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Research article

Some fixed point results for α-admissible extended $\mathcal{Z}$-contraction mappings in extended rectangular b-metric spaces

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Abstract: In this paper, we introduce α-admissible extended $\mathcal{Z}$-contraction in the extended rectangular b-metric spaces, then we provide some other conditions in Theorem 3.1, which are different from that in Chifu et al. [1], and obtain the existence and uniqueness of fixed point in such spaces. Moreover, some examples are given to show the validity of our main theorems, and we give some corollaries related to our main results. As an application, we apply our main results to solve the existence of solutions for a class of boundary value problems of second order ordinary differential equations.

Keywords: fixed point; extended $\mathcal{Z}$-contraction mappings; extended rectangular b-metric
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Over the past decades, several generalizations of the standard metric have been made by researchers in the field of fixed point theory. Distinct types generalized metric spaces possess different properties, these spaces play important roles for the corresponding fixed point theorems by all kinds of contractions. In 1993, Czerwik [2] introduced and studied b-metric spaces, which is an interesting metric-type space. Indeed, this notion was considered earlier by different authors, e.g. Bakhtin [3], Berinde [4] and so on. Some fixed point results in b-metric spaces were studied by many investigators (for example, see [5–7] and references therein). In 2000, Branciari [8] defined a generalized metric by replacing the triangular inequality with more general inequality, namely, quadrilateral inequality. Note that the generalized metric space is also announced as rectangular metric space or Branciari distance space. In 2017, Kamran et al. [9] presented a type of generalized b-metric space and termed it as extended b-metric spaces, by replacing the coefficient of b-metric with binary function. These kinds of new spaces have inspired many authors, they got several fixed point results via certain contractive conditions in extended b-metric spaces (for example, see [10–13] and references therein). In 2019, Asim et al. [14] proposed a new type of metric space namely extended rectangular b-metric space, this idea originated from combining [8] with [9]. They got some fixed point results in such spaces and applied these results to solving the Fredholm integral equation. Recently, a few scholars have studied extended rectangular b-metric spaces (for example, see [15–19] and references therein).

In 2015, an interesting results were raised by Khojasteh et al. [20] in which the notion of the $\mathcal{Z}$-contraction produced by simulation functions was defined. They proved the existence and uniqueness of the fixed point of $\mathcal{Z}$-contraction mappings in complete metric spaces. As a new generalization of Banach contraction, it unified lots of famous nonlinear contractions in the fixed point theory. In recent years, many investigators have studied such contraction conditions (for example, see [21–24] and references therein). Very recently, Chifu et al. [1] presented the notion of an admissible extended $\mathcal{Z}$-contraction mappings and attained the fixed point results of an admissible extended $\mathcal{Z}$-contraction mappings in the setting of extended b-metric spaces.

Inspired by the above research results, we decided to further investigate the fixed point theory in extended rectangular b-metric spaces. In this paper, we introduce α-admissible extended $\mathcal{Z}$-contraction mappings in extended rectangular b-metric spaces, we utilize conditions different from Chifu et al. [1] and get some fixed point results in such spaces. Finally, we give some examples and corollaries related to our main results.

2. Preliminaries

In the beginning, we recall the basic definitions of some metric-type spaces, which will be used in the following.

In 1993, Czerwik [2] formally proposed b-metric spaces.

Definition 2.1. [2] Let X be a non-empty set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if it satisfies the following conditions:
$(d1)$ $d(x, y) = 0 \Leftrightarrow x = y$;
$(d2)$ $d(x, y) = d(y, x)$, for all $x, y \in X$;
$(d3)$ $d(x, y) \leqslant s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.
Then (X, d) is said to be a b-metric space, and s is called the coefficient of b-metric.

In 2000, Branciari [8] defined a generalized metric by replacing the triangular inequality with quadrilateral inequality.

Definition 2.2. [8] Let X be a non-empty set. A function $d_r : X \times X \to [0, \infty)$ is said to be a rectangular metric, for all $x, y \in X$ and all distinct $z, w \in X \setminus \{x, y\}$, if it satisfies the following conditions:
$(d1)$ $d_r(x, y) = 0 \Leftrightarrow x = y$;
$(d2)$ $d_r(x, y) = d_r(y, x)$, for all $x, y \in X$;
$(d3)$ $d_r(x, y) \leqslant d_r(x, z) + d_r(z, w) + d_r(w, y)$.
Then (X, d_r) is said to be a rectangular metric space.

In 2017, Kamran et al. [9] presented extended b-metric space by replacing the coefficient of b-metric with binary function θ.

Definition 2.3. [9] Let X be a non-empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is said to be an extended b-metric, if it satisfies the following conditions:
$(d_\theta 1)$ $d_\theta(x, y) = 0 \Leftrightarrow x = y$;
$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$, for all $x, y \in X$;
$(d_\theta 3)$ $d_\theta(x, y) \leqslant \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)]$, for all $x, y, z \in X$.
Then (X, d_θ) is said to be an extended b-metric space with $\theta(x, y)$.

In 2019, Asim et al. [14] defined a new type of metric space namely extended rectangular b-metric space, which can be considered as a generalization of extended b-metric space, this idea originated from combining [8] with [9].

Definition 2.4. [14] Let X be a non-empty set and $\xi : X \times X \to [1, \infty)$. A function $d_\xi : X \times X \to [0, \infty)$ is said to be an extended rectangular b-metric, for all $x, y \in X$ and all distinct $z, w \in X \setminus \{x, y\}$, if d_ξ satisfies the following conditions:
$(d_\xi 1)$ $d_\xi(x, y) = 0 \Leftrightarrow x = y$;
$(d_\xi 2)$ $d_\xi(x, y) = d_\xi(y, x)$;
$(d_\xi 3)$ $d_\xi(x, y) \leqslant \xi(x, y)[d_\xi(x, z) + d_\xi(z, w) + d_\xi(w, y)]$.
Then (X, d_ξ) is said to be an extended rectangular b-metric space.

Remark 2.1. If $\theta(x, y) = s$ for $s \geqslant 1$, then an extended b-metric reduced to a b-metric space [2]. If $\xi(x, y) = s$ for $s \geqslant 1$, then an extended rectangular b-metric becomes a rectangular b-metric [25]. The relationship between these types of metric spaces can be found in [14].

Example 2.1. [14] Let $X = \{1, 2, 3, 4, 5\}$. Define $\xi : X \times X \to [1, \infty)$ by

$$\xi(x, y) = x + y + 1, \text{for all } x, y \in X.$$

Define $d_\xi : X \times X \to [0, \infty)$ by

$$\begin{aligned} d_\xi(x, x) &= 0, \text{ for all } x \in X; \\ d_\xi(x, y) &= d_\xi(y, x), \text{ for all } x, y \in X; \\ d_\xi(1, 3) &= d_\xi(2, 5) = 70, \ d_\xi(1, 4) = 1000 \text{ and } d_\xi(1, 5) = 2000; \\ d_\xi(1, 2) &= d_\xi(2, 3) = d_\xi(3, 4) = 60, \ d_\xi(3, 5) = d_\xi(4, 5) = d_\xi(2, 4) = 400. \end{aligned}$$

It's obvious that d_ξ is an extended rectangular b-metric.

Remark 2.2. Note that, a b-metric is not a continuous function in general. For more details, refer to Kamran et al. [9]. So, any combination of b-metric, including the extended b-metric, extended rectangular b-metric is not continuous. For this reason, in the proofs, we do not use the continuity of the distance function.

Some topological properties of extended rectangular b-metric spaces, such as Cauchy sequence, convergence and completeness can be defined as follows:

Definition 2.5. [14] Let (X, d_ξ) be an extended rectangular b-metric space.
(1) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} d_\xi(x_n, x_{n+p}) = 0$, for all $p \in \mathbb{N}$;
(2) A sequence $\{x_n\}$ in X is said to be convergent to x if $\lim_{n\to\infty} d_\xi(x_n, x) = 0$;
(3) (X, d_ξ) is said to be complete if every Cauchy sequence in X convergent to some point in X.

Lemma 2.1. [14] Let (X, d_ξ) be an extended rectangular b-metric space and $\{x_n\}$ be a Cauchy sequence such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ converges at most one point.

In 2015, Khojasteh et al. [20] defined a new family of contractions by the following simulation functions.

Definition 2.6. [20] Let $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a given mapping. Then ζ is called a simulation function if it satisfies the following conditions:
(ζ1) $\zeta(0, 0) = 0$;
(ζ2) $\zeta(u, v) < v - u$, for $u, v > 0$;
(ζ3) If $\{u_n\}, \{v_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 0$, then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Example 2.2. [20] Let $\zeta_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, 3$, be defined as follows:

(1) $\zeta_1(u, v) = \lambda v - u$ for all $u, v \in \mathbb{R}_+$, where $\lambda \in [0, 1)$;

(2) $\zeta_2(u, v) = \eta(v) - \omega(u)$ for all $u, v \in \mathbb{R}_+$, where $\eta, \omega : [0, \infty)$ are two continuous functions such that $\eta(u) = \omega(u) = 0$ if and only if $u = 0$ and $\eta(u) < u \leqslant \omega(u)$ for all $u > 0$;

(3) $\zeta_3(u, v) = v - \varphi(v) - u$ for all $u, v \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\varphi(u) = 0$ if and only if $u = 0$.

Definition 2.7. [20] Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\zeta(d(Tx, Ty), d(x, y)) \geqslant 0,$$

where $x, y \in X$, with $x \neq y$.

Theorem 2.1. [20] Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.

In 2012, α-admissible mappings firstly introduced by Samet et al. [26]. In 2014, Popescu [27] raised the concepts of α-orbit admissible mappings and triangular α-orbit admissible mappings based on α-admissible mappings. Later, some authors studied some fixed point results by using the concept of triangular α-orbit admissible mappings (for example, see [7, 28]).

Definition 2.8. [27] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}_+$ be a function. Then T is said to be α-orbit admissible if

$$\alpha(x, Tx) \geqslant 1 \Rightarrow \alpha(Tx, T^2x) \geqslant 1, \text{ for all } x \in X.$$

Definition 2.9. [27] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}_+$ be a function. Then T is said to be triangular α-orbit admissible if it satisfies the following conditions:
(i) T is α-orbit admissible;
(ii) $\alpha(x, y) \geqslant 1$ and $\alpha(y, Ty) \geqslant 1 \Rightarrow \alpha(x, Ty) \geqslant 1$, for all $x, y \in X$.
Obviously, every triangular α-orbit admissible mapping is also α-orbit admissible mapping.

In 2020, Chifu et al. [1] attained the fixed point results of α-admissible extended $\mathcal{Z}$-contraction mappings in extended b-metric spaces. Now, we recall the definition of (b)-comparison functions.

Definition 2.10. [29] Let $s \geqslant 1$ be a real number. A function $\phi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function, if it satisfies the following conditions:
(i) ϕ is increasing;
(ii) There exists a convergent nonnegative series $\sum\limits_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leqslant as^k\phi^k(t)+v_k$, for all $k \geqslant k_0$ and $t \geqslant 0$, where $k_0 \in \mathbb{N}, a \in [0, 1)$.

We denotes the collection of all (b)-comparison functions by Φ_b.

Definition 2.11. [1] Let (X, d_θ) be an extended b-metric space and $\theta : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called an admissible extended $\mathcal{Z}$-contraction if there exists a $\zeta \in \mathcal{Z}$ such that

$$\zeta(\alpha(x, y)d_\theta(Tx, Ty), \phi(M(x, y))) \geqslant 0, \text{ for all } x, y \in X,$$

where $\phi \in \Phi_b$ and

$$M(x, y) = \max\{d_\theta(x, y), d_\theta(x, Tx), d_\theta(y, Ty)\}.$$

Theorem 2.2. [1] Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be a mapping. Suppose there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$, $q_n > 1$ for all $n \in \mathbb{N}$ such that $\theta(x_n, x_m) < q_n$ for all $m > n$. If T is an admissible extended $\mathcal{Z}$-contraction satisfying
(i) T is triangular α-orbital admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;
(iii) T is continuous,
then T has a fixed point x, and we have $T^n y \to x$ for each $y \in X$.

In 1974, Ćirić [30] considered the concept of orbit and proved some fixed point results.

Definition 2.12. [30] Let (X, d_ξ) be an extended rectangular b-metric space and T be a given mapping. We define

$$O(x; \infty) = \{x, Tx, \cdots, T^n x, \cdots\}, \text{ for all } x \in X,\ n \in \mathbb{N}.$$

Then we call the set $O(x; \infty)$ is the orbit of T at x, in short $O(x)$.

Definition 2.13. [14] Let (X, d_ξ) be an extended rectangular b-metric space, $\{x_n\} \subset O(x_0)$. A mapping $T : X \to X$ is called orbitally continuous if $\lim\limits_{k\to\infty} x_{n_k} = x$ for some $x \in X$ implies that $\lim\limits_{k\to\infty} T(x_{n_k}) = Tx$. Moreover, (X, d_ξ) is called T-orbitally complete if every Cauchy sequence which is obtained in $O(x)$ converges to some point in X.

3. Main results

In this section, we prove that the results of Chifu et al. [1] are still available if we replace conditions of Theorem 2.2 with some new conditions, and we extend the notion of admissible extended $\mathcal{Z}$-contraction to extended rectangular b-metric spaces.

Let Ψ be the set of all increasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following condition:

$$\lim_{n\to\infty} \psi^n(t) = 0, \text{ for all } t > 0.$$

Remark 3.1. It is obvious that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Definition 3.1. Let (X, d_ξ) be an extended rectangular b-metric space and $\xi : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called an α-admissible extended $\mathcal{Z}$-contraction, if there exists a $\zeta \in \mathcal{Z}$ such that

$$\zeta(\alpha(x, y)d_\xi(Tx, Ty), \psi(M(x, y))) \geqslant 0 \tag{3.1}$$

for $x, y \in X$, where $\psi \in \Psi$ and

$$M(x, y) = \max\{d_\xi(x, y), d_\xi(Tx, x), d_\xi(y, Ty)\}.$$

Remark 3.2. (a) Note that ψ in (3.1) is weaker than ϕ in Definition 2.11. Clearly, every (b)-comparison function is a member of Ψ, so there must be $\Phi_b \subseteq \Psi$, i.e., we expand the scope of the set Φ_b;
(b) If T is an α-admissible extended $\mathcal{Z}$-contraction in an extended rectangular b-metric space, then we have

$$\alpha(x, y)d_\xi(Tx, Ty) \leqslant \psi(M(x, y)), \text{ for all } x, y \in X. \tag{3.2}$$

In what follows we shall express the main theorem of this paper.

Theorem 3.1. Let (X, d_ξ) be an extended rectangular b-metric space and $T : X \to X$ be an α-admissible extended $\mathcal{Z}$-contraction mappings. If the following conditions hold:
(i) T is an $\alpha-$orbital admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(x_0, T^2x_0) \geqslant 1$;
(iii) (X, d_ξ) is T-orbitally complete;
(iv) T is orbitally continuous;
(v) For $x_0 \in X$ in (ii), we have $\limsup_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)}\xi(x_{n+1}, x_{n+p}) < 1$, where $t > 0$, $p \in \mathbb{N}_+$ and $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$,
then T has a fixed point.

Proof. From (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(x_0, T^2x_0) \geqslant 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. By (i) and mathematical induction, it easily follows that

$$\alpha(x_n, x_{n+1}) \geqslant 1, \text{ for all } n \in \mathbb{N}, \tag{3.3}$$

and

$$\alpha(x_n, x_{n+2}) \geqslant 1, \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

If there exist some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $Tx_{n_0} = x_{n_0}$, so x_{n_0} is a fixed point of T, the proof is completed. So, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, i.e., $d_\xi(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Now, we show that $x_m \neq x_n$ for all $m \neq n$. Suppose that $x_m = x_n$ for some $m < n$, so we get $x_{m+1} = Tx_m = Tx_n = x_{n+1}$. By this way, we can easily obtain $x_{n+k} = x_{m+k}$ for all $k \in \mathbb{N}$. Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}0 < d_\xi(x_m, x_{m+1}) = d_\xi(x_n, x_{n+1}) = d_\xi(Tx_{n-1}, Tx_n) &\leqslant \alpha(x_{n-1}, x_n) d_\xi(Tx_{n-1}, Tx_n)\\ &\leqslant \psi(M(x_{n-1}, x_n)) < d_\xi((x_{n-1}, x_n)) < d_\xi(x_{n-2}, x_{n-1}) < \cdots\\ &< d_\xi(x_m, x_{m+1}),\end{aligned}$$

this is a contradiction. So next, we can assume that $x_m \neq x_n$ for all $m \neq n$. Set $x = x_n$, $y = x_{n+1}$ in (3.2) and by (3.3), we have

$$d_\xi(x_{n+1}, x_{n+2}) \leqslant \alpha(x_n, x_{n+1}) d_\xi(Tx_n, Tx_{n+1}) \leqslant \psi(M(x_n, x_{n+1})), \tag{3.5}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\{d_\xi(x_n, x_{n+1}), d_\xi(Tx_n, x_n), d_\xi(x_{n+1}, Tx_{n+1})\}\\ &= \max\{d_\xi(x_n, x_{n+1}), d_\xi(x_{n+1}, x_{n+2})\}.\end{aligned}$$

If $M(x_n, x_{n+1}) = d_\xi(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$, by (3.5) and the property of ψ, it follows that

$$d_\xi(x_{n+1}, x_{n+2}) \leqslant \psi(d_\xi(x_{n+1}, x_{n+2})) < d_\xi(x_{n+1}, x_{n+2}), \tag{3.6}$$

which is a contradiction. So $M(x_n, x_{n+1}) = d_\xi(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then we get

$$d_\xi(x_{n+1}, x_{n+2}) \leqslant \psi(d_\xi(x_n, x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

By induced iteration, we get

$$d_\xi(x_{n+1}, x_{n+2}) \leqslant \psi^{n+1}(d_\xi(x_0, x_1)), \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

Now we apply $x = x_n$, $y = x_{n+2}$ to (3.2) and by (3.4), we have

$$d_\xi(x_{n+1}, x_{n+3}) \leqslant \alpha(x_n, x_{n+2}) d_\xi(Tx_n, Tx_{n+2}) \leqslant \psi(M(x_n, x_{n+2})), \tag{3.8}$$

where

$$\begin{aligned}M(x_n, x_{n+2}) &= \max\{d_\xi(x_n, x_{n+2}), d_\xi(Tx_n, x_n), d_\xi(x_{n+2}, Tx_{n+2})\}\\ &= \max\{d_\xi(x_n, x_{n+2}), d_\xi(x_n, x_{n+1})\}.\end{aligned}$$

Now if $M(x_n, x_{n+2}) = d_\xi(x_n, x_{n+1})$ for some $n \in \mathbb{N}$, then by (3.7) and (3.8), we get

$$d_\xi(x_{n+1}, x_{n+3}) \leqslant \psi(d_\xi(x_n, x_{n+1})) \leqslant \psi^{n+1}(d_\xi(x_0, x_1)) < \psi^n(d_\xi(x_0, x_1)). \tag{3.9}$$

If $M(x_n, x_{n+2}) = d_\xi(x_n, x_{n+2})$, then by (3.8), it follows that

$$d_\xi(x_{n+1}, x_{n+3}) \leqslant \psi(d_\xi(x_n, x_{n+2})),$$

so

$$d_\xi(x_{n+1}, x_{n+3}) \leqslant \psi^{n+1}(d_\xi(x_0, x_2)) < \psi^n(d_\xi(x_0, x_2)). \tag{3.10}$$

Let $d^* = \max\{d_\xi(x_0, x_1), d_\xi(x_0, x_2)\}$, combine (3.9) with (3.10), then

$$d_\xi(x_{n+1}, x_{n+3}) < \psi^n(d^*). \tag{3.11}$$

Now we claim that $\{x_n\}$ is a Cauchy sequence. We discuss the following two cases:
Case 1. $p = 2k + 1$ (p is odd), for any $k \geqslant 1$. Now by ($d_\xi 3$) and (3.7), for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
d_\xi(x_n, x_{n+2k+1}) &\leqslant \xi(x_n, x_{n+2k+1})[(d_\xi(x_n, x_{n+1}) + d_\xi(x_{n+1}, x_{n+2}) + d_\xi(x_{n+2}, x_{n+2k+1})] \\
&= \xi(x_n, x_{n+2k+1})[d_\xi(x_n, x_{n+1}) + d_\xi(x_{n+1}, x_{n+2})] + \xi(x_n, x_{n+2k+1})d_\xi(x_{n+2}, x_{n+2k+1}) \\
&\leqslant \xi(x_n, x_{n+2k+1})(d_n + d_{n+1}) + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1})(d_{n+2} + d_{n+3}) \\
&\quad + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1})d_\xi(x_{n+2}, x_{n+2k+1}) \\
&\leqslant \xi(x_n, x_{n+2k+1})(d_n + d_{n+1}) + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1})(d_{n+2} + d_{n+3}) + \cdots \\
&\quad + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1}) \cdots \xi(x_{n+2k-2}, x_{n+2k+1})(d_{n+2k-2} + d_{n+2k-1}) \\
&\quad + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1}) \cdots \xi(x_{n+2k-2}, x_{n+2k+1})d_\xi(x_{n+2k}, x_{n+2k+1}) \\
&\leqslant \xi(x_n, x_{n+2k+1})(\psi^n(G_0) + \psi^{n+1}(G_0)) + \xi(x_{n+2}, x_{n+2m+1})(\psi^{n+2}(G_0) + \psi^{n+3}(G_0)) + \cdots \\
&\quad + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1}) \cdots \xi(x_{n+2k-2}, x_{n+2k+1})(\psi^{n+2k-2}(G_0) + \psi^{n+2k-1}(G_0)) \\
&\quad + \xi(x_n, x_{n+2k+1})\xi(x_{n+2}, x_{n+2k+1}) \cdots \xi(x_{n+2k-2}, x_{n+2k+1})\psi^{n+2k}(G_0) \\
&\leqslant \xi(x_0, x_{n+2k+1})\xi(x_1, x_{n+2k+1})\xi(x_2, x_{n+2k+1}) \cdots \xi(x_n, x_{n+2k+1})[\psi^n(G_0) \\
&\quad + \xi(x_{n+1}, x_{n+2k+1})\psi^{n+1}(G_0)] + \xi(x_0, x_{n+2k+1})\xi(x_1, x_{n+2k+1})\xi(x_2, x_{n+2k+1}) \cdots \\
&\quad \times \xi(x_{n+2}, x_{n+2k+1})[\psi^{n+2}(G_0) + \xi(x_{n+3}, x_{n+2k+1})\psi^{n+3}(G_0)] + \cdots + \xi(x_0, x_{n+2k+1}) \\
&\quad \times \xi(x_1, x_{n+2k+1})\xi(x_2, x_{n+2k+1}) \cdots \xi(x_{n+2k-2}, x_{n+2k+1})[\psi^{n+2k-2}(G_0) \\
&\quad + \xi(x_{n+2k-1}, x_{n+2k+1})\psi^{n+2k-1}(G_0)] + \xi(x_0, x_{n+2k+1})\xi(x_1, x_{n+2k+1})\xi(x_2, x_{n+2k+1}) \\
&\quad \times \cdots \xi(x_{n+2k}, x_{n+2k+1})\psi^{n+2k}(G_0) \\
&= \sum_{i=n}^{n+2k} \psi^i(G_0) \prod_{j=0}^{i} \xi(x_j, x_{n+2k+1}),
\end{aligned} \tag{3.12}$$

where $d_n = d_\xi(x_n, x_{n+1})$ and $\psi^n(G_0) = \psi^n(d_\xi(x_0, x_1))$ for some $G_0 = d_\xi(x_0, x_1)$. Let

$$S_n = \sum_{i=0}^{n} \psi^i(G_0) \prod_{j=0}^{i} \xi(x_j, x_{n+2k+1})$$

for all $n \in \mathbb{N}$. From (3.12), we can deduce that

$$d_\xi(x_n, x_{n+2k+1}) \leqslant S_{n+2k} - S_{n-1}. \tag{3.13}$$

Consider the series $\sum_{i=0}^{\infty} \psi^i(G_0) \prod_{j=0}^{i} \xi(x_j, x_{n+2k+1})$. Let $u_n = \psi^n(G_0) \prod_{j=0}^{n} \xi(x_j, x_{n+2k+1})$, then we have

$$\frac{u_{n+1}}{u_n} = \frac{\psi^{n+1}(G_0) \prod_{j=0}^{n+1} \xi(x_j, x_{n+2k+1})}{\psi^n(G_0) \prod_{j=0}^{n} \xi(x_j, x_{n+2k+1})} = \frac{\psi^{n+1}(G_0)}{\psi^n(G_0)} \xi(x_{n+1}, x_{n+2k+1}).$$

In view of (v), by the ratio test of positive series, we conclude that the series $\sum_{i=0}^{\infty} \psi^i(G_0) \prod_{j=0}^{i} \xi(x_j, x_{n+2k+1})$ is convergent. Consequently, let $n \to \infty$ in (3.13), we get that

$$d_\xi(x_n, x_{n+p}) \to 0.$$

Case 2. $p = 2k$ (p is even) for any $k \geqslant 1$. Combine $(d_\xi 3)$, (3.7) with (3.11), we obtain

$$\begin{aligned}
d_\xi(x_n, x_{n+2k}) &\leqslant \xi(x_n, x_{n+2k})[(d_\xi(x_n, x_{n+1}) + d_\xi(x_{n+1}, x_{n+2}) + d_\xi(x_{n+2}, x_{n+2k})] \\
&= \xi(x_n, x_{n+2k})[d_\xi(x_n, x_{n+1}) + d_\xi(x_{n+1}, x_{n+2})] + \xi(x_n, x_{n+2k}) d_\xi(x_{n+2}, x_{n+2k}) \\
&\leqslant \xi(x_n, x_{n+2k})(d_n + d_{n+1}) + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k})(d_{n+2} + d_{n+3}) \\
&\quad + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k}) d_\xi(x_{n+3}, x_{n+2k}) \\
&\leqslant \xi(x_n, x_{n+2k})(d_n + d_{n+1}) + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k})(d_{n+2} + d_{n+3}) + \cdots \\
&\quad + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k}) \cdots \xi(x_{n+2k-3}, x_{n+2k})(d_{n+2k-3} + d_{n+2k-2}) \\
&\quad + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k}) \cdots \xi(x_{n+2k-3}, x_{n+2k}) d_\xi(x_{n+2k-2}, x_{n+2k}) \\
&< \xi(x_n, x_{n+2k})(\psi^n(G_0) + \psi^{n+1}(G_0)) + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k})(\psi^{n+2}(G_0) + \psi^{n+3}(G_0)) + \cdots \\
&\quad + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k}) \cdots \xi(x_{n+2k-3}, x_{n+2k})(\psi^{n+2k-3}(G_0) + \psi^{n+2k-2}(G_0)) \\
&\quad + \xi(x_n, x_{n+2k})\xi(x_{n+2}, x_{n+2k}) \cdots \xi(x_{n+2k-3}, x_{n+2k})\psi^{n+2k-1}(d^*) \\
&\leqslant \xi(x_0, x_{n+2k})\xi(x_1, x_{n+2k})\xi(x_2, x_{n+2k}) \cdots \xi(x_n, x_{n+2k})[\psi^n(G_0) \\
&\quad + \xi(x_{n+1}, x_{n+2k})\psi^{n+1}(G_0)] + \xi(x_0, x_{n+2k})\xi(x_1, x_{n+2k})\xi(x_2, x_{n+2k}) \cdots \\
&\quad \times \xi(x_{n+2}, x_{n+2k})[\psi^{n+2}(G_0) + \xi(x_{n+3}, x_{n+2k})\psi^{n+3}(G_0)] + \cdots + \xi(x_0, x_{n+2k}) \\
&\quad \times \xi(x_1, x_{n+2k})\xi(x_2, x_{n+2k}) \cdots \xi(x_{n+2k-3}, x_{n+2k})[\psi^{n+2k-3}(G_0) \\
&\quad + \xi(x_{n+2k-2}, x_{n+2k})\psi^{n+2k-2}(G_0)] + \xi(x_0, x_{n+2k})\xi(x_1, x_{n+2k})\xi(x_2, x_{n+2k}) \\
&\quad \times \cdots \xi(x_{n+2k-2}, x_{n+2k})\psi^{n+2k-1}(d^*) \\
&< \sum_{i=n}^{n+2k-1} \psi^i(d^*) \prod_{j=0}^{i} \xi(x_j, x_{n+2k}),
\end{aligned} \tag{3.14}$$

where $\psi^n(G_0) = \psi^n(d_\xi(x_0, x_1))$, $d_n = d_\xi(x_n, x_{n+1})$ and $d^* = \max\{d_\xi(x_0, x_1), d_\xi(x_0, x_2)\}$. Choose for all $n \in \mathbb{N}$,

$$R_n = \sum_{i=0}^{n} \psi^i(d^*) \prod_{j=0}^{i} \xi(x_j, x_{n+2k}).$$

From (3.14), we can deduce that

$$d_\xi(x_n, x_{n+2k}) < R_{n+2k-1} - R_{n-1}. \tag{3.15}$$

Consider the series $\sum_{i=0}^{\infty} \psi^i(d^*) \prod_{j=0}^{i} \xi(x_j, x_{n+2k})$. Let $v_n = \psi^n(d^*) \prod_{j=0}^{n} \xi(x_j, x_{n+2k})$, then we have

$$\frac{v_{n+1}}{v_n} = \frac{\psi^{n+1}(d^*) \prod_{j=0}^{n+1} \xi(x_j, x_{n+2k})}{\psi^n(d^*) \prod_{j=0}^{n} \xi(x_j, x_{n+2k})} = \frac{\psi^{n+1}(d^*)}{\psi^n(d^*)} \xi(x_{n+1}, x_{n+2k}).$$

In similar way, we get that the series $\sum_{i=0}^{\infty}\psi^i(d^*)\prod_{j=0}^{i}\xi(x_j,x_{n+2k})$ is convergent. So, take $n\to\infty$ in (3.15), it must be

$$d_\xi(x_n,x_{n+p})\to 0.$$

In the both cases, we have

$$\lim_{n\to\infty}d_\xi(x_n,x_{n+p})=0,\ \text{for all } p\in\mathbb{N}.$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since (X,d_ξ) is T-orbitally complete, then there exists $x\in X$ such that

$$\lim_{n\to\infty}d_\xi(x_n,x)=0. \tag{3.16}$$

In view of the orbitally continuity of T, thus, there exists $\{n_k\}\subset\mathbb{N}$ such that $\lim_{k\to\infty}d_\xi(Tx_{n_k},Tx)\to 0$, and by $(d_\xi3)$, it follows that

$$d_\xi(x,Tx)\leqslant\xi(x,Tx)[d_\xi(x,x_{n_k})+d_\xi(x_{n_k},x_{n_k+1})+d_\xi(Tx_{n_k},Tx)].$$

Taking $k\to\infty$ in the both sides of this inequality, then we have

$$d_\xi(x,Tx)=0.$$

So $Tx=x$, that is T has a fixed point. □

Remark 3.3. We improve, extend the previous results of Chifu et al. [1], and get the same results in an extended rectangular b-metric space by weaker conditions than some conditions in Theorem 2.2. Notice that triangular α-orbital admissible is replaced by α-orbital admissible and $\alpha(x_0,T^2x_0)\geqslant 1$, the latter is weaker than the former. Moreover, we utilize T-orbitally completeness and orbitally continuity instead of completeness and continuity, respectively. Meanwhile, we give a new condition (v), which is different from the original condition in Theorem 2.2.

Now, we give some examples to demonstrate the validity of Theorem 3.1.

Example 3.1. Let $X=\{1,2,3,4,5\}$. Defined $\xi:X\times X\to[0,\infty)$ and d_ξ by

$$\xi(x,y)=x+y+1,\text{for all } x,y\in X$$

and

$$\begin{aligned}
d_\xi(x,x)&=0,\ \text{for all } x\in X;\\
d_\xi(x,y)&=d_\xi(y,x),\ \text{for all } x,y\in X;\\
d_\xi(1,3)&=d_\xi(2,5)=70,\ d_\xi(1,4)=1000 \text{ and } d_\xi(1,5)=2000;\\
d_\xi(1,2)&=d_\xi(2,3)=d_\xi(3,4)=60,\ d_\xi(3,5)=d_\xi(4,5)=d_\xi(2,4)=400.
\end{aligned}$$

In Example 2.1, we already know that d_ξ is an extended rectangular b-metric. Next, take $T:X\to X$ and $\alpha(x,y)$ as follows:

$$Tx=\begin{cases}1, & \text{if } x=1,3,5;\\ 2, & \text{otherwise}\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}\dfrac{3}{2}, & \text{if } (x,y)\in A;\\ 0, & \text{otherwise,}\end{cases}$$

where $A=\{(1,1),(2,2),(4,2),(5,1),(5,2)\}$. Let $\zeta_1(u,v)=\frac{3}{5}v-u$ and $\psi(t)=\frac{1}{5}t$. We show that T is an α-admissible extended $\mathcal{Z}$-contraction mapping. Indeed, if $(x,y)\notin A$, then (3.1) holds. We consider $(x,y)\in A$, then we have

$$\begin{aligned}\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))&=\frac{3}{25}M(x,y)-\alpha(x,y)d_\xi(Tx,Ty)\\&=\frac{3}{25}\max\{d_\xi(x,y),d_\xi(Tx,x),d_\xi(y,Ty)\}-\frac{3}{2}d_\xi(Tx,Ty).\end{aligned}\tag{3.17}$$

Case 1. $(x,y)=(1,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=0$;
Case 2. $(x,y)=(2,2)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=0$;
Case 3. $(x,y)=(4,2)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=48$;
Case 4. $(x,y)=(5,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=240$;
Case 5. $(x,y)=(5,2)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=150$.

So, it must be $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))\geqslant 0$. Thus, T is an α-admissible extended $\mathcal{Z}$-contraction. T is an $\alpha-$orbital admissible, in fact, if $\alpha(x,Tx)\geqslant 1$, then $(x,Tx)\in A$, by definition of T, it easily follows that $(Tx,T^2x)\in A$, that is $\alpha(Tx,T^2x)\geqslant 1$. Clearly,

Case 1. $x_0=1$, $\alpha(1,T1)=\alpha(1,1)\geqslant 1$, $\alpha(T1,T^21)=\alpha(1,1)\geqslant 1$;
Case 2. $x_0=2$, $\alpha(2,T2)=\alpha(2,2)\geqslant 1$, $\alpha(T2,T^22)=\alpha(2,2)\geqslant 1$;
Case 3. $x_0=4$, $\alpha(4,T4)=\alpha(4,2)\geqslant 1$, $\alpha(T4,T^24)=\alpha(2,2)\geqslant 1$;
Case 4. $x_0=5$, $\alpha(5,T5)=\alpha(5,1)\geqslant 1$, $\alpha(T5,T^25)=\alpha(1,1)\geqslant 1$.

At the same time, there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geqslant 1$, $\alpha(x_0,T^2x_0)\geqslant 1$ and $x_n=1$ for all $n\geqslant 2$. We have

$$\limsup_{n\to\infty}\frac{\psi^{n+1}(t)}{\psi^n(t)}\xi(x_{n+1},x_{n+p})=\limsup_{n\to\infty}\frac{1}{5}[x_{n+1}+x_{n+p}+1]=\frac{3}{5}<1,\ \text{for all } t>0,$$

we can easily prove (X,d_ξ) is T-orbitally complete and T is orbitally continuous. All conditions of Theorem 3.1 are satisfied, 1 and 2 are two fixed points of T.

Example 3.2. Let $X=\mathbb{N}-\{0\}$. Define $d_\xi: X\times X\to[0,\infty)$ by

$$d_\xi(x,y)=\begin{cases}0, & \text{if } x=y;\\ \dfrac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd};\\ \dfrac{1}{y}, & \text{if } y \text{ is even and } x \text{ is odd};\\ 1, & \text{otherwise.}\end{cases}$$

Then (X,d_ξ) is a complete extended rectangular b-metric space with $\xi(x,y)=x+y+1$. It is apparent that (X,d_ξ) satisfies the condition (iii) in Theorem 3.1.

Define the mapping $T : X \to X$ and the function $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \text{ is even;} \\ \dfrac{x+1}{2}, & \text{otherwise} \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} x, & \text{if } (x, y) \in Q; \\ 0, & \text{otherwise,} \end{cases}$$

where $Q = \{(x, y) : |x - y| \leqslant 1\} - \{(x, y) : |x - y| \leqslant 1, \text{if } y \text{ is even and } x \text{ is odd}\}$. Then T satisfies the conditions (i) and (iv) in Theorem 3.1. Take $\psi(t) = \frac{t}{4}$ and $\zeta(u, v) = \frac{1}{2}v - u$, note that there exists $x_0 = 1$ such that T satisfies the conditions (ii) and (v) in Theorem 3.1. Furthermore, we show that T is an α-admissible extended $\mathcal{Z}$-contraction. Indeed, $\alpha(x, y)d_\xi(Tx, Ty) = 0$ for all $x, y \in X$. So, by Theorem 3.1, T has a fixed point $x = 1$.

Now, we provide sufficient conditions for the existence of fixed points of T in Theorem 3.1, but it can't guarantee the uniqueness of the fixed point. In order to assure the uniqueness of the fixed point, consider the following condition:
(vi) For all $x, y \in \text{Fix}(T) \Rightarrow \alpha(x, y) \geqslant 1$, where $\text{Fix}(T)$ is the set of fixed points of T.

Theorem 3.2. Adding (vi) to the conditions of Theorem 3.1, we can obtain the uniqueness of the fixed point of T.

Proof. If there exist two different fixed points x^* and y^* in X, that is $Tx^* = x^*, Ty^* = y^*$. Applying $x = x^*, y = y^*$ to (3.2), and by (vi), then we have

$$d_\xi(x^*, y^*) \leqslant \alpha(x^*, y^*)d_\xi(Tx^*, Ty^*) \leqslant \psi(M(x^*, y^*)) = \psi(d_\xi(x^*, y^*)) < d_\xi(x^*, y^*),$$

this is a contradiction. So $x^* = y^*$, i.e., T has a unique fixed point. □

Now, we use Example 2.1 again to demonstrate the validity of Theorem 3.2.

Example 3.3. Let $X = \{1, 2, 3, 4, 5\}$. Defined $\xi : X \times X \to [0, \infty)$ and d_ξ by

$$\xi(x, y) = x + y + 1, \text{for all } x, y \in X$$

and

$$\begin{aligned} d_\xi(x, x) &= 0, \text{ for all } x \in X; \\ d_\xi(x, y) &= d_\xi(y, x), \text{ for all } x, y \in X; \\ d_\xi(1, 3) &= d_\xi(2, 5) = 70, \ d_\xi(1, 4) = 1000 \text{ and } d_\xi(1, 5) = 2000; \\ d_\xi(1, 2) &= d_\xi(2, 3) = d_\xi(3, 4) = 60, \ d_\xi(3, 5) = d_\xi(4, 5) = d_\xi(2, 4) = 400. \end{aligned}$$

In Example 2.1, we already know that d_ξ is an extended rectangular b-metric. Next, take $T : X \to X$ and $\alpha(x, y)$ as follows:

$$Tx = \begin{cases} 1, & \text{if } x = 1, 2; \\ 2, & \text{otherwise} \end{cases}$$

and

$$\alpha(x,y)=\begin{cases}2, & \text{if } (x,y)\in B;\\ 0, & \text{otherwise,}\end{cases}$$

where $B=\{(1,1),(2,1),(4,1),(5,1)\}$. Let $\zeta_1(u,v)=\frac{1}{2}v-u$ and $\psi(t)=\frac{1}{4}t$. We show that T is an α-admissible extended $\mathcal{Z}$-contraction mapping. Indeed, if $(x,y)\notin B$, then (3.1) holds. we consider $(x,y)\in B$, then we have

$$\begin{aligned}\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))&=\frac{1}{8}M(x,y)-\alpha(x,y)d_\xi(Tx,Ty)\\&=\frac{1}{8}\max\{d_\xi(x,y),d_\xi(Tx,x),d_\xi(y,Ty)\}-2d_\xi(Tx,Ty).\end{aligned}\tag{3.18}$$

Case 1. $(x,y)=(1,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=0$;
Case 2. $(x,y)=(2,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=\frac{15}{2}$;
Case 3. $(x,y)=(4,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=5$;
Case 4. $(x,y)=(5,1)$, then $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))=130$.

So, it must be $\zeta(\alpha(x,y)d_\xi(Tx,Ty),\psi(M(x,y)))\geqslant 0$. Thus, T is an α-admissible extended $\mathcal{Z}$-contraction mapping. T is an $\alpha-$orbital admissible mapping, in fact, if $\alpha(x,Tx)\geqslant 1$, then $(x,Tx)\in B$, by definition of T, it easily follows that $(Tx,T^2x)\in B$, that is $\alpha(Tx,T^2x)\geqslant 1$. Clearly,

Case 1. $x_0=1$, $\alpha(1,T1)=\alpha(1,1)\geqslant 1$, $\alpha(T1,T^21)=\alpha(1,1)\geqslant 1$;
Case 2. $x_0=2$, $\alpha(2,T2)=\alpha(2,1)\geqslant 1$, $\alpha(T2,T^22)=\alpha(1,1)\geqslant 1$.

While $x_0\in X$, $x_n=1$ for all $n\geqslant 2$, $p>0$, we can easily prove (X,d_ξ) is T-orbitally complete and T is orbitally continuous. Meanwhile, for $x_0\in X$, we have

$$\limsup_{n\to\infty}\frac{\psi^{n+1}(t)}{\psi^n(t)}\xi(x_{n+1},x_{n+p})=\limsup_{n\to\infty}\frac{1}{4}[x_{n+1}+x_{n+p}+1]=\frac{3}{4}<1,\ \text{for all } t>0.$$

There exists $x_0=1$ or $x_0=2$, satisfying $\alpha(x_0,Tx_0)\geqslant 1$ and $\alpha(x_0,T^2x_0)\geqslant 1$. All conditions of Theorem 3.2 are satisfied, so T has a unique fixed point $x=1$.

Corollary 3.1. If we replace extended rectangular b-metric space with extend b-metric space in Theorem 3.1, then the conclusion still holds.

Proof. Every extended b-metric space is also an extended rectangular b-metric space, so the Corollary 3.1 holds clearly. □

In Theorem 3.1 and Corollary 3.1, if we replace (*iii*) and (*iv*) with the completeness of (X,d_ξ) and the continuity of T respectively, then we can acquire the following results.

Corollary 3.2. Let (X,d_ξ) be a complete rectangular extended b-metric space and $T:X\to X$ be an α-admissible extended $\mathcal{Z}$-contraction. If the following conditions hold:
(*i*) T is an α-orbital admissible;
(*ii*) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geqslant 1$ and $\alpha(x_0,T^2x_0)\geqslant 1$;
(*iii*) T is continuous;
(*iv*) For $x_0\in X$ in (*ii*), we have $\limsup_{n\to\infty}\frac{\psi^{n+1}(t)}{\psi^n(t)}\xi(x_{n+1},x_{n+p})<1$, where $t>0$, $p\in\mathbb{N}_+$ and $x_n=T^n(x_0)$, for all $n\in\mathbb{N}$,
then T has a fixed point.

Corollary 3.3. Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be an α-admissible extended $\mathcal{Z}$-contraction. If the following conditions hold:
(i) T is an α-orbital admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(x_0, T^2x_0) \geqslant 1$;
(iii) T is continuous;
(iv) For $x_0 \in X$ in (ii), we have $\limsup\limits_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)}\theta(x_{n+1}, x_{n+p}) < 1$, where $t > 0$, $p \in \mathbb{N}_+$ and $x_n = T^n(x_0)$, for all $n \in \mathbb{N}$,
then T has a fixed point.

Corollary 3.4. Let (X, d) be a complete b-metric space with coefficient s and $T : X \to X$ be an α-admissible extended $\mathcal{Z}$-contraction. If the following conditions hold:
(i) T is an α-orbital admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(x_0, T^2x_0) \geqslant 1$;
(iii) T is continuous;
(iv) For $x_0 \in X$ in (ii). Take sequence $\{x_n = T^n x_0\}$, we have $\limsup\limits_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)} < \frac{1}{s}$, for all $t > 0$,
then T has a fixed point.

Corollary 3.5. Let (X, d_ξ) be a complete extended rectangular b-metric space and $T : X \to X$ be an α-admissible extended $\mathcal{Z}$-contraction mapping. If the following conditions hold:
(i) T is triangular $\alpha-$orbital admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;
(iii) T is continuous;
(iv) For $x_0 \in X$ in (ii) such that $\limsup\limits_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)}\xi(x_{n+1}, x_{n+p}) < 1$, where $t > 0$, $p \in \mathbb{N}_+$ and $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$,
then T has a fixed point.

Remark 3.4. Corollary 3.5 is an improvement of Theorem 2.2. On the one hand, extended rectangular b-metric space is a generalization of extended b-metric space; on the other hand, the condition (v) is different from the conditions in Theorem 2.2.

4. Application

In this section, we discuss an application that attributes the solvability of boundary value problem of second order ordinary differential equation:

$$\begin{cases} x''(u) = -f(u, x(u)), \quad u \in [0, 1], \\ x(0) = x(1) = 0. \end{cases} \tag{4.1}$$

Let $X = C([0, 1], \mathbb{R})$ be the set of all real continuous functions defined on $[0, 1]$, endowed with the extended rectangular b-metric

$$d_\xi(x, y) = \max_{u\in[0,1]} (|x(u) - y(u)|)^n, \quad \text{for all } x, y \in X.$$

It is evident that (X, d_ξ) is a complete extended rectangular b-metric space with $\xi(x, y) = 3^{n-1} + x + y$, where $n > 1$. The boundary value problem of (4.1) is equivalent to the following integral equation:

$$x(u) = \int_0^1 G(u,t)f(t,x(t))dt, \quad \text{for all } u \in [0,1],$$

where $G(u,t)$ is the Green function given as

$$G(u,t) = \begin{cases} t(1-u), & 0 \leqslant t \leqslant u \leqslant 1, \\ u(1-t), & 0 \leqslant u \leqslant t \leqslant 1. \end{cases}$$

Define the mapping $T : X \to X$ by

$$Tx(u) = \int_0^1 G(u,t)f(t,x(t))dt, \quad \text{for all } u \in [0,1]$$

and let $\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function.

Theorem 4.1. Consider the integral equation (4.1). Suppose that the following assertions hold:
(i) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous;
(ii) There exist $x_0 \in X$ and $\psi(t) \in \Psi$ such that $\delta(x_0, Tx_0) \geqslant 0$, $\delta(x_0, T^2x_0) \geqslant 0$ and

$$\limsup_{n\to\infty} \frac{\psi^{n+1}(t)}{\psi^n(t)} \xi(x_{n+1}, x_{n+p}) < 1,$$

where $t > 0$, $p \in \mathbb{N}_+$ and $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$;
(iii) For all $u \in [0,1]$ and $x \in X$, $\delta(x, Tx) \geqslant 0$ implies that $\delta(Tx, T^2x) \geqslant 0$;
(iv) If $\delta(x,y) \geqslant 0$, then $|f(t,x(t)) - f(t,y(t))| \leqslant \sqrt[n]{8^{n-1}M(x,y)}$, where

$$M(x,y) = \max\{d_\xi(x,y), d_\xi(x,Tx), d_\xi(y,Ty)\}.$$

Then the integral equation (4.1) has a solution in X.

Proof. We define the function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \delta(x,y) \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

By (i)–(iii), it is clear that the conditions (i)–(iv) in Theorem 3.1 hold.

Case 1. If $\delta(x,y) < 0$, i.e., $\alpha(x,y) = 0$, then $\frac{1}{8}M(x,y) - \alpha(x,y)d_\xi(Tx,Ty) \geqslant 0$.

Case 2. If $\delta(x,y) \geqslant 0$, i.e., $\alpha(x,y) = 1$, by (iv), we have

$$\begin{aligned} \frac{1}{8}M(x,y) - \alpha(x,y)d_\xi(Tx,Ty) &= \frac{1}{8}M(x,y) - d_\xi(Tx,Ty) \\ &= \frac{1}{8}M(x,y) - \max_{u\in[0,1]}(|Tx(u) - Ty(u)|)^n \\ &= \frac{1}{8}M(x,y) - \max_{u\in[0,1]}(|\int_0^1 G(u,t)(f(t,x(t)) - f(t,y(t)))dt|)^n \end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{1}{8}M(x,y) - \max_{u\in[0,1]}\Big(\int_0^1 G(u,t)|f(t,x(t)) - f(t,y(t))|dt\Big)^n \\
&\geqslant \frac{1}{8}M(x,y) - \max_{u\in[0,1]}\Big(\int_0^1 G(u,t)\sqrt[n]{8^{n-1}M(x,y)}dt\Big)^n \\
&\geqslant \frac{1}{8}M(x,y) - \frac{1}{8^{1-n}}M(x,y)\max_{u\in[0,1]}\Big(\int_0^1 G(u,t)dt\Big)^n \\
&= \frac{1}{8}M(x,y) - \frac{1}{8^{1-n}}M(x,y)\Big(\frac{1}{8}\Big)^n \\
&= 0.
\end{aligned}
$$

In the both cases, it must be $\frac{1}{8}M(x,y)-\alpha(x,y)d_\xi(Tx,Ty) \geqslant 0$ for all $x, y \in X$. Now, we take $\psi(t) = \frac{1}{4}t$ and $\zeta(u,v) = \frac{1}{2}v - u$, it is easy to verify that T is an α-admissible extended $\mathcal{Z}$-contraction mapping. By (iv), it is clear the conditions (v) in Theorem 3.1 holds. Therefore, by Theorem 3.1, we can make sure that the boundary value problems (4.1) has a solution. □

Eventually, we give an example to show the applicability of Theorem 4.1.

Example 4.1. Let $f(u, x(u)) = \frac{u}{2}$ for all $u \in [0,1]$. It is obvious that there exist $x_0 = 0$ and $\psi(t) = \frac{t}{4}$ such that the all assertions in Theorems 4.1 hold. So, by Theorem 4.1, the boundary value problem of (4.1) has a solution, which is $x(u) = -\frac{u^3}{12} + \frac{u}{12}$.

5. Conclusions

In this paper, we acquire the existence and uniqueness of the fixed point of α-admissible extended $\mathcal{Z}$-contraction mappings in extended rectangular b-metric spaces and provide some examples to show the validity of our main results. It is obvious that we can take the reasonable auxiliary functions α, ψ, ζ to give some corollary in the various settings (in the context of partially ordered set endowed with a metric, orthogonal set endowed with a metric, cyclic contraction, etc.). Finally, we give an application to the boundary value problems of second order ordinary differential equation.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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