$

by scaling we have

$$U_1(x) = \sqrt{\lambda_1}\, w(\sqrt[4]{\lambda_1}\, x), \quad V_2(x) = \sqrt{\lambda_2}\, w(\sqrt[4]{\lambda_2}\, x). \tag{4.2}$$

Thus two kinds of semi-trivial solutions of (1.3) are respectively $\mathbf{u}_1 = (U_1,0)$ and $\mathbf{v}_2 = (0,V_2)$.

Definition 4.1. *We define the two constants related to U_1 and V_2 as follows:*

$$S_1^2 := \inf_{\varphi\in H^2(\mathbb{R}^N)\backslash\{0\}} \frac{\|\varphi\|_2^2}{\int_{\mathbb{R}^N} U_1^2\varphi^2}, \quad S_2^2 := \inf_{\varphi\in H^2(\mathbb{R}^N)\backslash\{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}^N} V_2^2\varphi^2}, \tag{4.3}$$

and

$$\Lambda^+ = \max\{S_1^2, S_2^2\}, \ \Lambda^- = \min\{S_1^2, S_2^2\}.$$

Proposition 4.1. *i). If $0 < \beta < \Lambda^-$, then $\mathbf{u}_1, \mathbf{v}_2$ are strict local minimum elements of Φ constrained on $\mathcal{N}$.*

ii). If $\beta > \Lambda^+$, then $\mathbf{u}_1, \mathbf{v}_2$ are saddle points of Φ constrained on $\mathcal{N}$. Moreover

$$\inf_{\mathcal{N}} \Phi(\mathbf{u}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}, \tag{4.4}$$

Proof. Since the proof is similar to [5], we omit it. □

Next, we will see that the infimum of Φ constrained on the Nehari manifold $\mathcal{N}$ is attained under appropriate parameter conditions. We also give the existence of a mountain pass critical point.

Proof. We first give the proof of Theorem 2.1 (*i*).

By Lemma 3.1, there exists a bounded PS sequence $\{\mathbf{u}_n\} \subset \mathcal{N}$ of Φ, i.e.

$$\Phi(\mathbf{u}_n) \to c := \inf_{\mathcal{N}} \Phi \ \text{ and } \ \Phi'_{\mathcal{N}}(\mathbf{u}_n) \to 0.$$

We can assume that the sequence $\{\mathbf{u}_n\}$ possesses a subsequence such that

$$\begin{aligned} &\mathbf{u}_n \rightharpoonup \widetilde{\mathbf{u}} \ \text{ in } \mathbb{H}, \\ &\mathbf{u}_n \to \widetilde{\mathbf{u}} \ \text{ in } L^p_{loc}(\mathbb{R}^N) \times L^p_{loc}(\mathbb{R}^N) \text{ for } 2 \le p < 2^*, \\ &\mathbf{u}_n \to \widetilde{\mathbf{u}} \ \text{ for a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Suppose that

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\mathbf{u}_n|^2 dx = \sup_{z\in\mathbb{R}^N} (\int_{B(z,1)} |u_n|^2 dx + \int_{B(z,1)} |v_n|^2 dx) \to 0.$$

From Lemma 3.4, we have $\mathbf{u}_n \to 0$. This contradicts with $\mathbf{u}_n \in \mathcal{N}$. In view of Lions' Lemma, there exists $y_n \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2 dx > \delta \ \text{ or } \ \liminf_{n\to\infty} \int_{B(y_n,1)} |v_n|^2 dx > \delta.$$

Without loss of generality, we assume that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2 dx > \delta.$$

For each $y_n \subset \mathbb{R}^N$, we can find $z_n \subset \mathbb{Z}^N$ such that $B(y_n, 1) \subset B(z_n, 1 + \sqrt{N})$, and thus

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} |u_n|^2 dx \geq \liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2 dx > \delta. \tag{4.5}$$

If z_n is bounded in $\mathbb{Z}^N$, by $u_n \to \widetilde{u}$ in $L^2_{loc}(\mathbb{R}^N)$, it follows that $\widetilde{u} \neq 0$. We assume that z_n is unbounded in $\mathbb{Z}^N$. Define $\overline{u}_n = u_n(\cdot + z_n)$ and $\overline{v}_n = v_n(\cdot + z_n)$. For any compact set K, up to a subsequence, we have

$$\begin{aligned}
&\overline{\mathbf{u}}_n \rightharpoonup \overline{\mathbf{u}} \ \text{ in } \mathbb{H},\\
&\overline{\mathbf{u}}_n \to \overline{\mathbf{u}} \ \text{ in } L^p(K) \times L^p(K) \text{ for } 2 \leq p < 2^*,\\
&\overline{\mathbf{u}}_n \to \overline{\mathbf{u}} \ \text{ for a.e. } x \in \mathbb{R}^N,
\end{aligned}$$

where $\overline{\mathbf{u}} = (\overline{u}, \overline{v})$. From (4.5), we have that

$$\liminf_{n\to\infty} \int_{B(0,1+\sqrt{N})} |\overline{u}_n|^2 dx > \delta,$$

and thus $\overline{\mathbf{u}} = (\overline{u}, \overline{v}) \neq (0, 0)$.

From Lemmas 3.1 and 3.2, we notice that $\overline{\mathbf{u}}_n, \overline{\mathbf{u}} \in \mathcal{N}$ and $\mathbf{u}_n$ is PS sequence for Φ on $\mathcal{N}$. Moreover, by Fatou's Lemma, we obtain the following:

$$c = \liminf_{n\to\infty} \Phi(\mathbf{u}_n) = \liminf_{n\to\infty} \Phi_{\mathcal{N}}(\mathbf{u}_n) \geq \Phi_{\mathcal{N}}(\overline{\mathbf{u}}) = \Phi(\overline{\mathbf{u}}).$$

Hence $\Phi(\overline{u}, \overline{v}) = c$ and $(\overline{u}, \overline{v}) \neq (0, 0)$ is a ground state solution of the system (1.3).

In addition, we can conclude that both components of $\overline{\mathbf{u}}$ are non-trivial. In fact, if the second component $\overline{v} \equiv 0$, then $\overline{\mathbf{u}} = (\overline{u}, 0)$. So $\overline{\mathbf{u}} = (\overline{u}, 0)$ is the non-trivial solution of the system (1.3). Hence, we have

$$I_1(\overline{u}) = \Phi(\overline{\mathbf{u}}) < \Phi(\mathbf{u}_1) = I_1(U_1).$$

However, this is a contradiction due to the fact that U_1 is a ground state solution of $\triangle^2 u + \lambda u = u^3$. Similarly, we conclude that the first component $\overline{u} \neq 0$. From Proposition 4.1-(ii) and $\beta > \Lambda^+$, we have

$$\Phi(\overline{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{v}_2)\}. \tag{4.6}$$

Next we give the proof of Theorem 2.1 (*ii*).

From Proposition 4.1-(i), we obtain that $\mathbf{u}_1, \mathbf{v}_2$ are strict local minima Φ of on $\mathcal{N}$. Under this condition, we are able to apply the mountain pass theorem to Φ on $\mathcal{N}$ that provide us with a PS sequence $\mathbf{v}_n \in \mathcal{N}$ such that

$$\Phi(\mathbf{v}_n) \to c := \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \Phi(\gamma(t)),$$

where

$$\Gamma := \{\gamma : [0, 1] \to \mathcal{N} \mid \gamma \text{ is continuous and } \gamma(0) = \mathbf{u}_1, \gamma(1) = \mathbf{v}_2\}.$$

From Lemmas 3.1 and 3.2, we have that $c = \Phi(\mathbf{u}^*)$ and thus $\mathbf{u}^*$ is a critical point of Φ. □

5. Conclusions

In this paper, using Nehari manifold method and concentration compactness theorem, we prove the existence of ground state solution for a coupled system of biharmonic Schrödinger equations. Previous results on ground state solutions are obtained in radially symmetric Sobolev space. We consider ground state solutions in the space without radially symmetric restriction, which can be viewed as extension of previous one.

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Conflict of interest

There is no conflict of interest of the authors.

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