Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 3648–3652.
DOI:10.3934/math.2022201
Received: 17 September 2021
Accepted: 25 November 2021
Published: 06 December 2021

Research article

A note on PM-compact K_4-free bricks

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Abstract: A 3-connected graph is a *brick* if the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of the matching covered graphs. Lovász (Combinatorica, 3 (1983), 105-117) showed that every brick is K_4-based or $\overline{C}_6$-based. A brick is K_4*-free* (respectively, $\overline{C}_6$*-free*) if it is not K_4-based (respectively, $\overline{C}_6$-based). Recently, Carvalho, Lucchesi and Murty (SIAM Journal on Discrete Mathematics, 34(3) (2020), 1769-1790) characterised the PM-compact $\overline{C}_6$-free bricks. In this note, we show that, by using the brick generation procedure established by Norine and Thomas (J Combin Theory Ser B, 97 (2007), 769-817), the only PM-compact K_4-free brick is $\overline{C}_6$, up to multiple edges.

Keywords: matching covered graph; perfect matching and brick
Mathematics Subject Classification: 05C70, 05C75

1. Introduction

Graphs considered in this paper are simple graphs. We will use generally Bondy and Murty [1] for the notation and terminology not defined here.

A connected graph is called *matching covered* if it has at least one edge and each of its edges is contained in some perfect matching. For the terminology that is specific to matching covered graphs, we will generally follow Lovász and Plummer [9]. Edmonds et al. [6] (also see Lovász [8], Szigeti [11] and Carvalho et al. [4]) showed that a graph G is a brick if and only if G is 3-connected and $G - x - y$ has a perfect matching for any two distinct vertices $x, y \in V(G)$.

Let G be a matching covered graph, and let M_1 and M_2 be two perfect matchings of G. We denote by $PM(G)$ the *perfect-matching graph* of G, which is the graph whose vertices are the perfect matchings of G, with two vertices M_1 and M_2 are adjacent in $PM(G)$ if $M_1 \oplus M_2$ is a cycle, where $\oplus$ denotes the symmetric difference operation. We say that G is *PM-compact* if $PM(G)$ is complete. Clearly, each conformal subgraph of a PM-compact graph is also PM-compact. Since an even cycle contains two perfect matchings, the following simple observation is an immediate consequence.

Lemma 1. *A matching covered graph G is PM-compact if and only if for any two vertex-disjoint even cycles C_1 and C_2 of G, $G - V(C_1 \cup C_2)$ has no perfect matchings.*

Moreover, Wang et al. [12] characterized PM-compact bipartite and near-bipartite graphs. Wang et al. [13] characterized all claw-free PM-compact cubic graphs. Wang et al. [14] characterized all Hamiltonian PM-compact bipartite graphs.

A *bisubdivision* of a graph H is a graph obtained from H by inserting an even number of vertices to some of its edges. A matching covered graph H is a *conformal minor* of a matching covered graph G if there exists a bisubdivision J of H which is a subgraph of G such that $G - V(J)$ has a perfect matching. A matching covered graph is K_4-*free* if it contains no K_4 as a conformal minor, otherwise it is K_4-*based*. $\overline{C}_6$-*free* and $\overline{C}_6$-*based* graphs are analogously defined ($\overline{C}_6$ is the complement graph of the cycle with six vertices). Recently, Carvalho et al. [2] characterised the PM-compact $\overline{C}_6$-free bricks. In this note, we characterize the PM-compact K_4-free bricks as follows.

Theorem 2. *Let G be a K_4-free brick. If G is PM-compact, then G is isomorphic to $\overline{C}_6$, up to multiple edges.*

The proof of Theorem 2 will be given in Section 3 following some properties concerning matching covered graphs given in Section 2.

2. Preliminaries

In this section, we present some properties of matching covered graphs which will be used in the proof of the main result. We start with some basic definitions. Let G be a matching covered graph, v a vertex of degree two of G. The *bicontraction* of v is the operation of contracting the two edges incident with v. The *retract* of G is the graph obtained by bicontracting all its vertices of degree two. An edge e in a brick G is *thin* if the retract of $G - e$ is also a brick. A thin edge e of a simple brick G is *strictly thin* if the retract of $G - e$ is a simple brick. Carvalho, Lucchesi, and Murty [3] proved that every brick distinct from $K_4, \overline{C}_6$ and the Petersen graph has a thin edge. We say that a brick is a *Norine-Thomas brick* if it is the Petersen graph or it belongs to any of the following described five well-defined infinite families of graphs:

Odd Wheel W_{2k+1}. For each integer $k \geq 1$, W_{2k+1} is defined to be the join graph of an odd cycle C_{2k+1} and a new vertex. See Figure 1 (a).

Truncated biwheel T_{2k+2}. For each integer $k \geq 2$, let $P = v_1 v_2 \cdots v_{2k}$ be an odd path, u and v be two new vertices. Let T_{2k+2} be obtained from $P \cup u \cup v$ by adding edges uv_i and vv_j for $i = 1, 2k$ and all even $i \in \{1, 2, \ldots, 2k\}$ and $j = 1, 2k$ and all odd $j \in \{1, 2, \ldots, 2k\}$. See Figure 1 (b).

Möbius ladder M_{4k}. For each integer $k \geq 1$, let $P_1 = u_1 u_2 \cdots u_{2k}$ and $P_2 = v_1 v_2 \cdots v_{2k}$ be two odd paths. Let M_{4k} be obtained from $P_1 \cup P_2$ by adding edges $u_i v_i$ for $i = 1, 2, \ldots, 2k$ and $u_1 v_{2k}$ and $v_1 u_{2k}$. See Figure 1 (c).

Prism Pr_{4k+2}. For each integer $k \geq 1$, let $P_1 = u_1 u_2 \cdots u_{2k+1}$ and $P_2 = v_1 v_2 \cdots v_{2k+1}$ be two odd paths. Let Pr_{4k+2} be obtained from $P_1 \cup P_2$ by adding edges $u_i v_i$ for $i = 1, 2, \ldots, 2k + 1$ and $u_1 u_{2k+1}$ and $v_1 v_{2k+1}$. See Figure 1 (d).

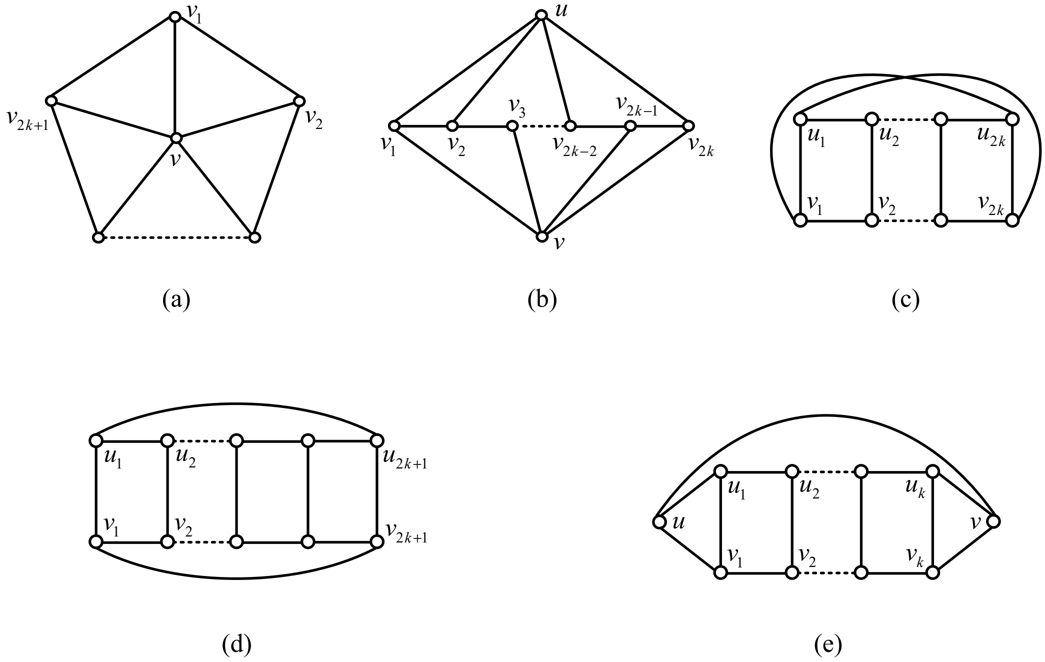

Figure 1. (a) Odd wheel W_{2k+1}, $(k \geq 1)$; (b) Truncated biwheel T_{2k+2}, $(k \geq 2)$; (c) Möbius ladder M_{4k}, $(k \geq 1)$; (d) Prism Pr_{4k+2}, $(k \geq 1)$; (e) Staircase St_{2k+2}, $(k \geq 2)$.

Staircase St_{2k+2}. For each integer $k \geq 2$, let $P_1 = u_1u_2 \cdots u_k$ and $P_2 = v_1v_2 \cdots v_k$ be two paths, u and v be two new vertices. Let St_{2k+2} be obtained from $P_1 \cup P_2 \cup u \cup v$ by adding edges u_iv_i for $i = 1, 2, \ldots, k$ and uu_1, uv_1, vu_k, vv_k and uv. See Figure 1 (e).

Norine and Thomas [10] showed that every simple brick distinct from the Norine-Thomas bricks has a strictly thin edge. This implies the following result.

Theorem 3. [5, 10] *Given any brick G, there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that (i) G_1 is a Norine-Thomas brick; (ii) $G_r := G$; and (iii) for $2 \leq i \leq r$, there exists a strictly thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$.*

Since a matching covered graph is PM-compact if and only if its retract is PM-compact [2], the following consequence holds immediately.

Corollary 4. *Let e be a thin edge of a brick G and let H be the retract of $G - e$. If G is PM-compact, then H is also PM-compact.*

Recently, Kothari and Murty [7] proved that, for any thin edge e of a brick G and any cubic brick J, the retract of $G - e$ is J-free if G is J-free. This implies the following simple observation immediately.

Lemma 5. [7] *Let e be a thin edge of a brick G and let H be the retract of $G - e$. If G is K_4-free, then H is also K_4-free.*

3. Proof of Theorem 2

Suppose, to the contrary, that there exists a K_4-free PM-compact brick that is not isomorphic to $\overline{C}_6$, up to multiple edges. Among all such bricks, we choose a brick G containing minimum number of edges. If G contains multiple edges, then the underlying simple graph of G is also a counterexample to Theorem 2. Thus, we may assume that G is a simple brick. By Theorem 3, there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that (i) G_1 is a Norine-Thomas brick; (ii) $G_r := G$; and (iii) for $2 \leq i \leq r$, there exists a strictly thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$. Since G is PM-compact, each brick G_i in the above sequence is PM-compact as well by Corollary 4. In particular, the base brick G_1 is PM-compact. Now, we will shall that G_1 is $\overline{C}_6$, namely, G_1 is the prism or the stair case or the truncated biwheel, with order 6. Since each odd wheel, the staircase R_8 with order 8 and the Petersen graph P are K_4-based, it suffices to consider the four possible alternatives separately, by Lemma 5, as follows.

If G_1 is a truncated biwheel T_{2k+2} with $k \geq 3$, then $C_1 = vv_1v_2v_3v$ and $C_2 = uv_4v_5v_6u$. Clearly, C_1 and C_2 are two disjoint even cycles of G_1 and $G_1 - V(C_1 \cup C_2)$ has a perfect matching $\{v_{2i-1}v_{2i} | 4 \leq i \leq k\}$ (see Figure1 (b)). Thus, G_1 is not PM-compact by Lemma 1.

If G_1 is a Möbius ladder M_{4k} with $k \geq 2$, then let $C_1 = u_1v_1v_2u_2u_1$ and $C_2 = u_3v_3v_4u_4u_3$. Clearly, C_1 and C_2 are two disjoint even cycles of G_1 and $G_1 - V(C_1 \cup C_2)$ has a perfect matching $\{u_iv_i | 5 \leq i \leq 2k\}$ (see Figure1 (c)). Thus, G_1 is not PM-compact by Lemma 1.

If G_1 is a prism Pr_{4k+2} with $k \geq 2$, then let $C_1 = u_1v_1v_2u_2u_1$ and $C_2 = u_3v_3v_4u_4u_3$. Clearly, C_1 and C_2 are two disjoint even cycles of G_1 and $G_1 - V(C_1 \cup C_2)$ has a perfect matching $\{u_iv_i | 5 \leq i \leq 2k+1\}$ (see Figure1 (d)). Thus, G_1 is not PM-compact by Lemma 1.

If G_1 is a staircase St_{2k+2} with $k \geq 4$, then let $C_1 = u_1v_1v_2u_2u_1$ and $C_2 = u_3v_3v_4u_4u_3$. Clearly, C_1 and C_2 are two disjoint even cycles of G_1 and $G_1 - V(C_1 \cup C_2)$ has a perfect matching $\{uv, u_iv_i | 5 \leq i \leq 2k\}$ (see Figure1 (e)). Thus, G_1 is not PM-compact by Lemma 1.

Finally, we claim that G is $\overline{C}_6$. Otherwise, the length r of the brick sequence greater than one. Since G_1 is cubic (i.e., 3-regular), G_2 must be obtained from G_1 by adding an edge. Hence, G_2 is K_4-based. It is a contradiction by Lemma 5. Thus, $G = G_1$. This completes the proof. □

4. Concluding remarks

In this work, we obtained a characterization of PM-compact K_4-free bricks. Various similar or analogous problems may also be of interest and worthy of study. We propose two problems as follows.

Problem 1. *Characterize PM-compact K_4-based bricks.*

Problem 2. *Characterize PM-compact K_4-free matching covered graphs.*

Acknowledgments

The authors would like to thank the anonymous referees and the academic editor for their careful reading and valuable suggestions. The research is supported by National Natural Science Foundation of China (Grant No. 12001250), the Program of Qingjiang Excellent Young Talents, Jiangxi University of Science and Technology, the Foundation of Jiangxi Provincial Education Department of China (No. GJJ190490) and the Doctoral Fund Project of Doctoral Fund Project of Jiangxi University of Science and Technology (No. jxxjbs18063).

Conflict of interest

The authors declare no conflicts of interest in this paper.

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