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Research article

Further characterizations of the weak core inverse of matrices and the weak core matrix

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Abstract: The present paper is devoted to characterizing the weak core inverse and the weak core matrix using the core-EP decomposition. Some new characterizations of the weak core inverse are presented by using its range space, null space and matrix equations. Additionally, we give several new representations and properties of the weak core inverse. Finally, we consider several equivalent conditions for a matrix to be a weak core matrix.

Keywords: weak core inverse; weak core matrices; core-EP decomposition; range space; null space
Mathematics Subject Classification: 15A09

1. Introduction

The weak core inverse was introduced in [1] where the authors presented some characterizations and properties. In [2], the authors introduced an extension of the weak core inverse. Continuing previous research about the weak core inverse, our purpose is to present new characterizations and representations of the weak core inverse. Additionally, we also give several equivalent conditions for a matrix to be a weak core matrix.

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices and $\mathbb{Z}^+$ denotes the set of all positive integers. The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, $r(A)$ and I_n will denote the range space, null space, conjugate transpose, rank of $A \in \mathbb{C}^{m\times n}$ and the identity matrix of order n. $\mathrm{Ind}(A)$ means the index of $A \in \mathbb{C}^{n\times n}$. Let $\mathbb{C}_k^{n\times n}$ be the set consisting of all $n \times n$ complex matrices with index k. The symbol $\dim(S)$ represents the dimension of a subspace $S \subseteq \mathbb{C}^n$. P_L stands for the orthogonal projection onto the subspace L. P_A, P_{A^*} respectively denote the orthogonal projection onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, i.e., $P_A = AA^\dagger$, $P_{A^*} = A^\dagger A$.

We will now introduce definitions of several generalized inverses that will be used throughout the paper. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined as the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying [3]:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

In particular, X is an outer inverse of A which is denoted as $A^{(2)}$ if $XAX = X$. For any matrix $A \in \mathbb{C}^{m\times n}$ with $r(A) = r$, let $T \subseteq \mathbb{C}^n$, $S \subseteq \mathbb{C}^m$ be two subspaces such that $\dim(T) = t \leq r$ and $\dim(S) = m - t$. Then A has an outer inverse X that satisfies $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$. In that case X is unique and denoted by $A^{(2)}_{T,S}$ [4].

The Drazin inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by A^D, is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying [5]: $XAX = X$, $AX = XA$, $XA^{k+1} = A^k$.

For any matrix $A \in \mathbb{C}^{n\times n}_1$, a new generalized inverse, which is called core inverse [6] was introduced. Two other generalizations of the core inverse for $A \in \mathbb{C}^{n\times n}_k$ such as core-EP inverse [7], DMP inverse [8] were also introduced.

In 2018, Wang and Chen [9] defined the weak group inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by $A^{\textcircled{w}}$, as the unique matrix $X \in \mathbb{C}^{n\times n}$ such that [9]: $AX^2 = X$, $AX = A^{\textcircled{\dagger}}A$. Moreover, it was verified that $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2A$.

Recently, Ferreyra et al. introduced a new generalization of core inverse called the weak core inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by $A^{\textcircled{w},\dagger}$(or, in short, WC inverse). It is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying [1]:

$$XAX = X, \quad AX = CA^\dagger, \quad XA = A^DC,$$

where $C = AA^{\textcircled{w}}A$. Moreover, it is proved that $A^{\textcircled{w},\dagger} = A^DCA^\dagger = A^{\textcircled{w}}AA^\dagger$.

The structure of this paper is as follows: In Section 2, we give some preliminaries which will be made use of later in this paper. In Section 3, we discuss some characterizations of the WC inverse based on its range space, null space and matrix equations. In Section 4, several new representations of the WC inverse are proposed. Section 5 is devoted to deriving some properties of the WC inverse by the core-EP decomposition. Moreover, in Section 6, we present several equivalent conditions for a matrix to be a weak core matrix.

2. Preliminaries

For convenience, we will use the following notations: $\mathbb{C}^{\mathrm{CM}}_n$, $\mathbb{C}^{\mathrm{EP}}_n$, $\mathbb{C}^{\mathrm{P}}_n$ and $\mathbb{C}^{\mathrm{OP}}_n$ will denote the subsets of $\mathbb{C}^{n\times n}$ consisting of core matrices, EP matrices, idempotent matrices and Hermitian idempotent matrices, respectively, i.e.,

- $\mathbb{C}^{\mathrm{CM}}_n = \{A \mid A \in \mathbb{C}^{n\times n}, r(A^2) = r(A)\}$;
- $\mathbb{C}^{\mathrm{EP}}_n = \{A \mid A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\}$;
- $\mathbb{C}^{\mathrm{P}}_n = \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A\}$;
- $\mathbb{C}^{\mathrm{OP}}_n = \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\}$.

Before giving characterizations of the WC inverse, we first present the following auxiliary lemmas which will be repeatedly used throughout this paper.

Lemma 2.1. [10] *Let $A \in \mathbb{C}^{n\times n}_k$. Then A can be represented as*

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{2.1}$$

where $T \in \mathbb{C}^{t\times t}$ is nonsingular and $t = r(T) = r(A^k)$, N is nilpotent with index k, and $U \in \mathbb{C}^{n\times n}$ is unitary.

Moreover, the representation of A given by (2.1) is unique [10, Theorem 2.4]. In that case, we have that

$$A^k = U\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix} U^*, \tag{2.2}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

Lemma 2.2. [1,9–12] *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *Then :*

$$A^{\dagger} = U\begin{bmatrix} T^*\triangle & -T^*\triangle S N^{\dagger} \\ (I_{n-t} - N^{\dagger}N)S^*\triangle & N^{\dagger} - (I_{n-t} - N^{\dagger}N)S^*\triangle S N^{\dagger} \end{bmatrix} U^*, \tag{2.3}$$

$$A^D = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{bmatrix} U^*, \tag{2.4}$$

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.5}$$

$$A^{D,\dagger} = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^{\dagger} \\ 0 & 0 \end{bmatrix} U^*, \tag{2.6}$$

$$A^{\dagger,D} = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T} \\ (I_{n-t} - N^{\dagger}N)S^*\triangle & (I_{n-t} - N^{\dagger}N)S^*\triangle T^{-k}\widetilde{T} \end{bmatrix} U^*, \tag{2.7}$$

$$A^{\textcircled{W}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \tag{2.8}$$

.

$$A^{\textcircled{W},\dagger} = U\begin{bmatrix} T^{-1} & T^{-2}SNN^{\dagger} \\ 0 & 0 \end{bmatrix} U^*, \tag{2.9}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$ *and* $\triangle = [TT^* + S(I_{n-t} - N^{\dagger}N)S^*]^{-1}$.

$\widetilde{T}$ and $\triangle$ will be often used throughout this paper.

Lemma 2.3. [1] *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

(a) $A^{\textcircled{W},\dagger} = A^{(2)}_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^{\dagger})}$;
(b) $AA^{\textcircled{W},\dagger} = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^{\dagger})}$;
(c) $A^{\textcircled{W},\dagger}A = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$.

Lemma 2.4. *Let $A \in \mathbb{C}_k^{n\times n}$ and $C = AA^{\textcircled{W}}A$. The following conditions hold:*

(a) [1] $A^{\textcircled{W},\dagger} = (A^{\textcircled{\dagger}})^2AP_A$;
(b) [1] $r(A^{\textcircled{W},\dagger}) = r(A^k)$;
(c) [1]$CA^{\dagger}C = C$;
(d) [9] $A^{\textcircled{W}}A^{k+1} = A^k$;
(e) $C^k = A^kA^{\textcircled{W}}A$.

Proof. Item (e) can be directly verified by (2.1), (2.2) and (2.8). □

3. Some characterizations of the WC inverse

Applying existing results for the WC inverse with respect to $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$, some new results can be obtained for the WC inverse in the next result.

Theorem 3.1. *Let $A \in \mathbb{C}_k^{n\times n}$ and $C = AA^{ⓦ}A$. The following statements are equivalent:*

(a) $X = A^{ⓦ,\dagger}$;
(b) $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$ *and* $XAA^* = A^{ⓦ}AA^*$;
(c) $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$ *and* $XA = A^{ⓦ}A$;
(d) $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $A^*AX = A^*CA^\dagger$;
(e) $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $AX = CA^\dagger$;
(f) $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $A^kX = C^kA^\dagger$;
(g) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$ *and* $XAA^{ⓦ} = A^{ⓦ}$;
(h) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$ *and* $XA^{k+1} = A^k$.

Proof. $(a) \Rightarrow (b)$. By the definition of $A^{ⓦ,\dagger}$, we have that $XAA^* = A^DCA^* = A^DAA^{ⓦ}AA^* = A^{ⓦ}AA^*$. Hence, by (a) of Lemma 2.3, we now obtain that (b) holds.

$(b) \Rightarrow (c)$. Postmultiplying $XAA^* = A^{ⓦ}AA^*$ by $(A^\dagger)^*$, we obtain that $XA = A^{ⓦ}A$.

$(c) \Rightarrow (d)$. From $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$, we have that $\mathcal{N}(AA^\dagger) \subseteq \mathcal{N}((A^k)^*A^2A^\dagger) = \mathcal{N}(X)$, which leads to $X = XAA^\dagger$. Thus we get that $X = XAA^\dagger = A^{ⓦ}AA^\dagger = A^{ⓦ,\dagger}$ by $XA = A^{ⓦ}A$. Hence, by the definition of $A^{ⓦ,\dagger}$ and (a) of Lemma 2.3, we have that (d) holds.

$(d) \Rightarrow (e)$. Evidently.

$(e) \Rightarrow (f)$. Since $C = AA^{ⓦ}A$ and $C^k = A^kA^{ⓦ}A$, premultiplying $AX = CA^\dagger$ by A^{k-1}, we have that $A^kX = C^kA^\dagger$.

$(f) \Rightarrow (g)$. From (2.2) and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we can set $X = U\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}U^*$, where $X_1 \in \mathbb{C}^{t\times t}$, $X_2 \in \mathbb{C}^{t\times(n-t)}$ and $t = r(A^k)$. Furthermore, it follows from $A^kX = C^kA^\dagger$ and (2.9) that $X = A^{ⓦ,\dagger}$. Therefore, by the definition of $A^{ⓦ,\dagger}$ and (a) of Lemma 2.3, we obtain that (g) holds.

$(g) \Rightarrow (h)$. It follows from $A^{ⓦ}A^{k+1} = A^k$ and $XAA^{ⓦ} = A^{ⓦ}$ that $XA^{k+1} = XAA^{ⓦ}A^{k+1} = A^{ⓦ}A^{k+1} = A^k$.

$(h) \Rightarrow (a)$. By $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XA^{k+1} = A^k$, we get that $XAX = X$. Hence, by (a) of Lemma 2.3, we get that $X = A^{ⓦ,\dagger}$. □

Now we will consider other characterizations of the WC inverse by the fact that $A^{ⓦ,\dagger}AA^{ⓦ,\dagger} = A^{ⓦ,\dagger}$.

Theorem 3.2. *Let $A \in \mathbb{C}_k^{n\times n}$ and $C = AA^{ⓦ}A$. The following statements are equivalent:*

(a) $X = A^{ⓦ,\dagger}$;
(b) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$;
(c) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $AX = CA^\dagger$;
(d) $XAX = X$, $AX = CA^\dagger$ *and* $XA^k = A^{ⓦ}A^k$;
(e) $XAX = X$, $XA = A^{ⓦ}A$ *and* $A^kX = C^kA^\dagger$;
(f) $XAX = X$, $XA = A^{ⓦ}A$ *and* $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger)$.

Proof. $(a) \Rightarrow (b)$. The proof can be demonstrated by (a) of Lemma 2.3.

$(b) \Rightarrow (c)$. By the definition of $A^{\textcircled{w},\dagger}$ and (b) of Lemma 2.3, we get that $AX \in \mathbb{C}_n^{\mathrm{P}}$, $\mathcal{R}(AX) = A\mathcal{R}(X) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k) = \mathcal{R}(AA^{\textcircled{w},\dagger}) = \mathcal{R}(CA^\dagger)$ and $\mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}((A^k)^*A^2A^\dagger) = \mathcal{N}(AA^{\textcircled{w},\dagger}) = \mathcal{N}(CA^\dagger)$. On the other hand, Lemma 2.4 (c) implies $CA^\dagger \in \mathbb{C}_n^{\mathrm{P}}$, hence $AX = CA^\dagger$.

$(c) \Rightarrow (d)$. By item (c) of Lemma 2.3, we obtain that $\mathcal{R}(X) = \mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{w},\dagger}A)$. So we get that $A^{\textcircled{w},\dagger}AX = X$, which implies that $XA^k = A^{\textcircled{w},\dagger}AXA^k = A^{\textcircled{w},\dagger}CA^\dagger A^k = A^{\textcircled{w},\dagger}AA^{\textcircled{w}}AA^\dagger A^k = A^{\textcircled{w}}A^k$.

$(d) \Rightarrow (e)$. By conditions and $AA^{\textcircled{w}} = A^k(A^{\textcircled{w}})^k$, we can infer that $X = XCA^\dagger = XAA^{\textcircled{w}}AA^\dagger = XA^k(A^{\textcircled{w}})^kAA^\dagger = A^{\textcircled{w}}A^k(A^{\textcircled{w}})^kAA^\dagger = A^{\textcircled{w},\dagger}$. Hence, by $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$ and $A^kA^{\textcircled{w}}A = C^k$, we obtain that (e) holds.

$(e) \Rightarrow (f)$. Since $XAX = X$, $XA = A^{\textcircled{w}}A$, we have that $\mathcal{R}(X) = \mathcal{R}(XA) = \mathcal{R}(A^{\textcircled{w}}A) = \mathcal{R}(A^k)$. We now obtain that $X = A^{\textcircled{w},\dagger}$ by (f) of Theorem 3.1. Hence (f) holds by (a) of Lemma 2.3.

$(f) \Rightarrow (a)$. It follows from $XAX = X$ that $\mathcal{N}(AX) = \mathcal{N}(X)$, by conditions and (a) of Lemma 2.3. We now obtain that $X = XAX = A^{\textcircled{w}}AX = A^{\textcircled{w}}AA^\dagger AX = A^{\textcircled{w},\dagger}P_{\mathcal{R}(AX),\ \mathcal{N}(AX)} = A^{\textcircled{w},\dagger}$. □

Notice the fact that $XA^{k+1} = A^k$ if $X = A^{\textcircled{w},\dagger}$. Therefore, we will characterize the WC inverse in terms of $A^{\textcircled{w},\dagger}A^{k+1} = A^k$.

Theorem 3.3. *Let $A \in \mathbb{C}_k^{n\times n}$ and $C = AA^{\textcircled{w}}A$. The following statements are equivalent:*

(a) $X = A^{\textcircled{w},\dagger}$;
(b) $XA^{k+1} = A^k$, $A^*AX = A^*CA^\dagger$ *and* $r(X) = r(A^k)$;
(c) $XA^{k+1} = A^k$, $AX = CA^\dagger$ *and* $r(X) = r(A^k)$;
(d) $XA^{k+1} = A^k$, $A^kX = C^kA^\dagger$ *and* $r(X) = r(A^k)$.

Proof. $(a) \Rightarrow (b)$. Since $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$, we can show that $XA^{k+1} = A^k$, $A^*AX = A^*CA^\dagger$. Then, by (b) of Lemma 2.4, we get that (b) holds.

$(b) \Rightarrow (c)$. Obviously.

$(c) \Rightarrow (d)$. Premultiplying $AX = CA^\dagger$ by A^{k-1}, we have that $A^kX = C^kA^\dagger$ from $A^kA^{\textcircled{w}}A = C^k$.

$(d) \Rightarrow (a)$. It follows from $XA^{k+1} = A^k$ and $r(X) = r(A^k)$ that $\mathcal{R}(X) = \mathcal{R}(A^k)$. Hence, we obtain that $X = A^{\textcircled{w},\dagger}$ from (f) of Theorem 3.1. □

In the following example, we show that the condition $r(X) = r(A^k)$ in Theorem 3.3 is necessary.

Example 3.4. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{Ind}(A) = 2$,

$$A^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A^{\textcircled{w},\dagger} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be directly verified that $XA^3 = A^2$, $A^*AX = A^*CA^\dagger$ and $r(X) \neq r(A^2)$, but $X \neq A^{\textcircled{w},\dagger}$. The other cases follow similarly.

By Lemma 2.3, it is clear that $AX = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)}$ and $XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$ if $X = A^{\textcircled{w},\dagger}$. However, the converse is invalid as shown in the next example:

Example 3.5. *Let A, X be the same as in Example 3.4. Then*

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad XA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{and } A^{\textcircled{w},\dagger} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be directly verified that $AX = P_{\mathcal{R}(A^2),\ \mathcal{N}((A^2)^*A^2A^\dagger)}$ *and* $XA = P_{\mathcal{R}(A^2),\ \mathcal{N}((A^2)^*A^2)}$, *but* $X \neq A^{\textcircled{w},\dagger}$.

In the next result, we will present some new equivalent conditions for the converse implication:

Theorem 3.6. *Let* $A \in \mathbb{C}_k^{n\times n}$ *and* $X \in \mathbb{C}^{n\times n}$. *The following statements are equivalent:*

(a) $X = A^{\textcircled{w},\dagger}$;
(b) $AX = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)}$, $XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$ *and* $r(X) = r(A^k)$;
(c) $AX = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)}$, $XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$ *and* $XAX = X$;
(d) $AX = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)}$, $XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$ *and* $AX^2 = X$.

Proof. $(a) \Rightarrow (b)$. The proof can be demonstrated by (b) and (c) of Lemma 2.3 and (b) of Lemma 2.4.

$(b) \Rightarrow (c)$. By $\mathcal{R}(XA) = \mathcal{R}(A^k)$ and $r(X) = r(A^k)$, we obtain that $\mathcal{R}(X) = \mathcal{R}(XA) = \mathcal{R}(A^k)$, hence we further derive that $XAX = X$.

$(c) \Rightarrow (d)$. By conditions and (a) of Lemma 2.3, we have that $X = A^{\textcircled{w},\dagger}$. Therefore, by (2.9), it can be directly verified that $AX^2 = X$.

$(d) \Rightarrow (a)$. From $AX^2 = X$, we have that $X = AX^2 = A^2X^3 = \cdots = A^kX^{k+1}$, which implies $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Combining with the condition $\mathcal{R}(A^k) = \mathcal{R}(XA) \subseteq \mathcal{R}(X)$, we get that $\mathcal{R}(X) = \mathcal{R}(A^k)$. From (2.2), we now set $X = U\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}U^*$, where $X_1 \in \mathbb{C}^{t\times t}$, $X_2 \in \mathbb{C}^{t\times(n-t)}$ and $t = r(A^k)$. On the other hand, it follows from $\mathcal{N}(AX) = \mathcal{N}((A^k)^*A^2A^\dagger)$ that $(A^k)^*A^2A^\dagger = (A^k)^*A^2X$, which yields $X_1 = T^{-1}$ and $X_2 = T^{-2}SNN^\dagger$. Therefore, by (2.9), we obtain that $X = A^{\textcircled{w},\dagger}$. □

In [1], the authors introduced the definition of $A^{\textcircled{w},\dagger}$ with an algebraic approach. In the next result, we will consider characterization of $A^{\textcircled{w},\dagger}$ with a geometrical point of view.

Theorem 3.7. *Let* $A \in \mathbb{C}_k^{n\times n}$. *Then:*

(a) $A^{\textcircled{w},\dagger}$ *is the unique matrix X that satisfies:*

$$AX = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)}, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k). \tag{3.1}$$

(b) $A^{\textcircled{w},\dagger}$ *is the unique matrix X that satisfies:*

$$XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}, \quad \mathcal{N}(A^*) \subseteq \mathcal{N}(X). \tag{3.2}$$

Proof. (a). Since $\mathcal{R}(A^D) = \mathcal{R}(A^k)$, it is a consequence of [2, Corollary 3.2] by properities of Drazin and MP inverse.

(b). Since items (c) of Lemma 2.3, $A^{\textcircled{w},\dagger}$ satisfies $XA = P_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2)}$. Additionally, we derive that $\mathcal{N}(A^*) = \mathcal{N}(A^\dagger) \subseteq \mathcal{N}(A^{\textcircled{w}}AA^\dagger) = \mathcal{N}(X)$. Now it remains to prove that X is unique.

Assume that X_1, X_2 satisfy (3.2), then $X_1A = X_2A$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(X_1)$ and $\mathcal{N}(A^*) \subseteq \mathcal{N}(X_2)$. Furthermore, we get that $(X_1 - X_2)A = 0$ and $\mathcal{R}(X_i^*) \subseteq \mathcal{R}(A)$ for $i = 1, 2$, which further imply that $A^*(X_1^* - X_2^*) = 0$ and $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{R}(A)$. Therefore we have that $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{R}(A) \cap \mathcal{N}(A^*) = \mathcal{R}(A) \cap \mathcal{R}(A)^\perp = \{0\}$. Thus, $X_1^* = X_2^*$, i.e., $X_1 = X_2$. □

Remark 3.8. In Theorem 3.7, $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$ in (3.1) can be replaced by $\mathcal{R}(X) = \mathcal{R}(A^k)$. However, if we replace $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$ with $\mathcal{N}(A^*) = \mathcal{N}(X)$ in (3.2), item (*b*) of Theorem 3.7 does not hold.

Characterizations of some generalized inverses by using its block matrices have been investigated in [13–17]. In [18, Theorem 3.2], the authors presented a characterization for the WC inverse using its block matrices. Next we will give another proof of it by using characterization of projection operator.

Theorem 3.9. *Let $A \in \mathbb{C}_k^{n\times n}$ and $r(A^k) = t$. Then there exist a unique matrix P such that*

$$P^2 = P, \;\; PA^k = 0, \;\; (A^k)^* A^2 P = 0, \;\; r(P) = n - t, \tag{3.3}$$

a unique matrix Q such that

$$Q^2 = Q, \;\; QA^k = 0, \;\; (A^k)^* A^2 A^\dagger Q = 0, \;\; r(Q) = n - t, \tag{3.4}$$

and a unique matrix X such that

$$r\left(\begin{bmatrix} A & I-Q \\ I-P & X \end{bmatrix}\right) = r(A). \tag{3.5}$$

Furthermore, X is the WC inverse $A^{\textcircled{w},\dagger}$ of A and

$$P = P_{\mathcal{N}((A^k)^*A^2),\, \mathcal{R}(A^k)}, \;\; Q = P_{\mathcal{N}((A^k)^*A^2A^\dagger),\, \mathcal{R}(A^k)}. \tag{3.6}$$

Proof. It is not difficult to prove that

$$\begin{aligned} \text{the condition (3.3) hold} \iff & (I-P)^2 = I-P, \; (I-P)A^k = A^k, \\ & (A^k)^* A^2 (I-P) = (A^k)^* A^2, \; r(P) = n-t \\ \iff & I - P = P_{\mathcal{R}(A^k),\, \mathcal{N}((A^k)^*A^2)} \\ \iff & P = P_{\mathcal{N}((A^k)^*A^2),\, \mathcal{R}(A^k)}. \end{aligned}$$

Similarly, we can show that (3.4) have the unique solution $Q = P_{\mathcal{N}((A^k)^*A^2A^\dagger),\, \mathcal{R}(A^k)}$.

Furthermore, comparing (3.6) and items (*b*) and (*c*) of Lemma 2.3 immediately leads to the conclusion that

$$r\left(\begin{bmatrix} A & I-Q \\ I-P & X \end{bmatrix}\right) = r\left(\begin{bmatrix} A & AA^{\textcircled{w},\dagger} \\ A^{\textcircled{w},\dagger}A & X \end{bmatrix}\right) = r(A) + r(X - A^{\textcircled{w},\dagger}).$$

By (3.5), we obtain that $X = A^{\textcircled{w},\dagger}$. □

4. Some representations of the WC inverse

In [19], Drazin introduced the (b, c)-inverse in semigroup. In [20], Benítez et al. investigated the (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$, as the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying [20]:

$$CAX = C, \quad XAB = B, \quad \mathcal{R}(X) = \mathcal{R}(B), \quad \mathcal{N}(X) = \mathcal{N}(C),$$

where $B, C \in \mathbb{C}^{n\times m}$. In the next result, we will show that the WC inverse is a special (B, C)-inverse.

Theorem 4.1. *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$A^{\text{Ⓦ},\dagger} = A^{(A^k,(A^k)^*A^2A^\dagger)}.$$

Proof. According to Lemma 2.3, we get that

$$\mathcal{R}(A^{\text{Ⓦ},\dagger}) = \mathcal{R}(A^k), \quad \mathcal{N}(A^{\text{Ⓦ},\dagger}) = \mathcal{N}((A^k)^*A^2A^\dagger)).$$

Observe that $A^{\text{Ⓦ},\dagger}AA^k = A^{\text{Ⓦ}}A^{k+1} = A^k$ and $(A^k)^*A^2A^\dagger AA^{\text{Ⓦ},\dagger} = (A^k)^*A^2A^{\text{Ⓦ}}AA^\dagger = (A^k)^*A^2A^\dagger$. Thus, we obtain $A^{\text{Ⓦ},\dagger} = A^{(A^k,(A^k)^*A^2A^\dagger)}$. □

In [21], the authors introduced the Bott-Duffin inverse of $A \in \mathbb{C}^{n\times n}$ when $AP_L + P_{L^\perp}$ is nonsingular, i.e., $A_L^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1} = P_L(AP_L + I - P_L)^{-1}$. In [22], the authors showed the weak group inverse by a special Bott-Duffin inverse. Next we will show that the WC inverse of A is indeed the Bott-Duffin inverse of A^2 with respect to $\mathcal{R}(A^k)$.

Theorem 4.2. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *Then*

$$A^{\text{Ⓦ},\dagger} = (A^2)^{(-1)}_{(\mathcal{R}(A^k))}AP_A = (P_{A^k}A^2P_{A^k})^\dagger AP_A. \tag{4.1}$$

Proof. It follows from (2.3) and (2.2) that

$$P_A = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^*, \tag{4.2}$$

$$P_{A^k} = U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{4.3}$$

We now obtain that

$$\begin{aligned}
(A^2)^{(-1)}_{(\mathcal{R}(A^k))}AP_A &= P_{A^k}(A^2P_{A^k} + I - P_{A^k})^{-1}AP_A \\
&= U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^2 & 0 \\ 0 & I_{n-t} \end{bmatrix}^{-1}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^* \\
&= U\begin{bmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* \\
&= A^{\text{Ⓦ},\dagger}.
\end{aligned}$$

Similarly, by a direct calculation, we can derive that $A^{\text{Ⓦ},\dagger} = (P_{A^k}A^2P_{A^k})^\dagger AP_A$. □

Working with the fact that $P = P_{\mathcal{N}((A^k)^*A^2),\, \mathcal{R}(A^k)}$ and $Q = P_{\mathcal{N}((A^k)^*A^2A^\dagger),\, \mathcal{R}(A^k)}$ in Theorem 3.9, we will consider other representations of $A^{\text{Ⓦ},\dagger}$ in the next theorem.

Theorem 4.3. *Let $A \in \mathbb{C}_k^{n\times n}$ and $P = P_{\mathcal{N}((A^k)^*A^2),\, \mathcal{R}(A^k)}$, $Q = P_{\mathcal{N}((A^k)^*A^2A^\dagger),\, \mathcal{R}(A^k)}$. Then for any $a, b \neq 0$, we have*

$$A^{\text{Ⓦ},\dagger} = (A + aP)^{-1}(I - Q) = (I - P)(A + bQ)^{-1}. \tag{4.4}$$

Proof. From items (b) and (c) of Lemma 2.3, it is not difficult to conclude that

$$(A + aP)A^{Ⓦ,\dagger} = I - Q.$$

Now we only need to show the invertibility of $A + aP$. Assume that $\alpha = U\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \mathbb{C}^n$ such that $(A + aP)\alpha = 0$, i.e., $A\alpha = -aP\alpha$, where $\alpha_1 \in \mathbb{C}^p$, $\alpha_2 \in \mathbb{C}^{n-p}$. Now it follows from condition (c) of Lemma 2.3 and (3.6) that

$$\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = -a\begin{bmatrix} 0 & -T^{-1}S - T^{-2}SN \\ 0 & I \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},$$

implying $\alpha_1 = 0$ and $\alpha_2 = 0$ since $a \neq 0$, N is nilpotent and T is non singular. Thus $A + aP$ is nonsingular.

Analogously, we can prove that $A + bQ$ is invertible and $A^{Ⓦ,\dagger} = (I - P)(A + bQ)^{-1}$. □

The limit expressions for some generalized inverses of matrices have been given in [14–17, 23, 24]. Similarly, the WC inverse can also be characterized as limit value as shown in the next result:

Theorem 4.4. *Let $A \in \mathbb{C}_k^{n\times n}$. Then:*

(a) $A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^*A^{k+2})^{-1}(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}$;
(b) $A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(A^k)^*A(\lambda I_n + A^{k+1}(A^k)^*A)^{-1}AA^*(\lambda I_n + AA^*)^{-1}$;
(c) $A^{Ⓦ,\dagger} = \lim_{\lambda\to 0}(\lambda I_n + A^k(A^k)^*A^2)^{-1}A^k(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}$;
(d) $A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}(\lambda I_n + A^{k+1}(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1})^{-1}$.

Proof. According to condition (a) of Lemma 2.3, it is not hard to show that

$$A^{Ⓦ,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A^2A^\dagger)} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A^2A^\dagger),\ \mathcal{N}(A^k(A^k)^*A^2A^\dagger)}.$$

Thus, by [25, Theorem 2.1], we have the following results:

(a) Let $X = A^k$, $Y = (A^k)^*A^2A^\dagger$ and by $A^\dagger = \lim_{\lambda\to 0} A^*(\lambda I_n + AA^*)^{-1}$. We have

$$A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^*A^{k+2})^{-1}(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}.$$

(b) Let $X = A^k(A^k)^*A$, $Y = AA^\dagger$ and by $A^\dagger = \lim_{\lambda\to 0} A^*(\lambda I_n + AA^*)^{-1}$. We have

$$A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(A^k)^*A(\lambda I_n + A^{k+1}(A^k)^*A)^{-1}AA^*(\lambda I_n + AA^*)^{-1}.$$

(c) Let $X = I_n$, $Y = A^k(A^k)^*A^2A^\dagger$ and by $A^\dagger = \lim_{\lambda\to 0} A^*(\lambda I_n + AA^*)^{-1}$. We have

$$A^{Ⓦ,\dagger} = \lim_{\lambda\to 0}(\lambda I_n + A^k(A^k)^*A^2)^{-1}A^k(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}.$$

(d) Let $X = A^k(A^k)^*A^2A^\dagger$, $Y = I_n$ and by $A^\dagger = \lim_{\lambda\to 0} A^*(\lambda I_n + AA^*)^{-1}$. We have

$$A^{Ⓦ,\dagger} = \lim_{\lambda\to 0} A^k(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1}(\lambda I_n + A^{k+1}(A^k)^*A^2A^*(\lambda I_n + AA^*)^{-1})^{-1}.$$

□

We end up this section with three examples of computing the WC inverse of a matrix using three different expressions in Theorems 4.2–4.4.

Example 4.5. *Let*

$$A=\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.5}$$

Then $\mathrm{Ind}(A)=2$ *and the weak core inverse of A is*

$$A^{\textcircled{w},\dagger}=A^2(A^4)^\dagger A^2A^\dagger=\begin{bmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Firstly, using the expression (4.1) *to compute the WC inverse of A. Then*

$$(A^2P_{A^2}+I-P_{A^2})^{-1}=\begin{bmatrix} 11/2 & -3 & 0 & 0 \\ -9/2 & 5/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } (P_{A^2}A^2P_{A^2})^\dagger=\begin{bmatrix} 11/2 & -3 & 0 & 0 \\ -9/2 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

After simplification, it follows that $(A^2)^{(-1)}_{(\mathcal{R}(A^2))}AP_A=A^{\textcircled{w},\dagger}$ *and* $(P_{A^2}A^2P_{A^2})^\dagger AP_A=A^{\textcircled{w},\dagger}$.
Secondly, using the expression (4.4), *we obtain*

$$(A-6P)^{-1}=\begin{bmatrix} 2 & -1 & -1/2 & -19/12 \\ -3/2 & 1 & 7/12 & 97/72 \\ 0 & 0 & -1/6 & -1/36 \\ 0 & 0 & 0 & -1/6 \end{bmatrix} \text{ and } (A+\frac{1}{5}Q)^{-1}=\begin{bmatrix} 2 & -1 & -1/2 & 5/2 \\ -3/2 & 1 & -2 & 10 \\ 0 & 0 & 5 & -25 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Therefore, it can be directly verified $(A-6P)^{-1}(I-Q)=A^{\textcircled{w},\dagger}$, $(I-P)(A+\frac{1}{5}Q)^{-1}=A^{\textcircled{w},\dagger}$.
Finally, using the limit expressions of item (*a*) *in Theorem 4.4.*
Let $B=A^2(\lambda I_n+(A^2)^*A^4)^{-1}(A^2)^*A^2A^*(\lambda I_n+AA^*)^{-1}$, *then*

$$\begin{aligned} B &= A^2(\lambda I_n+(A^2)^*A^4)^{-1}(A^2)^*A^2A^*(\lambda I_n+AA^*)^{-1} \\ &= \begin{bmatrix} 2(30321\lambda^2+5361\lambda+580)/\lambda_1 & 2(54629\lambda^2+6699\lambda-290)/\lambda_1 & (1305\lambda-58)/\lambda_2 & 0 \\ (110503\lambda^2+19405\lambda-870)/\lambda_1 & 4(49773\lambda^2+6090\lambda+145)/\lambda_1 & (2378\lambda+58)/\lambda_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

where $\lambda_1=\lambda^4+38734\lambda^3+1470569\lambda^2+197888\lambda+580$, $\lambda_2=\lambda^3+38697\lambda^2+38812\lambda+116$.
After simplification, it follows that

$$\lim_{\lambda\to 0}B=\lim_{\lambda\to 0}A^2(\lambda I_n+(A^2)^*A^4)^{-1}(A^2)^*A^2A^*(\lambda I_n+AA^*)^{-1}=A^{\textcircled{w},\dagger}.$$

The other cases in Theorem 4.4 can be similarly verified.

5. Some properties of the WC inverse

In this section, we discuss some properties of the WC inverse and consider the connection between the WC inverse and other known classes of matrices.

Lemma 5.1. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *Then:*

(a) $A \in \mathbb{C}_n^{\mathrm{EP}} \Leftrightarrow S = 0$ *and* $N = 0$*;*
(b) $A \in \mathbb{C}_n^{\mathrm{P}} \Leftrightarrow T = I_t$ *and* $N = 0$*;*
(c) $A \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow T = I_t,\ \ S = 0$ *and* $N = 0$*.*

Proof. (a) The proof can be easily verified from (2.1) and (2.3).

(b) By (2.1), we obtain that $A \in \mathbb{C}_n^{\mathrm{P}}$ is equivalent with

$$U\begin{bmatrix} T^2 & TS+SN \\ 0 & N^2 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*,$$

which is further equivalent with $T^2 = T$, $TS + SN = S$ and $N^2 = N$. Hence, by non singularity of T and $N^k = 0$, we can conclude that $A \in \mathbb{C}_n^{\mathrm{P}}$ if and only if $T = I_t$ and $N = 0$.

(c) Since $\mathbb{C}_n^{\mathrm{OP}} \subseteq \mathbb{C}_n^{\mathrm{P}}$, it is a direct consequence of item (b) and (2.1). □

Theorem 5.2. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *The following statements hold:*

(a) $A^{ⓦ,\dagger} = 0 \Leftrightarrow A$ is nilpotent;
(b) $A^{ⓦ,\dagger} = A \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$ and $A^3 = A$;
(c) $A^{ⓦ,\dagger} = A^* \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$ and $AA^* = P_{A^k}$;
(d) $A^{ⓦ,\dagger} = P_A \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{P}}$;
(e) $A^{ⓦ,\dagger} = P_{A^*} \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{OP}}$.

Proof. (a) By (2.1) and (2.9), we directly get that

$$\begin{aligned} A^{ⓦ,\dagger} = 0 \quad &\Longleftrightarrow \quad r(A^k) = t = 0 \\ &\Longleftrightarrow \quad A \ \text{ is nilpotent.} \end{aligned}$$

(b) It follows (2.1), (2.9) and (a) of lemma 5.1 that

$$\begin{aligned} A^{ⓦ,\dagger} = A \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &\Longleftrightarrow \quad S = 0\ ,\ \ N = 0 \ \ and \ \ T^3 = T \\ &\Longleftrightarrow \quad A \in \mathbb{C}_n^{\mathrm{EP}} \ \ and \ \ A^3 = A. \end{aligned}$$

(c) By (2.1), (2.9) and (a) of lemma 5.1, we have that

$$\begin{aligned} A^{ⓦ,\dagger} = A^* \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ &\Longleftrightarrow \quad S = 0\ ,\ \ N = 0 \ \ and \ \ TT^* = I_t \\ &\Longleftrightarrow \quad A \in \mathbb{C}_n^{\mathrm{EP}} \ \ and \ \ AA^* = P_{A^k}. \end{aligned}$$

(d) From (2.9), (4.2) and (b) of lemma 5.1, we obtain that

$$\begin{aligned} A^{\textcircled{W},\dagger} = P_A &\iff U\begin{bmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^* \\ &\iff T = I_t,\ \ N = 0 \\ &\iff A \in \mathbb{C}_n^{\mathrm{P}}. \end{aligned}$$

(e) It follows from (2.1) and (2.3) that

$$P_{A^*} = A^\dagger A = U\begin{bmatrix} T^*\triangle T & T^*\triangle S(I_{n-t} - N^\dagger N) \\ (I_{n-t} - N^\dagger N)S^*\triangle T & N^\dagger N + (I_{n-t} - N^\dagger N)S^*\triangle S(I_{n-t} - N^\dagger N) \end{bmatrix}U^*. \tag{5.1}$$

By (2.9) and (5.1), we now get that $A^{\textcircled{W},\dagger} = P_{A^*}$ is equivalent with

$$U\begin{bmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*\triangle T & T^*\triangle S(I_{n-t} - N^\dagger N) \\ (I_{n-t} - N^\dagger N)S^*\triangle T & N^\dagger N + (I_{n-t} - N^\dagger N)S^*\triangle S(I_{n-t} - N^\dagger N) \end{bmatrix}U^*,$$

which is further equivalent with $T^{-1} = T^*\triangle T$, $(I_{n-t}-N^\dagger N)S^*\triangle T = 0$ and $N^\dagger N+(I_{n-t}-N^\dagger N)S^*\triangle S(I_{n-t}-N^\dagger N) = 0$. Hence, by nonsingularity of $\triangle T$ and (c) of lemma 5.1, we can conclude that $A^{\textcircled{W},\dagger} = P_{A^*}$ if and only if $A \in \mathbb{C}_n^{\mathrm{OP}}$. □

From Lemma 2.3, we know that both $AA^{\textcircled{W},\dagger}$ and $A^{\textcircled{W},\dagger}A$ are oblique projectors. The next theorem will further discuss other characteriations for $AA^{\textcircled{W},\dagger}$ and $A^{\textcircled{W},\dagger}A$.

Theorem 5.3. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1)*. The following statements hold:*

(a) $AA^{\textcircled{W},\dagger} = P_A \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{CM}}$;
(b) $AA^{\textcircled{W},\dagger} = P_{A^*} \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$;
(c) $A^{\textcircled{W},\dagger}A = P_A \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$;
(d) $A^{\textcircled{W},\dagger}A = P_{A^*} \Leftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$.

Proof. It follows from (2.1) and (2.9) that

$$AA^{\textcircled{W},\dagger} = U\begin{bmatrix} I_t & T^{-1}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^*, \tag{5.2}$$

$$A^{\textcircled{W},\dagger}A = U\begin{bmatrix} I_t & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{bmatrix}U^*. \tag{5.3}$$

(a) By (4.2) and (5.2), the result can be directly verified.
(b) By (5.1) and (5.2), we can show that $AA^{\textcircled{W},\dagger} = P_{A^*}$ if and only if $(I_{n-t} - N^\dagger N)S^*\triangle T = 0$ and $N^\dagger N + (I_{n-t} - N^\dagger N)S^*\triangle S(I_{n-t} - N^\dagger N) = 0$, which is further equivalent with $S = 0$ and $N = 0$, i.e., $A \in \mathbb{C}_n^{\mathrm{EP}}$.
(c) It follows from (4.2) and (5.3) that $A^{\textcircled{W},\dagger}A = P_A$ is equivalent with $A \in \mathbb{C}_n^{\mathrm{EP}}$.
(d) From (5.1) and (5.3), it is similar to the proof of (b). □

Recall from [6] that the core inverse is necessarily EP. The next Theorem shows that this is not the case with the WC inverse.

Theorem 5.4. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and $t \in \mathbb{Z}^+$. The following statements are equivalent:*

(a) $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{EP}}$; (b) $SN = 0$;
(c) $A^{\textcircled{W},\dagger}A = A^{\textcircled{\dagger}}A$; (d) $A^tA^{\textcircled{W},\dagger} = A^tA^{\textcircled{\dagger}}$;
(e) $A^{\textcircled{W},\dagger}A^t = A^tA^{\textcircled{W}}$.

Proof. $(a) \Leftrightarrow (b)$. Since $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{EP}}$ is equivalent with $\mathcal{R}(A^{\textcircled{W},\dagger}) = \mathcal{R}((A^{\textcircled{W},\dagger})^*)$. Using (2.9), we have that $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{EP}}$ if and only if $SN = 0$.

$(c) \Leftrightarrow (b)$. By (2.5) and (5.3), it can be directly verified that $A^{\textcircled{W},\dagger}A = A^{\textcircled{\dagger}}A$ if and only if $SN = 0$.

$(d) \Leftrightarrow (b)$. By (2.5) and (2.9), it follows that

$$\begin{aligned} A^tA^{\textcircled{W},\dagger} = A^tA^{\textcircled{\dagger}} &\iff U\begin{bmatrix} T^{t-1} & T^{t-2}SNN^{\dagger} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{t-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &\iff SN = 0. \end{aligned}$$

$(e) \Leftrightarrow (b)$. From (2.8) and (2.9), it follows that

$$\begin{aligned} A^{\textcircled{W},\dagger}A^t = A^tA^{\textcircled{W}} &\iff U\begin{bmatrix} T^{t-1} & T^{t-2}S + T^{-2}T_tN \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{t-1} & T^{t-2}S \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{-2}T_tN = 0 \\ &\iff SN = 0. \end{aligned}$$

where $T_t = \sum_{j=0}^{t-1} T^jSN^{t-1-j}$. □

In [26], the authors introduced that a matrix A to be a weak group matrix if $A \in \mathbb{C}_n^{\mathrm{WG}}$, which is equivalent with $SN = 0$. Therefore, we have that following remark:

Remark 5.5. It is worth noting that conditions (a), (c)–(e) in Theorem 5.4 are equivalent with $A \in \mathbb{C}_n^{\mathrm{WG}}$.

The next theorems provide some equivalent conditions for $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}}$ and $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$.

Theorem 5.6. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *The following statements are equivalent:*

(a) $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}}$; (b) $T = I_t$;
(c) $AA^{\textcircled{W},\dagger} = A^{\textcircled{W},\dagger}$; (d) $A^{\textcircled{W},\dagger}A^k = A^k$;
(e) $A^k(A^{\textcircled{W},\dagger})^k = A^{\textcircled{W},\dagger}$; (f) $A(A^{\textcircled{W},\dagger})^k = (A^{\textcircled{W},\dagger})^k$.

Proof. $(a) \Leftrightarrow (b)$. From (2.9), it is not hard to prove that $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}}$ is equivalent with $T = I_t$.

$(c) \Leftrightarrow (b)$. From (2.9) and (5.3), it follows that $AA^{\textcircled{W},\dagger} = A^{\textcircled{W},\dagger}$ if and only if $T = I_t$.

$(d) \Leftrightarrow (b)$. By (2.2) and (2.9), it is easy to verify that $A^{\textcircled{W},\dagger}A^k = A^k$ if and only if $T = I_t$.

The proofs $(e) \Leftrightarrow (b)$ and $(f) \Leftrightarrow (b)$ are similar to the proof $(d) \Leftrightarrow (b)$. □

Theorem 5.7. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *The following statements are equivalent:*

(a) $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$; (b) $T = I_t$ *and* $SN = 0$;
(c) $AA^{\textcircled{W},\dagger} = (A^{\textcircled{W},\dagger})^*$; (d) $A^{\textcircled{W},\dagger}A = A^{\textcircled{W}}$;
(e) $(A^{\textcircled{W},\dagger})^kA^k = A^{\textcircled{W}}$; (f) $(A^{\textcircled{W},\dagger})^kA = (A^{\textcircled{W}})^k$.

Proof. $(a) \Leftrightarrow (b)$. From (2.9) and Theorem 5.6, we can show that $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$ is equivalent with $T = I_t$ and $SN = 0$.

$(c) \Leftrightarrow (b)$. By (2.9) and (5.3), it follows from $AA^{\textcircled{W},\dagger} = (A^{\textcircled{W},\dagger})^*$ that

$$\begin{bmatrix} I_t & T^{-1}SNN^\dagger \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T^{-1})^* & 0 \\ (T^{-2}SNN^\dagger)^* & 0 \end{bmatrix}.$$

Hence, we get that $AA^{\textcircled{W},\dagger} = (A^{\textcircled{W},\dagger})^*$ is equivalent with $T = I_t$ and $SN = 0$.

The proofs of $(d) \Leftrightarrow (b)$, $(e) \Leftrightarrow (b)$ and $(f) \Leftrightarrow (b)$ are similar to the proof of $(c) \Leftrightarrow (b)$. □

Corollary 5.8. *Let $A \in \mathbb{C}_k^{n\times n}$. Then $A \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if $A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}} \cap \mathbb{C}_n^{\mathrm{EP}}$.*

Proof. It is a direct consequence from Theorem 5.4 and Theorem 5.6. □

Working with Theorem 5.6 and Theorem 5.7, we have the following corollary.

Corollary 5.9. *Let $A \in \mathbb{C}_k^{n\times n}$ and for any $l \in \mathbb{N}$, $l \geq k$. The following statements statements hold:*

(a) $A \in \mathbb{C}_n^{\mathrm{P}} \Leftrightarrow A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}}$ *and* $A^l = A$;
(b) $A \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow A^{\textcircled{W},\dagger} \in \mathbb{C}_n^{\mathrm{P}}$ *and* $A^l = A^*$.

Proof. (a) The result can be easily derived by lemma 5.1 and Theorem 5.6,

(b) From Lemma 5.1 and Theorem 5.7, we can show that (b) holds. □

6. Different characterizations of weak core matrix

Ferreyra et al. [1] introduced the weak core matrix. The set of all $n\times n$ weak core matrices is denoted by $\mathbb{C}_n^{\mathrm{WC}}$, that is:

$$\mathbb{C}_n^{\mathrm{WC}} = \{A \mid A \in \mathbb{C}^{n\times n}, A^{\textcircled{W},\dagger} = A^{D,\dagger}\}.$$

In this section, we discuss some equivalent conditions satisfied by a matrix A such that $A \in \mathbb{C}_n^{WC}$ using the core-EP decomposition. For convenience, we introduce a necessary lemma.

Lemma 6.1. [1] *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *Then the following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{\mathrm{WC}}$;
(b) $SN^2 = 0$;
(c) $A^{\textcircled{W}} = A^D$.

Theorem 6.2. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and $t \in \mathbb{Z}^+$. The following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{\mathrm{WC}}$;
(b) $A^{\textcircled{W}}A = A^DA$;
(c) $A^tA^{\textcircled{W}}A = A^tA^DA$;
(d) $A^tA^{\textcircled{W},\dagger} = A^tA^{D,\dagger}$;
(e) $A^kA^{\textcircled{W},\dagger} = A^kA^\dagger$;
(f) $A^kA^{\textcircled{W}}A = A^k$.

Proof. $(a) \Rightarrow (b)$. It is a direct consequence from condition (c) of Lemma 6.1.

$(b) \Rightarrow (c)$. Evident.

$(c) \Rightarrow (a)$. By condition, it follows from (2.4) and (2.8) that

$$\begin{bmatrix} T^t & T^{t-1}S + T^{t-2}SN \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^t & T^{t-1}S + T^{t-k-1}\widetilde{T}N \\ 0 & 0 \end{bmatrix},$$

which implies $T^{t-k-1}(T^{k-2}SN^2 + \cdots + TSN^{k-1}) = 0$. We now obtain that $SN^2 = 0$ since T is invertible. By Lemma 6.1, we obtain that $A \in \mathbb{C}_n^{\mathrm{WC}}$.

$(a) \Leftrightarrow (d)$. It follows form (2.6), (2.9) and Lemma 6.1 that

$$\begin{aligned} A^t A^{Ⓦ,\dagger} = A^t A^{D,\dagger} &\iff U\begin{bmatrix} T^{t-1} & T^{t-2}SNN^\dagger \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{t-1} & T^{t-k-1}\widetilde{T}NN^\dagger \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{t-2}SNN^\dagger = T^{t-k-1}\widetilde{T}NN^\dagger \\ &\iff SN^2 = 0 \\ &\iff A \in \mathbb{C}_n^{\mathrm{WC}}. \end{aligned}$$

$(a) \Rightarrow (e)$. From the definition of the weak core matrix, we have that $A^k A^{Ⓦ,\dagger} = A^k A^{D,\dagger} = A^k A^\dagger$.

$(e) \Rightarrow (f)$. Evident.

$(f) \Rightarrow (a)$. If $A^k A^{Ⓦ} A = A^k$, by (2.2) and (2.8), we can conclude that SN^2. Hence item (a) holds. $\square$

Corollary 6.3. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1)*. Then $A \in \mathbb{C}_n^{\mathrm{WC}}$ if and only if $A^k = C^k$, where $C = AA^{Ⓦ}A$.*

Proof. Since $A^k A^{Ⓦ} A = C^k$, the result is a direct consequence of item (f) of Theorem 6.2. $\square$

Theorem 6.4. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and for some $t \in \mathbb{Z}^+$. The following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{\mathrm{WC}}$;
(b) $A(A^{Ⓦ})^t A = (A^{Ⓦ})^t A^2$;
(c) $(A^{Ⓦ})^t A = (A^{Ⓦ})^{t+1} A^2$;
(d) $A(A^{Ⓦ})^t A$ commutes with $(A^{Ⓦ})^t A^2$;
(e) $(A^{Ⓦ})^t A$ commutes with $(A^{Ⓦ})^{t+1} A^2$.

Proof. By (2.1) and (2.8), we get that

$$A(A^{Ⓦ})^t A = U\begin{bmatrix} T^{-t+2} & T^{-t+1}S + T^{-t}SN \\ 0 & 0 \end{bmatrix}U^*, \tag{6.1}$$

$$(A^{Ⓦ})^t A^2 = U\begin{bmatrix} T^{-t+2} & T^{-t+1}S + T^{-t}SN + T^{-t-1}SN^2 \\ 0 & 0 \end{bmatrix}U^*. \tag{6.2}$$

$(a) \Leftrightarrow (b)$. By (6.1), (6.2) and Lemma 6.1, we get that $A(A^{Ⓦ})^t A = (A^{Ⓦ})^t A^2$ if and only if $A \in \mathbb{C}_n^{\mathrm{WC}}$.

$(a) \Leftrightarrow (c)$. Similar to the part $(a) \Leftrightarrow (b)$.

$(a) \Leftrightarrow (d)$. It follows from (6.1), (6.2) and Lemma 6.1 that

$$A(A^{Ⓦ})^t A(A^{Ⓦ})^t A^2 - (A^{Ⓦ})^t A^2 A(A^{Ⓦ})^t A = U\begin{bmatrix} 0 & T^{-2t+1}SN^2 \\ 0 & 0 \end{bmatrix}U^*,$$

which implies that $A(A^{Ⓦ})^t A$ commutes with $(A^{Ⓦ})^t A^2$ if and only if $A \in \mathbb{C}_n^{\mathrm{WC}}$.

$(a) \Leftrightarrow (e)$. It is analogous to that of the part $(a) \Leftrightarrow (d)$. $\square$

Corollary 6.5. *Let $A \in \mathbb{C}_k^{n\times n}$ and $t \in \mathbb{Z}^+$. The following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{\mathrm{WC}}$;
(b) $A(A^{\textcircled{\dagger}})^t A^2 = (A^{\textcircled{\dagger}})^t A^3$;
(c) $(A^{\textcircled{\dagger}})^t A^2 = (A^{\textcircled{\dagger}})^{t+1} A^3$;
(d) $A(A^{\textcircled{\dagger}})^t A^2$ *commutes with* $(A^{\textcircled{\dagger}})^t A^3$;
(e) $(A^{\textcircled{\dagger}})^t A^2$ *commutes with* $(A^{\textcircled{\dagger}})^{t+1} A^3$.

Proof. Since $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$ and $A^{\textcircled{\dagger}} A A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$, Corollary 6.5 can be directly verified. □

7. Conclusions

In this paper, new characterizations and properties of the WC inverse are derived by using range, null space, matrix equations, respectively. Several expressions of the WC inverse are also given. Finally, we show various characterizations of the weak core matrix.

According to the current research background, more characterizations and applications for the WC inverse are worthy of further discussion which as follows:

1) Characterizing the WC inverse by maximal classes of matrices, full rank decomposition, integral expressions and so on;

2) New iterative algorithms and splitting methods for computing the WC inverse;

3) Using the WC inverse to solve appropriately constrained systems of linear equations;

4) Investigating the WC inverse of tensors.

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Conflict of interest

All authors read and approved the final manuscript. The authors declare no conflict of interest.

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