Mathematics
http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 3692–3700.
DOI: 10.3934/math.2022204
Received: 18 August 2021
Accepted: 12 November 2021
Published: 06 December 2021

Research article

Normwise condition numbers of the indefinite least squares problem with multiple right-hand sides

Limin Li*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: 1337488388@qq.com; Tel: +8618419702009.

Abstract: In this paper, we investigate the normwise condition numbers of the indefinite least squares problem with multiple right-hand sides with respect to the weighted Frobenius norm and 2-norm. The closed formulas or upper bounds for these condition numbers are presented, which extend the earlier work for the indefinite least squares problem with single right-hand side. Numerical experiments are performed to illustrate the tightness of the upper bounds.

Keywords: indefinite least squares problem; multiple right-hand sides; normwise condition number
Mathematics Subject Classification: 65F35, 65F20

1. Introduction

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $m \geq n$. The indefinite least squares (ILS) takes the form [3, 4]

$$\min_{x\in\mathbb{R}^n}(b - Ax)^T J(b - Ax), \tag{1.1}$$

where J is a signature matrix,

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = m.$$

In the above symbols, A^T, $\mathbb{R}^{m\times n}$, and I_p stands for the transpose of A, the set of $m \times n$ real matrices, and the identity matrix of order p, respectively. Obviously, the ILS problem reduces to the famous linear least squares problem when $q = 0$. The ILS problem has wide application background in the total least squares problem [18] and the area of optimization known as H^∞ smoothing [9, 16]. Therefore, some authors investigated its numerical algorithms, stability of algorithms, and perturbation analysis [3, 4, 6, 13, 14, 17].

Given a problem, the condition number measures the worst-case sensitivity of its solution to small perturbations in the input data. Combined with backward errors, it provides a (possibly approximate) linear upper bound for the forward error, i.e., the difference between a perturbed solution and the exact solution [10]. Bojanczyk et al. [3] studied the normwise condition number of the ILS problem and presented an upper bound. Li et al. [11] discussed the mixed and componentwise condition numbers of this problem, and derived their explicit expressions and the easily computable upper bounds. Recently, the condition numbers for a linear function of the solution for ILS problem also named as partial condition numbers are studied by Li and Wang [12]. Some results of the paper were recovered by Diao and Zhou [5] by using the dual technique of condition number theory.

The ILS problem with multiple right-hand sides (MILS) was first proposed by Ou and Peng [15] and its definition is

$$\min_{X\in\mathbb{R}^{n\times s}} \operatorname{trace}((B-AX)^T J(B-AX)), \tag{1.2}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times s}$, and J is the signature matrix defined above. Obviously, the MILS problem (1.2) reduces to the ILS problem (1.1) when $s = 1$. In [15], the authors presented the sufficient and necessary conditions of solvability of MILS problem. Yang and Li [19] studied the normwise, mixed and componentwise condition numbers and the corresponding structured condition numbers of the MILS problem. In this paper, we will study the normwise condition number of MILS problem under the weighted norm.

The following weighted Frobenius norm

$$\|(\alpha A, \beta b)\|_F = \sqrt{\alpha^2\|A\|_F^2 + \beta^2\|b\|_2^2}$$

was first used by Gratton for deriving the normwise condition number for the linear least squares problem [8]. Here $\|\diamond\|_F$ denotes the Frobenius norm of a matrix, and $\|\diamond\|_2$ denotes the spectral norm of a matrix or the Euclidean norm of a vector. We will call the latter 2-norm uniformly later in this paper. Subsequently, this kind of norm was used for the partial condition number for the linear least squares problem [1] and the normwise condition number of the truncated singular value solution of a linear ill-posed problem [2]. In this paper, we use the weighted Frobenius norm and weighted 2-norm which defined by

$$\|[A \;\; B]\|_{\mathcal{F}} = \sqrt{\alpha^2\|A\|_F^2 + \beta^2\|B\|_F^2}, \quad \alpha, \beta > 0,$$

and

$$\|[A \;\; B]\|_{\in} = \sqrt{\alpha^2\|A\|_2^2 + \beta^2\|B\|_2^2} \;\; \alpha, \beta > 0.$$

These norms are very flexible since they allow us to monitor the perturbations on A and B. For instance, large values of α (resp., β) enable us to obtain condition number problems where mainly B (resp., A) are perturbed.

2. Preliminaries

The operator vec and the Kronecker product will be of particular importance in what follows. The vec operator stacks the columns of the matrix argument into one long vector. For any matrices $X = [x_{ij}] \in R^{m\times n}$ and $Y \in R^{p\times q}$, the Kronecker product $X \otimes Y$ is defined by $X \otimes Y = [x_{ij}Y] \in R^{mp\times nq}$. It is easy to find that when A is a row vector and B is a column vector,

$$A \otimes B = BA.$$

The following results on Kronecker product are from [7]

$$(X \otimes Y)^T = X^T \otimes Y^T, \ \ \|X \otimes Y\|_2 = \|X\|_2 \|Y\|_2, \tag{2.1}$$

$$\mathrm{vec}(XZY) = (Y^T \otimes X)\mathrm{vec}(Z), \ \mathrm{vec}(X^T) = \Pi_{mn}\mathrm{vec}(X), \tag{2.2}$$

and

$$\Pi_{pm}(Y \otimes X) = (X \otimes Y)\Pi_{nq}, \tag{2.3}$$

where $Z \in R^{n\times p}$, $\Pi_{mn} \in \mathbb{R}^{mn\times mn}$ is the permutation matrix defined by

$$\Pi_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T.$$

Here each $E_{ij} \in \mathbb{R}^{m\times n}$ has entry 1 in position (i, j) and all other entries are zero. Furthermore, we have

$$(X \otimes Y)(C \otimes D) = (XC) \otimes (YD), \tag{2.4}$$

where the matrices C and D are of suitable orders.

3. Normwise condition numbers of MILS problem

In this section, we present two kinds of normwise condition numbers of (1.2) with respect to weighted Frobenius norm and weighted 2-norm. The MILS problem (1.2) has a unique solution if and only if $A^T JA > 0$, that is, it is positive definite. In this case, the unique solution can be expressed as [19]

$$X_{\mathrm{MILS}} = (A^T JA)^{-1}A^T JB.$$

Let $\Delta A \in \mathbb{R}^{m\times n}$, $\Delta B \in \mathbb{R}^{m\times s}$, and ΔA be sufficiently small such that $(A + \Delta A)^T J(A + \Delta A) > 0$. Hence, the perturbed MILS problem

$$\min_{X\in\mathbb{R}^{n\times s}} \mathrm{trace}\Big(((B + \Delta B) - (A + \Delta A)X)^T J((B + \Delta B) - (A + \Delta A)X)\Big)$$

has a unique solution

$$X_{\mathrm{MILS}} + \Delta X = [(A + \Delta A)^T J(A + \Delta A)]^{-1}(A + \Delta A)^T J(B + \Delta B).$$

The closed formula for the normwise condition number of MILS problem with respect to the weighted Frobenius norm is given first.

Theorem 3.1. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times s}$, and assume that $A^T JA > 0$. Then the normwise condition number*

$$\kappa_{\mathcal{F}}(A, B) = \lim_{\varepsilon\to 0} \sup_{\|[\Delta A \ \ \Delta B]\|_{\mathcal{F}} \le \varepsilon \|[A \ \ B]\|_{\mathcal{F}}} \frac{\|\Delta X\|_F}{\varepsilon \|X_{MILS}\|_F}$$

satisfies

$$\kappa_{\mathcal{F}}(A, B) = \frac{\left\| \begin{bmatrix} \frac{((JR)^T \otimes M^{-1})\Pi_{mn} - X_{MILS}^T \otimes A^{[\dagger]}}{\alpha} & \frac{I_s \otimes A^{[\dagger]}}{\beta} \end{bmatrix} \right\|_2 \|[A \ \ B]\|_{\mathcal{F}}}{\|X_{MILS}\|_F}, \tag{3.1}$$

where $R = B - AX_{MILS}$, $M = A^T JA$, and $A^{[\dagger]} = M^{-1}A^T J$.

Proof. It follows from the proof of Theorem 3.3 in [19] that

$$\Delta X = M^{-1}\Delta A^T JR - A^{[\dagger]}\Delta A X_{\mathrm{MILS}} + A^{[\dagger]}\Delta B + O(\varepsilon^2). \tag{3.2}$$

Omitting the second-order terms, applying the operator vec to (3.2) and using (2.2), we have

$$\begin{aligned} \mathrm{vec}(\Delta X) &= \left[\left((JR)^T \otimes M^{-1}\right)\Pi_{mn} - X_{\mathrm{MILS}}^T \otimes A^{[\dagger]}\right]\mathrm{vec}(\Delta A) + \left(I_s \otimes A^{[\dagger]}\right)\mathrm{vec}(\Delta B) \\ &= \begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix} \begin{bmatrix} \alpha\mathrm{vec}(\Delta A) \\ \beta\mathrm{vec}(\Delta B) \end{bmatrix}. \end{aligned} \tag{3.3}$$

Thus, a simple calculation yields

$$\begin{aligned} \frac{\|\Delta X\|_F}{\varepsilon\|X_{\mathrm{MILS}}\|_F} &\le \frac{\left\|\begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix}\right\|_2 \left\|\begin{bmatrix} \alpha\mathrm{vec}(\Delta A) \\ \beta\mathrm{vec}(\Delta B) \end{bmatrix}\right\|_2}{\varepsilon\|X_{\mathrm{MILS}}\|_F} \\ &= \frac{\left\|\begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix}\right\|_2 \|[\Delta A \ \ \Delta B]\|_{\mathcal{F}}}{\varepsilon\|X_{\mathrm{MILS}}\|_F} \\ &\le \frac{\left\|\begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix}\right\|_2 \|[A \ \ B]\|_{\mathcal{F}}}{\|X_{\mathrm{MILS}}\|_F}. \end{aligned}$$

The upper bound above is attainable according to the property of 2-norm. Consequently, (3.1) holds. □

Remark 1. When $\alpha = \beta = 1$, the normwise condition number $\kappa_{\mathcal{F}}(A, B)$ is reduced to the normwise condition number $\kappa(A, B)$ in Theorem 3.3 of [19].

Note that $\begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix} \in \mathbb{R}^{sn\times(mn+ms)}$, hence when m and n are very large, it is a very large matrix. Consequently, the storage requirements are very large. In the following theorem, we give a new expression of $\kappa_{\mathcal{F}}(A, B)$.

Theorem 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\kappa_{\mathcal{F}}(A, B) = \frac{\left\|\frac{N}{\alpha^2} + \frac{I_s\otimes\left(A^{[\dagger]}(A^{[\dagger]})^T\right)}{\beta^2}\right\|_2^{1/2} \|[A \ \ B]\|_{\mathcal{F}}}{\|X_{MILS}\|_F}, \tag{3.4}$$

where $N = (R^TR) \otimes M^{-2} + (X_{MILS}^T X_{MILS}) \otimes (A^{[\dagger]}(A^{[\dagger]})^T) - \left[(R^TAM^{-1}) \otimes (M^{-1}X_{MILS}) + (X_{MILS}^T M^{-1}) \otimes (M^{-1}A^TR)\right]\Pi_{ns}$.

Proof. Considering (2.1), (2.3) and (2.4), we have

$$\begin{aligned} &\begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix} \begin{bmatrix} \frac{((JR)^T\otimes M^{-1})\Pi_{mn} - X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}}{\alpha} & \frac{I_s\otimes A^{[\dagger]}}{\beta} \end{bmatrix}^T \\ = &\frac{\left((R^TJ)\otimes M^{-1}\right)\left((JR)\otimes M^{-1}\right) + \left(X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}\right)\left(X_{\mathrm{MILS}}\otimes (A^{[\dagger]})^T\right)}{\alpha^2} + \frac{\left(I_s\otimes A^{[\dagger]}\right)\left(I_s\otimes (A^{[\dagger]})^T\right)}{\beta^2} \\ &- \frac{\left((R^TJ)\otimes M^{-1}\right)\Pi_{mn}\left(X_{\mathrm{MILS}}\otimes (A^{[\dagger]})^T\right) + \left(X_{\mathrm{MILS}}^T\otimes A^{[\dagger]}\right)\Pi_{mn}^T\left((JR)\otimes M^{-1}\right)}{\alpha^2} \end{aligned}$$

$$= \frac{(R^TR)\otimes M^{-2} + (X_{\rm MILS}^T X_{\rm MILS})\otimes (A^{[\dagger]}(A^{[\dagger]})^T)}{\alpha^2} + \frac{I_s\otimes (A^{[\dagger]}(A^{[\dagger]})^T)}{\beta^2}$$

$$-\frac{\left((R^TJ)\otimes M^{-1}\right)\left((A^{[\dagger]})^T\otimes X_{\rm MILS}\right)\Pi_{ns} + \left(X^T\otimes A^{[\dagger]}\right)\left(M^{-1}\otimes (JR)\right)\Pi_{ns}}{\alpha^2}$$

$$= \frac{N}{\alpha^2} + \frac{I_s\otimes (A^{[\dagger]}(A^{[\dagger]})^T)}{\beta^2}.$$

It is well known that $\|C\|_2 = \|CC^T\|_2^{1/2}$ for any matrix $C\in\mathbb{R}^{m\times n}$, hence substituting the above equality into (3.1) gives (3.4). □

Since $\frac{N}{\alpha^2} + \frac{I_s\otimes(M^{-1}A^TAM^{-1})}{\beta^2}\in\mathbb{R}^{sn\times sn}$, the expression (3.4) reduces the storage requirements significantly when m and n are very large.

Remark 2. When $B\equiv b$ is an m-vector (i.e., $s=1$) and $A^TJA>0$, the ILS problem with a single right-hand side has a unique solution $x_{\rm ILS} = (A^TJA)^{-1}A^TJb$ and $r = b - Ax_{\rm ILS}$. It follows from Theorem 3.2 that

$$\kappa_{\mathcal{F}}(A,b) = \frac{\left\|\left(\frac{\|x_{\rm ILS}\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right)M^{-1}A^TAM^{-1} + \frac{\|r\|_2^2}{\alpha^2}M^{-2} - \frac{M^{-1}A^Trx_{\rm ILS}^TM^{-1} + M^{-1}x_{\rm ILS}r^TAM^{-1}}{\alpha^2}\right\|_2^{1/2}\|[A\ \ b]\|_{\mathcal{F}}}{\|x_{\rm ILS}\|_F}.$$

Denote $c_1 = \sqrt{\beta^2/\alpha^2 + 1/\|x_{\rm ILS}\|_2^2}$ and $c_2 = c_1 + 1/\|x_{\rm ILS}\|_2$, it can easily be verified that

$$\left(\frac{\|x_{\rm ILS}\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right)M^{-1}A^TAM^{-1} + \frac{\|r\|_2^2}{\alpha^2}M^{-2} - \frac{M^{-1}A^Trx_{\rm ILS}^TM^{-1} + M^{-1}x_{\rm ILS}r^TAM^{-1}}{\alpha^2}$$
$$= M^{-1}\begin{bmatrix} -\frac{\|x_{\rm ILS}\|_2}{\beta}A^TJ & \frac{\|r\|_2}{\alpha}I_n\end{bmatrix}\begin{bmatrix} c_1I_m - c_2\frac{Jrr^TJ}{\|r\|_2^2} & \frac{\beta}{\alpha}\frac{Jrx_{\rm ILS}^T}{\|x_{\rm ILS}\|_2\|r\|_2} \\ 0 & I_n\end{bmatrix}\begin{bmatrix} c_1I_m - c_2\frac{Jrr^TJ}{\|r\|_2^2} & 0 \\ \frac{\beta}{\alpha}\frac{x_{\rm ILS}r^TJ}{\|x_{\rm ILS}\|_2\|r\|_2} & I_n\end{bmatrix}\begin{bmatrix} -\frac{\|x_{\rm ILS}\|_2}{\beta}JA \\ \frac{\|r\|_2}{\alpha}I_n\end{bmatrix}M^{-1},$$

which together with the above equality gives

$$\kappa_{\mathcal{F}}(A,b) = \frac{\left\|M^{-1}\begin{bmatrix} -\frac{\|x_{\rm ILS}\|_2}{\beta}A^TJ & \frac{\|r\|_2}{\alpha}I_n\end{bmatrix}\begin{bmatrix} c_1I_m - c_2\frac{Jrr^TJ}{\|r\|_2^2} & \frac{\beta}{\alpha}\frac{Jrx_{\rm ILS}^T}{\|x_{\rm ILS}\|_2\|r\|_2} \\ 0 & I_n\end{bmatrix}\right\|_2\|[A\ \ b]\|_{\mathcal{F}}}{\|x_{\rm ILS}\|_F}.$$

The above closed formula of $\kappa_{\mathcal{F}}(A,b)$ appears in [5, Theorem 3.2] with $L = I_n$.

Although, Theorem 3.1 and Theorem 3.2 give the explicit expressions for the condition number $\kappa_{\mathcal{F}}(A,B)$, these expressions may not be computed easily for involving Kronecker products. Using the properties of 2-norm and (3.4), we can get the following easily computable upper bound for this condition number.

Theorem 3.3. *For the normwise condition number $\kappa_{\mathcal{F}}(A,B)$, we have*

$$\kappa_{\mathcal{F}}(A,B) \le \frac{\left(\frac{\|R\|_2^2\|M^{-1}\|_2^2 + \|X_{MILS}\|_2^2\left\|A^{[\dagger]}\right\|_2^2 + 2\|M^{-1}X_{MILS}\|_2\|M^{-1}A^TR\|_2}{\alpha^2} + \frac{\left\|A^{[\dagger]}\right\|_2^2}{\beta^2}\right)^{1/2}\|[A\ \ B]\|_{\mathcal{F}}}{\|X_{MILS}\|_F} := \overline{\kappa_{\mathcal{F}}(A,B)}. \quad (3.5)$$

In the following theorem, we will give the upper bound of the normwise condition number for the MILS problem under the weighted 2-norm.

Theorem 3.4. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times s}$, and assume that $A^T JA > 0$. Then the normwise condition number*

$$\kappa_{\in}(A, B) = \lim_{\varepsilon\to 0} \sup_{\|[\Delta A \quad \Delta B]\|_{\in} \le \varepsilon \|[A \quad B]\|_{\in}} \frac{\|\Delta X\|_2}{\varepsilon \|X_{MILS}\|_2}$$

satisfies

$$\kappa_{\in}(A, B) \le \frac{\left\| \begin{bmatrix} \frac{\|M^{-1}\|_2 \|R\|_2 + \left\|A^{[\dagger]}\right\|_2 \|X_{MILS}\|_2}{\alpha} & \frac{\left\|A^{[\dagger]}\right\|_2}{\beta} \end{bmatrix} \right\|_2 \|[A \quad B]\|_{\in}}{\|X_{MILS}\|_2} := \overline{\kappa_{\in}(A, B)}, \tag{3.6}$$

where $R = B - AX_{MILS}$, $M = A^T JA$, and $A^{[\dagger]} = M^{-1}A^T J$.

Proof. Omitting the second-order terms and taking 2-norm of (3.2), we obtain

$$\begin{aligned} \|\Delta X\|_2 &\le \left(\|M^{-1}\|_2 \|R\|_2 + \left\|A^{[\dagger]}\right\|_2 \|X_{\mathrm{MILS}}\|_2\right) \|\Delta A\|_2 + \left\|A^{[\dagger]}\right\|_2 \|\Delta B\|_2 \\ &= \begin{bmatrix} \frac{\|M^{-1}\|_2 \|R\|_2 + \left\|A^{[\dagger]}\right\|_2 \|X_{\mathrm{MILS}}\|_2}{\alpha} & \frac{\left\|A^{[\dagger]}\right\|_2}{\beta} \end{bmatrix} \begin{bmatrix} \alpha \|\Delta A\|_2 \\ \beta \|\Delta B\|_2 \end{bmatrix}. \end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\|\Delta X\|_2 \le \left\| \begin{bmatrix} \frac{\|M^{-1}\|_2 \|R\|_2 + \left\|A^{[\dagger]}\right\|_2 \|X_{\mathrm{MILS}}\|_2}{\alpha} & \frac{\left\|A^{[\dagger]}\right\|_2}{\beta} \end{bmatrix} \right\|_2 \|[\Delta A \quad \Delta B]\|_{\in}.$$

Thus, the inequality (3.6) follows. □

4. Numerical experiments

We consider the MILS problem (1.2) with [19]

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ A_1 = \begin{bmatrix} 101 & & & \\ 102 & 102 & & \\ \vdots & \vdots & \ddots & \\ n+100 & n+100 & \cdots & n+100 \end{bmatrix} \in \mathbb{R}^{n\times n},\ A_2 = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$J = \begin{bmatrix} I_{n+1} & 0 \\ 0 & -I_{n-1} \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & \cdots & s \\ 1 & 2 & \cdots & s \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & s \end{bmatrix} \in \mathbb{R}^{2n\times s}.$$

In specific numerical experiments, we set $s = 5$, $\alpha = \beta = 1$ and $n = 20, 30, 40$ or 50. In these cases, $A^T JA > 0$, which guarantees the MILS problem has the unique solution. All the computations were carried out in Matlab R2015b.

First, we report the computed results (denoted by 'CR') and the elapsed CPU times in seconds (denoted by 'CPU') for computing the normwise condition number $\kappa_{\mathcal{F}}(A, B)$ by using formulae (3.1) and (3.4) and the upper bound of $\kappa_{\mathcal{F}}(A, B)$ (i.e., $\overline{\kappa_{\mathcal{F}}(A, B)}$) in Table 1.

Table 1. Computed results and the elapsed CPU times.

n	Formula (3.1)		Formula (3.4)		$\overline{\kappa_{\mathcal{F}}(A,B)}$	
	CR	CPU	CR	CPU	CR	CPU
20	497.3247	1.1768	497.3247	0.0214	497.3377	0.0015
30	836.7598	15.0208	836.7598	0.0310	836.7791	0.0019
40	1.2435×10^3	93.8911	1.2435×10^3	0.0423	1.2436×10^3	0.0035
50	1.7206×10^3	345.5236	1.7206×10^3	0.0571	1.7206×10^3	0.0037

We can see from Table 1 that the computing results by using formulae (3.1) and (3.4) are asymptotically equal, but the elapsed CPU times by using (3.4) are much less than those by using formula (3.1). In addition, the upper bound $\overline{\kappa_{\mathcal{F}}(A,B)}$ is good estimate of the corresponding condition number $\kappa_{\mathcal{F}}(A,B)$ and the elapsed CPU times are minimum.

Now we show the tightness of the upper bound estimate on the normwise condition number $\kappa_{\in}(A,B)$ provided in Theorem 3.4. Let the perturbations be $\Delta A = 10^{-12} \times A$ and $\Delta B = 10^{-10} \times \mathrm{rand}(2n, s)$, where $\mathrm{rand}(\cdot)$ is the MATLAB function. Define $\varepsilon = |||[\Delta A \ \ \Delta B]|||_{\in} \,/\, |||[A \ \ B]|||_{\in}$. For small ε, it follows from the definition of $\kappa_{\in}(A,B)$ that

$$\frac{\|\Delta X\|_2}{\|X_{\mathrm{MILS}}\|_2} \leq \varepsilon\kappa_{\in}(A,B) + O(\varepsilon^2) \leq \varepsilon\overline{\kappa_{\in}(A,B)} + O(\varepsilon^2).$$

As shown in Table 2, the error bounds given by the upper bound of the condition number in Theorem 3.4 are at most two order of magnitude larger than the actual errors. This illustrates that, as the estimate of its corresponding condition number, the upper bound in Theorem 3.4 is tight.

Table 2. Comparisons of our estimated errors with the exact errors.

n	20	30	40	50
$\frac{\|\Delta X\|_2}{\|X_{\mathrm{MILS}}\|_2}$	3.7184×10^{-11}	3.3238×10^{-11}	3.2861×10^{-11}	4.4377×10^{-11}
$\varepsilon\overline{\kappa_{\in}(A,B)}$	5.1648×10^{-10}	8.4054×10^{-10}	1.2276×10^{-9}	1.6852×10^{-9}

5. Conclusions

In this paper, we investigate the normwise condition numbers of the indefinite least squares problem with multiple right-hand sides with respect to the weighted Frobenius norm and 2-norm. The closed formulas or upper bounds for these condition numbers are presented, which extend the earlier work for the indefinite least squares problem with single right-hand side.

Acknowledgments

This work was supported by the Foundation for Distinguished Young Scholars of Gansu Province (Grant No.20JR5RA540).

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Arioli, M. Baboulin, S. Gratton, A partial condition number for linear least squares problems, *SIAM J. Matrix Anal. Appl.*, **29** (2007), 413–433. doi: 10.1137/050643088.
2. E. H. Bergou, S. Gratton, J. Tshimanga, The exact condition number of the truncated singular value solution of a linear ill-posed problem, *SIAM J. Matrix Anal. Appl.*, **35** (2014), 1073–1085. doi: 10.1137/120869286.
3. A. Bojanczyk, N. J. Higham, H. Patel, Solving the indefinite least squares problem by hyperbolic QR factorization, *SIAM J. Matrix Anal. Appl.*, **24** (2003), 914–931. doi: 10.1137/S0895479802401497.
4. S. Chandrasekaran, M. Gu, A. H. Sayed, A stable and efficient algorithm for the indefinite linear least-squares problem, *SIAM J. Matrix Anal. Appl.*, **20** (1998), 354–362. doi: 10.1137/S0895479896302229.
5. H. Diao, T. Zhou, Backward error and condition number analysis for the indefinite linear least squares problem, *Int. J. Comput. Math.*, **96** (2019), 1603–1622. doi: 10.1080/00207160.2018.1467007.
6. J. I. Giribet, A. Maestripieri, F. M. Peria, A geometrical approach to indefinite least squares problem, *Acta Appl. Math.*, **111** (2011), 65–81. doi: 10.1007/s10440-009-9532-3.
7. A. Graham, *Kronecker Products and Matrix Calculus with Application*, Wiley, New York, 1981.
8. S. Gratton, On the condition number of linear least squares problems in a weighted Frobenius norm, *BIT*, **36** (1996), 523–530. doi: 10.1007/BF01731931.
9. B. Hassibi, A. H. Sayed, T. Kailath, Linear estimation in Krein spaces. Part I: Theory, *IEEE Trans. Automat. Contr.*, **41** (1996), 18–33. doi: 10.1109/9.481605.
10. N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2Eds., SIAM, Philadelphia, 2002.
11. H. Li, S. Wang, H. Yang, On mixed and componentwise condition numbers for indefinite least squares problem, *Linear Algebra Appl.*, **448** (2014), 104–129. doi: 10.1016/j.laa.2014.01.030.
12. H. Li, S. Wang, On the partial condition numbers for the indefinite least squares problem, *Appl. Numer. Math.*, **123** (2018), 200–220. doi: 10.1016/j.apnum.2017.09.006.
13. Q. Liu, X. Li, Preconditioned conjugate gradient methods for the solution of indefinite least squares problems, *Calcolo*, **48** (2011), 261–271. doi: 10.1007/s10092-011-0039-8.

14. Q. Liu, A. Liu, Block SOR methods for the solution of indefinite least squares problems, *Calcolo*, **51** (2014), 367–379. doi: 10.1007/s10092-013-0090-8.
15. Y. Ou, Z. Peng, The solvability conditions for one kind of indefinite problems, Proceedings of the ninth International Conference on Matrix Theory and its Applications, World Academic Union, UK, 2010, 142–145.
16. A. H. Sayed, B. Hassibi, T. Kailath, Inertia properties of indefinite quadratic forms, *IEEE. Signal Process. Lett.*, **3** (1996), 57–59. doi: 10.1109/97.484217.
17. J. Song, USSOR method for solving the indefinite least squares problem, *Int. J. Comput. Math.*, doi: 10.1080/00207160.2019.1658869.
18. S. Van Huffel, J. Vandewalle, The Total Least Squares Problem: Computational Aspects and Analysis, SIAM, Philadelphia, 1991. doi:10.2307/2153088.
19. L. Yang, H. Li, Condition numbers for indefinite least squares problem with multiple right-hand sides, *J. Math. Res. Appl.*, **37** (2017), 725–742.