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Research article

The least-squares solutions of the matrix equation $A^*XB + B^*X^*A = D$ and its optimal approximation

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Abstract: In this paper, the least-squares solutions to the linear matrix equation $A^*XB + B^*X^*A = D$ are discussed. By using the canonical correlation decomposition (CCD) of a pair of matrices, the general representation of the least-squares solutions to the matrix equation is derived. Moreover, the expression of the solution to the corresponding weighted optimal approximation problem is obtained.

Keywords: matrix equation; least-squares solution; canonical correlation decomposition; weighted optimal approximation
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1. Introduction

Throughout this paper, the complex $m \times n$ matrix space is denoted by $\mathbb{C}^{m\times n}$ and the set of all $n \times n$ unitary matrices is denoted by $\mathbb{UC}^{n\times n}$. The conjugate transpose and the Frobenius norm of a complex matrix A are denoted by $A^* \triangleq \bar{A}^\top$ and $\|A\|$, respectively. The Hermitian (skew-Hermitian) matrix A is denoted by $A = A^*(A = -A^*)$. The identity matrix of size n is represented by I_n. For matrices $A = (\alpha_{ij}) \in \mathbb{C}^{m\times n}, B = (\beta_{ij}) \in \mathbb{C}^{m\times n}$, $A * B$ is used to define the Hadamard product of A and B, that is, $A * B = (\alpha_{ij}\beta_{ij}) \in \mathbb{C}^{m\times n}$.

It is known that the matrix equation

$$A^*XB + B^*X^*A = D \tag{1.1}$$

plays an important role in automatic control. In 1991, Yasuda and Skelton [1] studied the assigning controllability and observability Gramians in feedback control by (1.1). In 1994, Fujioka and Hara [2] considered (1.1) in the context of studying state covariance assignment problem with measurement noise. Owing to its important applications, there has been an increased interest in solving (1.1). In

1980, Baksalary and Kala [3] established the solvability conditions and the representation of the general solution to the matrix equation $AXB + CYD = E$. In 1987, Chu [4] considered the compatibility of $AXB + CYD = E$ by using the generalized singular value decomposition (GSVD), and provided the least norm solution when the solution exists. After that, Chu [5] provided the solvability conditions of (1.1) by using the GSVD. In addition, some iterative methods [6–9] are also used to solve such matrix equations. There is no doubt that the researching of the least-squares solutions to this kind of matrix equation should also be significant and interesting. In 1998, Xu et al. [10] provided the least-squares Hermitian (skew-Hermitian) solutions of $AXA^{\mathrm{H}} + CYC^{\mathrm{H}} = F$ by using the canonical correlation decomposition (CCD). In 2006, Liao et al. [11] considered the least-squares solution with the minimum norm of $AXB^{\mathrm{H}} + CYD^{\mathrm{H}} = E$ by the CCD and GSVD. Yuan et al. [12, 13] considered the least-squares solutions with some constraints of the matrix equation $AXB + CXD = E$. Besides, other scholars [14–16] also have studied the least-squares problems of $AXB + CXD = E$. However, solving the least-squares solutions of (1.1) seems to be rarely considered in the literatures. Recently, Yuan [17] proposed the least-squares solutions to the matrix equation $A^{\top}XB - B^{\top}X^{\top}A = D$ by applying the canonical correlation decomposition of $[A^{\top}, B^{\top}]$. Subsequently, Yuan [18, 19] proposed the minimum norm solution of (1.1) by taking advantage of the generalized singular value decomposition of the matrix pair $[A^*, B^*]$, and provided the least-squares solution with the minimum norm of (1.1) by using the normal equation and singular value decompositions. Motivated by the work above, it occurred to us that can the least-squares solutions to (1.1) be derived in a similar way? The answer is affirmative. In this paper, we will consider the least-squares solutions of (1.1) and the associated weighted optimal approximation problem by utilizing the canonical correlation decomposition of a pair of matrices, which can be mathematically formulated as follows.

Problem I. Given $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{p\times m}, D \in \mathbb{C}^{m\times m}$ with $D = D^*$. Find $X \in \mathbb{C}^{n\times p}$ such that

$$\Phi_1 = \|A^*XB + B^*X^*A - D\| = \min. \tag{1.2}$$

Problem II. Given $F \in \mathbb{C}^{n\times p}$, find $\hat{X} \in \mathcal{S}_E$ such that

$$\|F - \hat{X}\|_W = \min_{X\in\mathcal{S}_E}\|F - X\|_W, \tag{1.3}$$

where $\|\cdot\|_W$ is the weighted norm will be defined below, $\mathcal{S}_E$ is the solution set of Problem I.

By using the canonical correlation decomposition, the explicit expression of the least-squares solutions to Problem I is derived. Also, the expression of the corresponding optimal approximation solution under a weighted Frobenius norm sense to Problem II is deduced. Further, numerical examples are provided to verify the correctness of our results.

2. Solution to Problem I

In order to solve Problem I, the following lemmas are needed.

Lemma 2.1. *[10] Let* $J_1 \in \mathbb{C}^{m\times n}, J_2 \in \mathbb{C}^{n\times m}$, $C_A = diag(\alpha_1, \alpha_2, \cdots, \alpha_m), S_A = diag(\beta_1, \beta_2, \cdots, \beta_m)$ *with* $\alpha_i > 0, \beta_i > 0$, *and* $\alpha_i^2 + \beta_i^2 = 1, (i = 1, 2, \cdots, m)$. *Then the following minimization problem with respect to* $X \in \mathbb{C}^{m\times n}$:

$$\phi_1 = \|C_AX - J_1\|^2 + \left\|S_AX - J_2^*\right\|^2 = \min$$

holds if and only if X can be expressed as $X = C_AJ_1 + S_AJ_2^*$.

Lemma 2.2. *Let* $J_1, J_2 \in \mathbb{C}^{m\times m}$ *with* $J_1 = J_1^*$, $C_A = diag(\alpha_1, \alpha_2, \cdots, \alpha_m), S_A = diag(\beta_1, \beta_2, \cdots, \beta_m)$ *with* $\alpha_i > 0, \beta_i > 0$, *and* $\alpha_i^2 + \beta_i^2 = 1, (i = 1, 2, \cdots, m)$. *Then the following minimization problem with respect to* $X \in \mathbb{C}^{m\times m}$:

$$\phi_2 = \|C_A X + X^* C_A - J_1\|^2 + \left\|S_A X - J_2^*\right\|^2 + \|X^* S_A - J_2\|^2 = \min$$

holds if and only if X can be expressed as

$$X + C_A X^* C_A = C_A J_1 + S_A J_2^*. \tag{2.1}$$

Proof. For $X = (x_{ij}) \in \mathbb{C}^{m\times m}$, $J_l = (J_{ij}^{(l)}) \in \mathbb{C}^{m\times m}$, $(l = 1, 2)$, we have

$$\phi_2 = \Sigma_{ij}\left(\left|\alpha_i x_{ij} + x_{ji}\alpha_j - J_{ij}^{(1)}\right|^2 + 2\left|\beta_i x_{ij} - J_{ji}^{(2)}\right|^2\right).$$

Clearly, ϕ_2 is a continuously differentiable function of $2m^2$ variables of $\mathrm{Re}(x_{ij})$, $\mathrm{Im}(x_{ij})$, $(i, j = 1, 2, \cdots, m)$. The function of x_{ij} is

$$\Omega = \left(\left|\alpha_i x_{ij} + x_{ji}\alpha_j - J_{ij}^{(1)}\right|^2 + \left|\alpha_j x_{ji} + x_{ij}\alpha_i - J_{ji}^{(1)}\right|^2 + 2\left|\beta_i x_{ij} - J_{ji}^{(2)}\right|^2\right).$$

According to the necessary condition of function which is minimizing at a point and $J_{ij}^{(1)} = J_{ji}^{(1)}$, we obtain the following expression:

$$x_{ij} + \alpha_i x_{ji}\alpha_j = \alpha_i J_{ij}^{(1)} + \beta_i J_{ji}^{(2)}, \ (i, j = 1, 2, \cdots, m). \tag{2.2}$$

Then the equation of (2.1) follows from (2.2). □

Lemma 2.3. *Let* $J \in \mathbb{C}^{m\times m}$, $C_A = diag(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *with* $0 < \alpha_i < 1$, $(i = 1, 2, \cdots, m)$. *Then the following equation with respect to* $X \in \mathbb{C}^{m\times m}$:

$$X + C_A X^* C_A = J \tag{2.3}$$

holds if and only if X can be expressed as

$$X = K * (J - C_A J^* C_A), \tag{2.4}$$

where $K = \left(k_{ij}\right) \in \mathbb{C}^{m\times m}$, $k_{ij} = \dfrac{1}{1 - (\alpha_i\alpha_j)^2}$, $(i, j = 1, 2, \cdots, m)$.

Proof. For $X = (x_{ij}) \in \mathbb{C}^{m\times m}, J = (J_{ij}) \in \mathbb{C}^{m\times m}$, (2.3) can be equivalently written as

$$x_{ij} + \alpha_i x_{ji}\alpha_j = J_{ij}, \ \ x_{ji} + \alpha_j x_{ij}\alpha_i = J_{ji}, \ (i, j = 1, 2, \cdots, m). \tag{2.5}$$

By (2.5), we can get

$$x_{ij} = \frac{J_{ij} - \alpha_i J_{ji}\alpha_j}{1 - \alpha_i^2\alpha_j^2}, \ (i, j = 1, 2, \cdots, m). \tag{2.6}$$

Then the equation of (2.4) follows from (2.6). □

Assume that the canonical correlation decomposition (CCD) [20] of the matrix pair $[A^*, B^*]$ is

$$A^* = Q[\Sigma_A, 0]E_A^{-1}, \qquad B^* = Q[\Sigma_B, 0]E_B^{-1}, \tag{2.7}$$

where $E_A \in \mathbb{C}^{n\times n}$ and $E_B \in \mathbb{C}^{p\times p}$ are nonsingular matrices, and

$$\Sigma_A = \begin{array}{cc} \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & C_A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_A & 0 \\ 0 & 0 & I \end{array}\right] & \begin{array}{c} q \\ s \\ h-q-s \\ m-h-s-t \\ s \\ t \end{array} \\ \begin{array}{ccc} q & s & t \end{array} & \end{array},$$

$$\Sigma_B = \begin{array}{cc} \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{c} q \\ s \\ h-q-s \\ m-h-s-t \\ s \\ t \end{array} \\ \begin{array}{ccc} q & s & h-q-s \end{array} & \end{array},$$

$$C_A = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_s), 1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0,$$
$$S_A = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_s), 0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_s < 1$$

with

$$\alpha_i^2 + \beta_i^2 = 1,\ (i = 1, 2, \cdots, s),$$

$q = \mathrm{rank}(A) + \mathrm{rank}(B) - \mathrm{rank}(A^*, B^*)$, $g = \mathrm{rank}(A) = q+s+t$, $h = \mathrm{rank}(B)$, and $Q = [Q_1, Q_2, Q_3, Q_4, Q_5, Q_6] \in \mathbb{UC}^{m\times m}$ with the partition of Q being compatible with those of Σ_A and Σ_B.

According to (2.7), (1.2) can be equivalently written as

$$\Phi_1 = \left\| [\Sigma_A, 0]E_A^{-1}X(E_B^{-1})^*[\Sigma_B, 0]^* + [\Sigma_B, 0]E_B^{-1}X^*(E_A^{-1})^*[\Sigma_A, 0]^* - Q^*DQ \right\|, \tag{2.8}$$

partition the matrices $E_A^{-1}X(E_B^{-1})^*$ and Q^*DQ into the following forms:

$$E_A^{-1}X(E_B^{-1})^* = \begin{array}{cc} \left[\begin{array}{cccc} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{array}\right] & \begin{array}{c} q \\ s \\ t \\ n-g \end{array} \\ \begin{array}{cccc} q & s & h-q-s & p-h \end{array} & \end{array}, \tag{2.9}$$

$$Q^*DQ = \begin{array}{cc} \left[\begin{array}{cccccc} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12}^* & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13}^* & D_{23}^* & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14}^* & D_{24}^* & D_{34}^* & D_{44} & D_{45} & D_{46} \\ D_{15}^* & D_{25}^* & D_{35}^* & D_{45}^* & D_{55} & D_{56} \\ D_{16}^* & D_{26}^* & D_{36}^* & D_{46}^* & D_{56}^* & D_{66} \end{array}\right] & \begin{array}{c} q \\ s \\ u \\ v \\ s \\ t \end{array} \\ \begin{array}{cccccc} q & s & u & v & s & t \end{array} & \end{array}, \tag{2.10}$$

where $u = h - q - s, v = m - h - s - t$. Inserting (2.9) and (2.10) into (2.8), we have

$$\Phi_1 = \left\| \begin{bmatrix} X_{11} + X_{11}^* - D_{11} & X_{12} + X_{21}^* C_A - D_{12} & X_{13} - D_{13} & -D_{14} & X_{21}^* S_A - D_{15} & X_{31}^* - D_{16} \\ C_A X_{21} + X_{12}^* - D_{12}^* & C_A X_{22} + X_{22}^* C_A - D_{22} & C_A X_{23} - D_{23} & -D_{24} & X_{22}^* S_A - D_{25} & X_{32}^* - D_{26} \\ X_{13}^* - D_{13}^* & X_{23}^* C_A - D_{23}^* & -D_{33} & -D_{34} & X_{23}^* S_A - D_{35} & X_{33}^* - D_{36} \\ -D_{14}^* & -D_{24}^* & -D_{34}^* & -D_{44} & -D_{45} & -D_{46} \\ S_A X_{21} - D_{15}^* & S_A X_{22} - D_{25}^* & S_A X_{23} - D_{35}^* & -D_{45}^* & -D_{55} & -D_{56} \\ X_{31} - D_{16}^* & X_{32} - D_{26}^* & X_{33} - D_{36}^* & -D_{46}^* & -D_{56}^* & -D_{66} \end{bmatrix} \right\|.$$

Apparently, $\Phi_1 = \min$ if and only if

$$\begin{gathered} \left\| X_{11} + X_{11}^* - D_{11} \right\| = \min, \\ \|X_{13} - D_{13}\|^2 + \left\| X_{13}^* - D_{13}^* \right\|^2 = \min, \quad \left\| X_{31} - D_{16}^* \right\|^2 + \left\| X_{31}^* - D_{16} \right\|^2 = \min, \\ \left\| X_{32} - D_{26}^* \right\|^2 + \left\| X_{32}^* - D_{26} \right\|^2 = \min, \quad \left\| X_{33} - D_{36}^* \right\|^2 + \left\| X_{33}^* - D_{36} \right\|^2 = \min, \\ \left\| C_A X_{21} + X_{12}^* - D_{12}^* \right\|^2 + \left\| S_A X_{21} - D_{15}^* \right\|^2 = \min, \end{gathered} \tag{2.11}$$

$$\|C_A X_{23} - D_{23}\|^2 + \left\| S_A X_{23} - D_{35}^* \right\|^2 = \min, \tag{2.12}$$

$$\left\| C_A X_{22} + X_{22}^* C_A - D_{22} \right\|^2 + \left\| S_A X_{22} - D_{25}^* \right\|^2 + \left\| X_{22}^* S_A - D_{25} \right\|^2 = \min. \tag{2.13}$$

By (2.11), we have

$$\begin{gathered} X_{11} = \frac{1}{2} D_{11} + N, \\ X_{13} = D_{13}, \quad X_{31} = D_{16}^*, \quad X_{32} = D_{26}^*, \quad X_{33} = D_{36}^*, \\ X_{12} = D_{12} - D_{15} S_A^{-1} C_A, \quad X_{21} = S_A^{-1} D_{15}^*, \end{gathered} \tag{2.14}$$

where $N \in \mathbb{C}^{q \times q}$ is some skew-Hermitian matrix. According to (2.12) and Lemma 2.1, we have

$$X_{23} = C_A D_{23} + S_A D_{35}^*. \tag{2.15}$$

By (2.13) and Lemmas 2.2 and 2.3, we have

$$X_{22} = K * (J - C_A J^* C_A), \tag{2.16}$$

where $K = \left(k_{ij}\right) \in \mathbb{C}^{s \times s}, k_{ij} = \dfrac{1}{1 - \left(\alpha_i \alpha_j\right)^2}$, $(i, j = 1, 2, \cdots, s)$, $J = C_A D_{22} + S_A D_{25}^*$. Inserting (2.14)–(2.16) into (2.9), we can get the following result.

Theorem 2.1. *Suppose that $A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{p \times m}, D \in \mathbb{C}^{m \times m}$ with $D = D^*$. Let the canonical correlation decomposition of the matrix pair $[A^*, B^*]$ be given by* (2.7), *the partition of the matrices $E_A^{-1} X (E_B^{-1})^*, Q^* D Q$ be given by* (2.9) *and* (2.10), *respectively. Then, the general solution of Problem I can be expressed as*

$$X = E_A \begin{bmatrix} \frac{1}{2} D_{11} + N & D_{12} - D_{15} S_A^{-1} C_A & D_{13} & X_{14} \\ S_A^{-1} D_{15}^* & X_{22} & C_A D_{23} + S_A D_{35}^* & X_{24} \\ D_{16}^* & D_{26}^* & D_{36}^* & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} E_B^*,$$

where X_{22} *is given by* (2.16)*,* $X_{i4}, X_{4j}, (i = 1,2,3,4; j = 1,2,3)$ *are arbitrary matrices and* $N \in \mathbb{C}^{q\times q}$ *is some skew-Hermitian matrix.*

Clearly, (1.1) with $D = D^*$ is sovable if and only if $\Phi_1 = \min = 0$, that is,

$$\begin{aligned}
&X_{11} + X_{11}^* - D_{11} = 0,\\
&X_{13} - D_{13} = 0,\ \ X_{31} - D_{16}^* = 0,\ \ X_{32} - D_{26}^* = 0,\ \ X_{33} - D_{36}^* = 0,\\
&C_AX_{21} + X_{12}^* - D_{12}^* = 0,\ S_AX_{21} - D_{15}^* = 0,\\
&C_AX_{23} - D_{23} = 0,\ S_AX_{23} - D_{35}^* = 0,\\
&C_AX_{22} + X_{22}^*C_A - D_{22} = 0,\ S_AX_{22} - D_{25}^* = 0,
\end{aligned} \tag{2.17}$$

$D_{i4} = 0, D_{4j} = 0,\ (i = 1,2,3,4; j = 5,6), D_{33} = 0, D_{55} = 0, D_{56} = 0, D_{66} = 0$. By (2.17), we obtain

$$X_{11} = \frac{1}{2}D_{11} + N, \tag{2.18}$$

$$\begin{aligned}
&X_{13} = D_{13},\ \ X_{31} = D_{16}^*,\ \ X_{32} = D_{26}^*,\ \ X_{33} = D_{36}^*,\\
&X_{12} = D_{12} - D_{15}S_A^{-1}C_A,\ \ X_{21} = S_A^{-1}D_{15}^*,\\
&X_{23} = C_A^{-1}D_{23} = S_A^{-1}D_{35}^*,\ \ C_A^{-1}D_{23} - S_A^{-1}D_{35}^* = 0,\\
&X_{22} = S_A^{-1}D_{25}^*,\ \ D_{22} = D_{25}S_A^{-1}C_A + C_AS_A^{-1}D_{25}^*,
\end{aligned} \tag{2.19}$$

where $N \in \mathbb{C}^{q\times q}$ is some skew-Hermitian matrix. Inserting (2.18) and (2.19) into (2.9), we have the following result.

Theorem 2.2. *Suppose that* $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{p\times m}, D \in \mathbb{C}^{m\times m}$ *with* $D = D^*$*. Let the canonical correlation decomposition(CCD) of the matrix pair* $[A^*, B^*]$ *be given by* (2.7)*, the partition of the matrices* $E_A^{-1}X(E_B^{-1})^*, Q^*DQ$ *be given by* (2.9) *and* (2.10)*, respectively. Then* (1.1) *has a solution if and only if*

$$\begin{aligned}
&D = D^*,\ DQ_4 = 0,\ D_{33} = 0,\ D_{55} = 0,\ D_{56} = 0,\ D_{66} = 0,\\
&C_A^{-1}D_{23} - S_A^{-1}D_{35}^* = 0,\ \ D_{22} = D_{25}S_A^{-1}C_A + C_AS_A^{-1}D_{25}^*.
\end{aligned} \tag{2.20}$$

In this case, the general solution of (1.1) *can be expressed as*

$$X = E_A\begin{bmatrix} \frac{1}{2}D_{11} + N & D_{12} - D_{15}S_A^{-1}C_A & D_{13} & X_{14}\\ S_A^{-1}D_{15}^* & S_A^{-1}D_{25}^* & C_A^{-1}D_{23} & X_{24}\\ D_{16}^* & D_{26}^* & D_{36}^* & X_{34}\\ X_{41} & X_{42} & X_{43} & X_{44}\end{bmatrix}E_B^*, \tag{2.21}$$

where $X_{i4}, X_{4j}, (i = 1,2,3,4; j = 1,2,3)$ *are arbitrary matrices and* $N \in \mathbb{C}^{q\times q}$ *is some skew-Hermitian matrix.*

Remark 2.1. *Note that* (1.1) *has a unique solution X if and only if*

$$n - g = 0,\quad p - h = 0,\quad q = 0,$$

which is equivalent to

$$rank(A^*,\ B^*) = n + p.$$

In this case, the unique solution of (1.1) *can be expressed as*

$$X = E_A \begin{bmatrix} S_A^{-1} D_{25}^* & C_A^{-1} D_{23} \\ D_{26}^* & D_{36}^* \end{bmatrix} E_B^*.$$

Based on Theorem 2.2, we can formulate the following algorithm 2.1 to solve (1.1).

Algorithm 2.1.

1) *Input matrices* A, B *and* D *with* $D = D^*$.

2) *Compute the canonical correlation decomposition of the matrix pair* $[A^*, B^*]$ *by* (2.7).

3) *Compute* $D_{11}, D_{12}, D_{13}, D_{15}, D_{16}, D_{22}, D_{23}, D_{25}, D_{26}, D_{35}$ *and* D_{36} *by* (2.10)*, respectively.*

4) *If the conditions* (2.20) *are satisfied, go to* 5)*; otherwise, Problem I has no solution, and stop.*

5) *Randomly choose skew-Hermitian matrix* $N \in \mathbb{C}^{q\times q}$ *and compute* X_{11} *by* (2.18).

6) *Compute* $X_{12}, X_{13}, X_{21}, X_{22}, X_{23}, X_{31}, X_{32}, X_{33}$ *by* (2.19)*, respectively.*

7) *Randomly choose* $X_{i4}, X_{4j}, (i = 1, 2, 3, 4; j = 1, 2, 3)$ *and compute* X *by* (2.21).

Example 2.1. *Let* $m = 10, n = 7, p = 6$. *Suppose that the matrices* A, B *and* D *are given by*

$$A = \begin{bmatrix} 0.89342 & 0.67419 & -0.80676 & -1.0582 & -0.42837 & 0.22173 & -1.3219 & 0.19747 & 0.36028 & -0.23965 \\ 1.0497 & 0.10238 & -0.39081 & -0.87045 & -0.47312 & 0.67326 & -1.5633 & -1.492 & -0.67715 & -0.08773 \\ 0.32308 & 0.075314 & -0.1318 & 0.13962 & -0.47165 & 0.26719 & -1.1466 & -1.6208 & -0.56962 & -0.23054 \\ 0.056371 & 0.36481 & -0.45522 & -0.45396 & -0.14119 & -0.04526 & -1.5865 & 0.19897 & -0.63784 & 0.84677 \\ 1.6889 & 1.3419 & -0.46073 & -0.11161 & -0.64301 & 1.0634 & -3.3326 & -1.4834 & -1.1785 & 1.2706 \\ -0.25273 & -0.11946 & -0.45172 & 0.064737 & -0.35219 & 0.4565 & 0.91037 & -0.0099442 & -0.39096 & 0.63466 \\ -1.8326 & -1.0145 & 1.1621 & 1.2864 & 0.94997 & -1.2306 & 2.6438 & 1.213 & 0.79133 & -0.67153 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.080912 & -0.15446 & 0.88071 & -1.4211 & -0.33307 & -0.87903 & -1.5 & -0.39493 & -0.84953 & 1.395 \\ 0.63295 & 0.58091 & 1.9653 & -1.6504 & -1.0121 & -0.90031 & -0.59117 & -0.080091 & -0.4563 & 2.4604 \\ -0.59487 & 0.52259 & 0.70132 & -0.24961 & 0.28837 & 1.2096 & 0.17834 & -0.086521 & 0.84257 & 0.53255 \\ 0.071586 & -0.58669 & -0.65556 & 0.24098 & -0.20074 & -0.79419 & -0.5282 & 0.046106 & -1.0188 & -1.0261 \\ 0.1335 & -0.43002 & -2.396 & 2.2382 & 1.3994 & 0.82002 & 1.024 & 0.51096 & 0.79283 & -2.3305 \\ -0.02447 & -0.033381 & -0.32694 & -0.014084 & 0.1577 & 0.0092585 & 0.74787 & -0.43361 & 0.8833 & 0.49608 \end{bmatrix},$$

$$D = \begin{bmatrix} -32.498 & -41.334 & -22.064 & 14.89 & 13.462 & -21.284 & -10.183 & -6.1465 & -27.471 & -65.9 \\ -41.334 & -35.07 & -28.227 & 34.054 & 21.937 & 0.88842 & 27.704 & 6.5266 & -12.587 & -48.428 \\ -22.064 & -28.227 & 43.379 & -7.7439 & -1.2298 & -26.915 & 9.5348 & -34.9 & -24.195 & 26.775 \\ 14.89 & 34.054 & -7.7439 & -11.873 & -5.0642 & 19.391 & 31.103 & 39.452 & 41.819 & 39.159 \\ 13.462 & 21.937 & -1.2298 & -5.0642 & -13.579 & 6.6957 & 21.593 & 17.629 & 17.442 & 29.843 \\ -21.284 & 0.88842 & -26.915 & 19.391 & 6.6957 & 18.154 & 35.902 & 36.906 & 2.051 & 4.0912 \\ -10.183 & 27.704 & 9.5348 & 31.103 & 21.593 & 35.902 & 182.31 & 61.244 & 120.04 & 58.104 \\ -6.1465 & 6.5266 & -34.9 & 39.452 & 17.629 & 36.906 & 61.244 & 2.0347 & 51.993 & -25.591 \\ -27.471 & -12.587 & -24.195 & 41.819 & 17.442 & 2.051 & 120.04 & 51.993 & 40.795 & -1.4555 \\ -65.9 & -48.428 & 26.775 & 39.159 & 29.843 & 4.0912 & 58.104 & -25.591 & -1.4555 & -44.385 \end{bmatrix}.$$

It is easy to verify the solvability conditions are satisfied:

$$Q_4^* D = 1.0 \times 10^{-13} \begin{bmatrix} -0.1776 & 0.0355 & -0.1243 & 0.1421 & 0.0711 & 0.0910 & 0.6395 & 0.1599 & 0.2665 & 0 \end{bmatrix},$$
$$D_{33} = 1.3831 \times 10^{-14}, \quad D_{55} = 1.0 \times 10^{-13} \begin{bmatrix} 0.0581 & -0.4387 \\ -0.3434 & -0.0124 \end{bmatrix},$$
$$D_{56} = 1.0 \times 10^{-13} \begin{bmatrix} -0.2143 & 0.0623 \\ -0.2106 & 0.0813 \end{bmatrix}, \; D_{66} = 1.0 \times 10^{-13} \begin{bmatrix} -0.2288 & -0.0117 \\ 0.0105 & 0.2715 \end{bmatrix},$$
$$C_A^{-1} D_{23} = S_A^{-1} D_{35}^* = \begin{bmatrix} 35.9983 \\ -22.9230 \end{bmatrix}, \; D_{22} = D_{25} S_A^{-1} C_A + C_A S_A^{-1} D_{25}^* = \begin{bmatrix} 3.5122 & 14.7861 \\ 14.7861 & 10.6574 \end{bmatrix}.$$

According to Algorithm 2.1 *above, if choose* $N = 0, X_{i4} = 0, X_{4j} = 0, (i = 1, 2, 3, 4; j = 1, 2, 3)$, *then we can obtain a feasible solution* X *of* (1.1) *as follows:*

$$X = \begin{bmatrix} 14.297 & -29.492 & -3.0526 & 10.77 & -2.5732 & -16.157 \\ 12.674 & -5.1615 & 4.4051 & 21.794 & -7.8158 & -20.305 \\ 6.4513 & 19.863 & 4.3724 & 5.6231 & 11.727 & -0.98557 \\ -2.9642 & 10.036 & 3.9324 & 3.2979 & -1.334 & -10.045 \\ 13.035 & -28.008 & -0.76611 & 3.0924 & -7.9222 & -13.129 \\ 3.5708 & -20.56 & 0.29646 & -13.416 & -2.922 & -13.579 \\ 18.501 & -21.876 & 2.6595 & 15.754 & -6.8772 & -25.22 \end{bmatrix}$$

with the corresponding residual estimated by

$$\|A^* X B + B^* X^* A - D\| = 8.8662 \times 10^{-13}.$$

3. Solution to Problem II

Suppose that the CCD of the matrix pair $[A^*, B^*]$ is of the form given by (2.7). For any $X \in \mathbb{C}^{n \times p}$, we define a weighted norm as follows [21–23] :

$$\|X\|_W \triangleq \|E_A^{-1} X (E_B^{-1})^*\|. \tag{3.1}$$

Write

$$E_A^{-1} F (E_B^{-1})^* = \begin{array}{c} \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{array}{c} q \\ s \\ t \\ n-g \end{array} \\ \begin{array}{cccc} q & s & h-q-s & p-h \end{array} \end{array}. \tag{3.2}$$

Therefore, for any $X \in \mathcal{S}_E$, (1.3) can be written as

$$\|F - X\|_W = \left\| E_A^{-1} F (E_B^{-1})^* - \begin{bmatrix} \frac{1}{2} D_{11} + N & D_{12} - D_{15} S_A^{-1} C_A & D_{13} & X_{14} \\ S_A^{-1} D_{15}^* & S_A^{-1} D_{25}^* & C_A^{-1} D_{23} & X_{24} \\ D_{16}^* & D_{26}^* & D_{36}^* & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} - \begin{bmatrix} \frac{1}{2}D_{11} + N & D_{12} - D_{15}S_A^{-1}C_A & D_{13} & X_{14} \\ S_A^{-1}D_{15}^* & S_A^{-1}D_{25}^* & C_A^{-1}D_{23} & X_{24} \\ D_{16}^* & D_{26}^* & D_{36}^* & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \right\|.$$

Obviously, $\|F - X\|_W = \min$, if and only if

$$\|M - N\| = \min, \tag{3.3}$$

$$\|F_{i4} - X_{i4}\| = \min, \ (i = 1, 2, 3, 4), \tag{3.4}$$

$$\left\|F_{4j} - X_{4j}\right\| = \min, \ (j = 1, 2, 3), \tag{3.5}$$

where $M = F_{11} - \frac{1}{2}D_{11}$. Note that $N \in \mathbb{C}^{q\times q}$ is some skew-Hermitian matrix, which implies that the relation of (3.3) is equivalent to

$$\|N - M\|^2 = \left\|N - \frac{1}{2}(M - M^*)\right\|^2 + \left\|\frac{1}{2}(M + M^*)\right\|^2,$$

therefore, we have

$$N = \frac{1}{2}(M - M^*), \ X_{11} = \frac{1}{2}(D_{11} + M - M^*), \tag{3.6}$$

where $M = F_{11} - \frac{1}{2}D_{11}$. Apparently, by (3.4) and (3.5), we have

$$X_{i4} = F_{i4}, \ (i = 1, 2, 3, 4), \ X_{4j} = F_{4j}, \ (j = 1, 2, 3). \tag{3.7}$$

Summing up the discussions above, we have the following result.

Theorem 3.1. *Given $F \in \mathbb{C}^{n\times p}$ and partition $E_A^{-1}F(E_B^{-1})^*$ as* (3.2)*. Let $M = F_{11} - \frac{1}{2}D_{11}$, then the unique solution of Problem II can be expressed as*

$$\hat{X} = E_A \begin{bmatrix} \frac{1}{2}(D_{11} + M - M^*) & D_{12} - D_{15}S_A^{-1}C_A & D_{13} & F_{14} \\ S_A^{-1}D_{15}^* & S_A^{-1}D_{25}^* & C_A^{-1}D_{23} & F_{24} \\ D_{16}^* & D_{26}^* & D_{36}^* & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} E_B^*. \tag{3.8}$$

Based on Theorem 3.1, we can formulate the following algorithm 3.1 to solve Problem II.

Algorithm 3.1.

1) *Input matrices A, B, F and D with $D = D^*$.*

2) *Compute the canonical correlation decomposition of the matrix pair $[A^*, B^*]$ by* (2.7)*.*

3) *Compute $D_{11}, D_{12}, D_{13}, D_{15}, D_{16}, D_{22}, D_{23}, D_{25}, D_{26}, D_{35}$ and D_{36} by* (2.10)*, respectively.*

4) *If the conditions* (2.20) *are satisfied, go to* 5)*; otherwise, Problem II has no solution, and stop.*

5) *Compute* F_{11}, F_{i4}, F_{4j} $(i = 1, 2, 3, 4; j = 1, 2, 3)$ *by* (3.2).

6) *Set* $M = F_{11} - \frac{1}{2}D_{11}$ *and compute* X_{11} *by* (3.6).

7) *Compute* X_{i4} *and* X_{4j} $(i = 1, 2, 3, 4; j = 1, 2, 3)$ *by* (3.7) *and compute* $\hat{X}$ *by* (3.8).

Example 3.1. *Let* $m = 10, n = 7, p = 6$. *Suppose that the matrices* A, B, D *and* F *are given by*

$$F = \begin{bmatrix} 0.9754 & -9.5717 & -9.5949 & -7.4313 & 0.46171 & -4.3874 \\ -2.785 & -4.8538 & 6.5574 & 3.9223 & -0.97132 & -3.8156 \\ 5.4688 & -8.0028 & -0.35712 & -6.5548 & -8.2346 & -7.6552 \\ -9.5751 & 1.4189 & -8.4913 & -1.7119 & -6.9483 & -7.952 \\ -9.6489 & -4.2176 & -9.3399 & 7.0605 & 3.171 & 1.8687 \\ 1.5761 & -9.1574 & -6.7874 & -0.31833 & -9.5022 & 4.8976 \\ -9.7059 & 7.9221 & 7.5774 & -2.7692 & -0.34446 & -4.4559 \end{bmatrix},$$

and the matrices A, B, D *are the same as those of Example* 2.1. *It is easy to verify the solvability conditions are satisfied by Example* 2.1. *According to Algorithm* 3.1 *above, we can obtain the unique solution* $\hat{X}$ *of Problem II as follows:*

$$\hat{X} = \begin{bmatrix} 22.453 & -27.405 & 2.902 & -33.42 & -13.843 & -20.599 \\ 20.879 & -12.679 & 20.666 & 1.2317 & -30.38 & -8.4501 \\ -12.383 & 21.633 & -5.6235 & 7.9966 & 11.409 & -0.83169 \\ -12.675 & -12.705 & -10.916 & -1.1556 & -23.496 & -1.0355 \\ 30.483 & -25.893 & 8.7596 & -21.928 & -10.622 & -17.599 \\ 9.0127 & -16.712 & 5.5371 & -46.447 & -9.6452 & -17.62 \\ 32.651 & -20.553 & 26.12 & -26.865 & -28.117 & -18.774 \end{bmatrix}$$

with the corresponding residual estimated by

$$\|A^*\hat{X}B + B^*\hat{X}^*A - D\| = 9.0698 \times 10^{-13}.$$

4. Conclusions

In this paper, we have obtained the expression of the least-squares solutions to Problem I and the unique solution $\hat{X}$ of Problem II by using the CCD of the matrix pair $[A^*, B^*]$. Two numerical examples verify the correctness of our results.

Conflict of interest

All authors declare that there is no conflict of interest in this paper.

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