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Research article

Multi-stability analysis of fractional-order quaternion-valued neural networks with time delay

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Abstract: This paper addresses the problem of multi-stability analysis for fractional-order quaternion-valued neural networks (QVNNs) with time delay. Based on the geometrical properties of activation functions and intermediate value theorem, some conditions are derived for the existence of at least $(2\mathcal{K}_p^R + 1)^n, (2\mathcal{K}_p^I + 1)^n, (2\mathcal{K}_p^J + 1)^n, (2\mathcal{K}_p^K + 1)^n$ equilibrium points, in which $[(\mathcal{K}_p^R + 1)]^n, [(\mathcal{K}_p^I + 1)]^n, [(\mathcal{K}_p^J + 1)]^n, [(\mathcal{K}_p^K + 1)]^n$ of them are uniformly stable while the other equilibrium points become unstable. Thus the developed results show that the QVNNs can have more generalized properties than the real-valued neural networks (RVNNs) or complex-valued neural networks (CVNNs). Finally, two simulation results are given to illustrate the effectiveness and validity of our obtained theoretical results.

Keywords: multiple stability; fractional-order; caputo fractional derivative; quaternion-valued neural networks; time delay
Mathematics Subject Classification: 34D05,34D06, 34D20,34D23

1. Introduction

In recent years, fast development of fractional calculus has found extensive applications in different fields that possess dynamical nature and uncertain behaviors. Such applications include biological modelling, control theory, engineering etc., see for example [1, 2] and references therein, which motivates the scientists and researchers to concentrate on the analysis of fractional-order characteristics into the systems. Specifically, for the dynamic neural network like systems, the incorporation of the fractional-order neural networks in [3, 4] has gained desired better results than the integer-order ones in [5, 6]. This is because, the fractional-order derivatives exhibit an excellent memory and hereditary properties in representing the network model. Following based on the theory of fractional-order differential systems with discontinuous right hand sides (RHS), there have been plenty of results made for stability and synchronization in the literature [7–9]. Fractional-order LANE-EMDEN systems have been solved using multiple techniques in [10–14] and the dynamic analysis of a novel discrete fractional model for COVID-19 was carried out by the authors of [15]. Caputo derivatives have been used widely for fractional-order derivatives. The major advantage of Caputo derivative when compared to the Grunwald-Letnikov derivatives and the Riemann-Liouville derivatives is that they consider the initial value similar to the integer order differential equations (i.e.,) they include the lowest terminal t=a limit values of integer-order derivatives of unknown functions [16, 17]. In [18], the left caputo fractional derivative has been considered to solve the fractional order problems. The authors of [19] used Caputo-Fabrizio derivative to handle fractional partial differential equations. The authors of [20] used the concept of caputo derivatives to evaluate fractional Burgers equation via the Crank-Nicolson finite difference method. Also, in [21, 22], the authors used caputo integrals to evaluate certain inequalities. The main advantage of the currently used operator than the previously used operators is that it can be used to solve fractional order QVNN problems. The problem of the RVNNs and the CVNNs have gained much interest, because of their wide applications in the field of science and engineering such as, optimization computation, image processing, parallel computation, pattern recognition, computational optimization (see for instance [23–27]) and were availed to investigate the dynamical properties of the fractional-order nonlinear systems. The main difference between them is that the CVNNs process information using complex-valued parameters and variables; whereas, the RVNNs use real valued parameters and variable to process information. The QVNNs are an extension of the RVNNs or complex-valued systems, and due to the non commutativity of the quaternion algebra (see [28–33]), the quaternion problems are more difficult than that of real-valued systems or complex-valued systems, which is the reason for the slow development of quaternion fields. Various techniques have been used to analyse varied dynamical properties of the QVNNs for instance semidiscretization technique was used in [28], the authors of [32] used the inequality technique and in [33], the authors used the Lyapunov-Krasovskii functional. This field has received more increasing interest recently, due to its applications in information processing, optimization and automatic control [34, 35].

Moreover, according to various topological structures, the dynamical properties of the neural networks systems may experience different dynamical behaviors and hence the problem of stability analysis (see [36–43]), synchronization and bifurcations, were derived in [3] and [44]. Various types of synchronization such as finite-time synchronization using non-separation method [45] and global asymptotical and Mittag-Leffler synchronization with delays [46] has been carried out for QVNN. In general, the considered quaternion-valued networks tend to converge to an equilibrium point resulting in a periodic orbit or a chaotic trajectory. As one of the classical phenomenon of dynamic neural networks, the multi-stability analysis has been extensively studied in [7, 42, 47, 48]. It is known, that the multi-stability of the designed neural networks is the great requirement in various applications, such as optimal computations, associative memory, and pattern recognition. Especially in pattern recognition, the system can converge to a certain stable equilibrium point for the process of memory attainment where the pattern is stored as binary vectors. Thus, it is more important to analyze the existence of multiple equilibria and their stable points. In [7], the sigmoid functions are employed in a class of recurrent neural networks for the existence of $(2K_i + 1)^n$ equilibrium points and stability of $(K_i + 1)^n$ equilibrium points. In [47], a class of integer-order recurrent neural networks with unbounded time varying delays were introduced, and by using the geometrical properties of non-monotonic activation functions it was showed that the addressed system have exactly $(2K_i + 1)^n$ equilibrium points, of which $(K_i + 1)$ were locally asymptotically stable while others are unstable. One of the main novelty in this paper is to extend the results of multi stability of integer-order RVNNs or CVNNs, to those of fractional-order QVNNs, in order to show that there exists more equilibrium points in QVNNs than RVNNs, thus making this work to be different from most existing works with integer-order ones. This paper is devoted to presenting a theoretical multi stability analysis for fractional-order QVNNs with time delay.

Motivated by the above discussion, we explore the study of multiple stability results on fractional-order QVNNs with time delay. First, an n-dimensional QVNNs can be converted into a $4n$-dimensional RVNNs system by using the decomposition and non commutativity properties of quaternions. Then some sufficient conditions are derived for the fractional-order nonlinear systems for the existence of equilibrium points $[(2\mathcal{K}_p^R + 1)]^n, [(2\mathcal{K}_p^I + 1)]^n, [(2\mathcal{K}_p^J + 1)]^n, [(2\mathcal{K}_p^K + 1)]^n$. Finally, two numerical examples are given to demonstrate the effectiveness of the theoretical results.

The major contributions of this article include conversion of a n-dimensional QVNNs with time delays into a 4n-dimensional RVNNs system with the help of decomposition and non commutativity properties. Also, multi-stability analysis for the same has been carried out and we have obtained a conclusion that $[(2\mathcal{K}_p^R + 1)]^n, [(2\mathcal{K}_p^I + 1)]^n, [(2\mathcal{K}_p^J + 1)]^n, [(2\mathcal{K}_p^K + 1)]^n$ of the equilibrium points are uniformly stable while the other equilibrium points among $(2\mathcal{K}_p^R+1)^n, (2\mathcal{K}_p^I+1)^n, (2\mathcal{K}_p^J+1)^n, (2\mathcal{K}_p^K+1)^n$ of the equilibrium points become unstable. Two numerical simulation results are also provided to validate the obtained results. To the best of our knowledge, this multiple stability analysis for fractional order QVNNs with time delays has not yet been presented which sums up the novelty of our work.

The paper is organised as follows: Section 2 details the preliminary definitions and assumptions required. In Section 3, the main results and theorems are proposed for the multi-stability condition. Numerical simulation to explain the effectiveness of the proposed results are provided in Section 4. The results of the paper are encapsulated in Section 5 to provide a proper conclusion.

1.1. Notations

The notations used in this paper are as follows: The real field, the complex field and the skew field of quaternions are denoted as $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$, respectively.

2. Preliminaries

In this section, we will present some important definitions and Lemmas of fractional calculus which helps to prove the main results.

Definition 2.1. *[17, 49] The fractional integral of order $\upsilon > 0$ for a function $f(t)$ is defined as*

$$I^{\upsilon} f(t) = \frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\upsilon}} \mathrm{d}s, t \geq t_0.$$

$\Gamma(\upsilon)$ *is Gamma function and it is denoted as* $\Gamma(\upsilon) = \int_0^{\infty} \frac{e^{-t}}{t^{1-\upsilon}} \mathrm{d}t.$

Definition 2.2. *[17, 49] The Caputo fractional derivative of function $f(t) \in C^n([t_0, +\infty], \mathbb{R})$ with order $\upsilon > 0$ is defined as*

$$ {}^{C}_{t_0}D^{\upsilon}_{t} f(t) = \frac{1}{\Gamma(n-\upsilon)} \int_{t_0}^{t} (t-s)^{n-\upsilon-1} \frac{\mathrm{d}^n f(s)}{\mathrm{d}s} \mathrm{d}s, $$

where $t \geq t_0$ and n is a positive integer such that $n-1 < \upsilon < n$. Especially, when $0 < \upsilon < 1$,

$$ {}^{C}_{t_0}D^{\upsilon}_{t} f(t) = \frac{1}{\Gamma(1-\upsilon)} \int_{t_0}^{t} (t-s)^{-\upsilon} \frac{\mathrm{d}^1 f(s)}{\mathrm{d}s} \mathrm{d}s. $$

For convenience, for rest of the paper, we adopt the notion D^{υ} to denote the Caputo fractional derivative operator ${}^{C}_{t_0}D^{\upsilon}_{t}$.

Consider the following fractional-order QVNNs with time delay as follows:

$$ D^{\upsilon} h_p(t) = -\, d_p h_p(t) + \sum_{q=1}^{n} a_{pq} f_q(h_q(t)) + \sum_{q=1}^{n} b_{pq} g_q(h_q(t-\tau)) + \mathcal{R}_p, \tag{2.1} $$

or equivalently

$$ D^{\upsilon} h(t) = -\mathcal{D}h(t) + \mathcal{A}f(h(t)) + \mathcal{B}g(h(t-\tau)) + \mathcal{R}, \tag{2.2} $$

where $p = 1, 2, \ldots, n$, $h(t) = (h_1(t), h_2(t), \ldots, h_n(t))^T \in \mathbb{Q}^n$ is the state vector of neurons at time t; $\mathcal{D} = \mathrm{diag}\{d_1, d_2, \ldots, d_n\} \in \mathbb{Q}^{n\times n}$ is a positive diagonal matrix; $f(h(t)), g(h(t-\tau))$ denotes neuron activation functions without and with time delays respectively; τ denotes the constant time delay and satisfies $\tau > 0$; $\mathcal{A} \in \mathbb{Q}^{n\times n}, \mathcal{B} \in \mathbb{Q}^{n\times n}$ are the interconnection matrices without and with time delay respectively; $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n)^T \in \mathbb{Q}^n$ is an external input. It follows from the non-commutativity of quaternion multiplication resulting from Hamilton rules: $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$, we can write (2.2) as the following four real valued neural networks (2.3).

$$ \begin{cases} D^{\upsilon}h^R(t) = & -\mathcal{D}h^R(t) + \mathcal{A}^R f^R(h^R(t)) - \mathcal{A}^I f^I(h^I(t)) - \mathcal{A}^J f^J(h^J(t)) - \mathcal{A}^K f^K(h^K(t)) + \mathcal{B}^R f^R(h^R(t-\tau)) \\ & -\mathcal{B}^I f^I(h^I(t-\tau)) - \mathcal{B}^J f^J(h^J(t-\tau)) - \mathcal{B}^K f^K(h^K(t-\tau)) + \mathcal{R}^R, \\ D^{\upsilon}h^I(t) = & -\mathcal{D}h^I(t) + \mathcal{A}^R f^I(h^I(t)) + \mathcal{A}^I f^R(h^R(t)) + \mathcal{A}^J f^K(h^K(t)) - \mathcal{A}^K f^J(h^J(t)) + \mathcal{B}^R f^I(h^I(t-\tau)) \\ & +\mathcal{B}^I f^R(h^R(t-\tau)) + \mathcal{B}^J f^K(h^K(t-\tau)) - \mathcal{B}^K f^J(h^J(t-\tau)) + \mathcal{R}^I, \\ D^{\upsilon}h^J(t) = & -\mathcal{D}h^J(t) + \mathcal{A}^R f^J(h^J(t)) - \mathcal{A}^I f^K(h^K(t)) + \mathcal{A}^J f^R(h^R(t)) + \mathcal{A}^K f^I(z^I(t)) + \mathcal{B}^R f^J(h^J(t-\tau)) \\ & -\mathcal{B}^I f^K(h^K(t-\tau)) + \mathcal{B}^J f^R(h^R(t-\tau)) + \mathcal{B}^K f^I(z^I(t-\tau)) + \mathcal{R}^J, \\ D^{\upsilon}h^K(t) = & -\mathcal{D}h^K(t) + \mathcal{A}^R f^K(h^K(t)) + \mathcal{A}^I f^J(h^J(t)) - \mathcal{A}^J f^I(h^I(t)) + \mathcal{A}^K f^R(h^R(t)) + \mathcal{B}^R f^K(h^K(t-\tau)) \\ & +\mathcal{B}^I f^J(h^J(t-\tau)) - \mathcal{B}^J f^I(h^I(t-\tau)) + \mathcal{B}^K f^R(h^R(t-\tau)) + \mathcal{R}^K. \end{cases} \tag{2.3} $$

Definition 2.3. *[2] The equilibrium point of (2.2) is said to be stable if for any $\varepsilon > 0$, there exists $\lambda(t_0, \varepsilon) > 0$, such that $\|\psi - h^*\| < \lambda$ implies $\|h(t) - h^*\| < \varepsilon$ for any $t \geq t_0 \geq 0$. It is stable if λ is independent of t_0.*

Assumption 2.4. *The activation functions* $f_p^R(h_p^R), f_p^I(h_p^I), f_p^J(h_p^J), f_p^K(h_p^K)$ *are continuous and differentiable and there exist constants* $m_p^R < M_p^R, m_p^I < M_p^I, m_p^J < M_p^J, m_p^K < M_p^K$*, for* $p = 1, 2, \dots, n, \wp = R, I, J, K$ *such that*

$$\lim_{h_p^\wp \to -\infty} f_p^\wp(h_p^\wp) = m_p^\wp, \lim_{h_p^\wp \to +\infty} f_p^\wp(h_p^\wp) = M_p^\wp.$$

Also, there exists constants
$-\infty \leq \bar{\sigma}_p^{(0)} < \bar{\beta}_p^{(0)} < \bar{\sigma}_p^{(1)} < \bar{\beta}_p^{(1)} < \dots \bar{\sigma}_p^{(\mathcal{K}_p^R-1)} < \bar{\beta}_p^{(\mathcal{K}_p^R-1)} < \bar{\sigma}_p^{(\mathcal{K}_p^R)} < \bar{\beta}_p^{(\mathcal{K}_p^R)} \leq +\infty,$
$-\infty \leq \tilde{\sigma}_p^{(0)} < \tilde{\beta}_p^{(0)} < \tilde{\sigma}_p^{(1)} < \tilde{\beta}_p^{(1)} < \dots \tilde{\sigma}_p^{(\mathcal{K}_p^I-1)} < \tilde{\beta}_p^{(\mathcal{K}_p^I-1)} < \tilde{\sigma}_p^{(\mathcal{K}_p^I)} < \tilde{\beta}_p^{(\mathcal{K}_p^I)} \leq +\infty,$
$-\infty \leq \check{\sigma}_p^{(0)} < \check{\beta}_p^{(0)} < \check{\sigma}_p^{(1)} < \check{\beta}_p^{(1)} < \dots \check{\sigma}_p^{(\mathcal{K}_p^J-1)} < \check{\beta}_p^{(\mathcal{K}_p^J-1)} < \check{\sigma}_p^{(\mathcal{K}_p^J)} < \check{\beta}_p^{(\mathcal{K}_p^J)} \leq +\infty,$
$-\infty \leq \hat{\sigma}_p^{(0)} < \hat{\beta}_p^{(0)} < \hat{\sigma}_p^{(1)} < \hat{\beta}_p^{(1)} < \dots \hat{\sigma}_p^{(\mathcal{K}_p^K-1)} < \hat{\beta}_p^{(\mathcal{K}_p^K-1)} < \hat{\sigma}_p^{(\mathcal{K}_p^K)} < \hat{\beta}_p^{(\mathcal{K}_p^K)} \leq +\infty,$
$\delta_q^{\nu R}, \delta_q^{\nu I}, \delta_q^{\nu J}, \delta_q^{\nu K}, \bar{\delta}_q^{\nu R}, \bar{\delta}_q^{\nu I}, \bar{\delta}_q^{\nu J}, \bar{\delta}_q^{\nu K}, r = 0, 1, 2, \dots, \mathcal{K}_p^\nu, s = 1, 2, \dots, \mathcal{K}_p^\nu$ *such that for* $p, q = 1, \dots, n$, $|f_q^\nu(h_1^R, h_1^I, h_1^J, h_1^K) - f_q^\nu(h_2^R, h_2^I, h_2^J, h_2^K)| \leq \delta_q^{\nu R}|h_1^R - h_2^R| + \delta_q^{\nu I}|h_1^I - h_2^I| + \delta_q^{\nu J}|h_1^J - h_2^J| + \delta_q^{\nu K}|h_1^K - h_2^K|$ *for any* $h_1^{(\nu)}, h_2^{(\nu)} \in \left(\sigma_p^{(r)}, \beta_p^{(r)}\right)$, $|f_q^\nu(h_1^R, h_1^I, h_1^J, h_1^K) - f_q^\nu(h_2^R, h_2^I, h_2^J, h_2^K)| \leq \bar{\delta}_q^{\nu R}|h_1^R - h_2^R| + \bar{\delta}_q^{\nu I}|h_1^I - h_2^I| + \bar{\delta}_q^{\nu J}|h_1^J - h_2^J| + \bar{\delta}_q^{\nu K}|h_1^K - h_2^K|$ *for any* $h_1^{(\nu)}, h_2^{(\nu)} \in \left[\beta_p^{(s-1)}, \sigma_p^{(s)}\right]$, *for* $(\nu = R, \sigma_p^{(r)} = \bar{\sigma}_p^{(r)}, \beta_p^{(r)} = \bar{\beta}_p^{(r)}, \sigma_p^{(s)} = \bar{\sigma}_p^{(s)}, \beta_p^{(s-1)} = \bar{\beta}_p^{(s-1)}), (\nu = I, \sigma_p^{(r)} = \tilde{\sigma}_p^{(r)}, \beta_p^{(r)} = \tilde{\beta}_p^{(r)}, \sigma_p^{(s)} = \tilde{\sigma}_p^{(s)}, \beta_p^{(s-1)} = \tilde{\beta}_p^{(s-1)}), (\nu = J, \sigma_p^{(r)} = \check{\sigma}_p^{(r)}, \beta_p^{(r)} = \check{\beta}_p^{(r)}, \sigma_p^{(s)} = \check{\sigma}_p^{(s)}, \beta_p^{(s-1)} = \check{\beta}_p^{(s-1)}), (\nu = K, \sigma_p^{(r)} = \hat{\sigma}_p^{(r)}, \beta_p^{(r)} = \hat{\beta}_p^{(r)}, \sigma_p^{(s)} = \hat{\sigma}_p^{(s)}, \beta_p^{(s-1)} = \hat{\beta}_p^{(s-1)})$. *Let us define the following subsets* $S_1 = \left(\bar{\sigma}_p^{(r)}, \bar{\beta}_p^{(r)}\right) \times \left(\tilde{\sigma}_p^{(r)}, \tilde{\beta}_p^{(r)}\right) \times \left(\check{\sigma}_p^{(r)}, \check{\beta}_p^{(r)}\right) \times \left(\hat{\sigma}_p^{(r)}, \hat{\beta}_p^{(r)}\right), S_2 = \left[\bar{\beta}_p^{(s-1)}, \bar{\sigma}_p^{(s)}\right] \times \left[\tilde{\beta}_p^{(s-1)}, \tilde{\sigma}_p^{(s)}\right] \times \left[\check{\beta}_p^{(s-1)}, \check{\sigma}_p^{(s)}\right] \times \left[\hat{\beta}_p^{(s-1)}, \hat{\sigma}_p^{(s)}\right]$.

To determine the number of multiple equilibrium points, we define the following bounding functions:

$$\begin{aligned}
\bar{F}_p^-(u^R) = & -d_p u^R + (a_{pp}^R + b_{pq}^R) f_p^R(u^R) + \mathcal{R}_p^R + \sum_{q=1, q\neq p}^n \min\left\{(a_{pq}^R + b_{pq}^R)m_q^R, (a_{pq}^R + b_{pq}^R)M_q^R\right\} \\
& - \sum_{q=1}^n \left\{\max\left\{(a_{pq}^I + b_{pq}^I)m_q^I, (a_{pq}^I + b_{pq}^I)M_q^I\right\} - \max\left\{(a_{pq}^J + b_{pq}^J)m_q^J, (a_{pq}^J + b_{pq}^J)M_q^J\right\}\right. \\
& \left. - \max\left\{(a_{pq}^K + b_{pq}^K)m_q^K, (a_{pq}^K + b_{pq}^K)M_q^K\right\}\right\}, \\
\bar{F}_p^+(u^R) = & -d_p u^R + (a_{pp}^R + b_{pq}^R) f_p^R(u^R) + \mathcal{R}_p^R + \sum_{q=1, q\neq p}^n \max\left\{(a_{pq}^R + b_{pq}^R)m_q^R, (a_{pq}^R + b_{pq}^R)M_q^R\right\} \\
& - \sum_{q=1}^n \left\{\min\left\{(a_{pq}^I + b_{pq}^I)m_q^I, (a_{pq}^I + b_{pq}^I)M_q^I\right\} - \min\left\{(a_{pq}^J + b_{pq}^J)m_q^J, (a_{pq}^J + b_{pq}^J)M_q^J\right\}\right. \\
& \left. - \min\left\{(a_{pq}^K + b_{pq}^K)m_q^K, (a_{pq} + b_{pq}^K)M_q^K\right\}\right\}.
\end{aligned}$$

and $\bar{F}_p^-(u^I), \bar{F}_p^-(u^J), \bar{F}_p^-(u^K), \bar{F}_p^+(u^I), \bar{F}_p^+(u^J), \bar{F}_p^+(u^K)$ are defined similarly.

From Assumption 2.4 and the conditions $\bar{F}_p^-(u^\wp)$ and $\bar{F}_p^+(u^\wp)$ are continuous, it follows that

$$\lim_{u^\wp \to -\infty} \bar{F}_p^-(u^\wp) = +\infty, \lim_{u^\wp \to +\infty} \bar{F}_p^+(u^\wp) = -\infty. \tag{2.4}$$

We consider the constants $\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}, \tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}, \check{\sigma}_p^{(0)}, \check{\beta}_p^{(\mathcal{K}_p^J)}, \hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}$ such that for $p = 1, 2, \dots, n$, $\bar{F}_p^+(u^R) \geq \bar{F}_p^-(u^R) > 0, \bar{F}_p^+(u^I) \geq \bar{F}_p^-(u^I) > 0, \bar{F}_p^+(u^J) \geq \bar{F}_p^-(u^J) > 0, \bar{F}_p^+(u^K) \geq \bar{F}_p^-(u^K) > 0, \bar{F}_p^-(v^R) \leq$

$\bar{F}_p^+(v^R) < 0, \bar{F}_p^-(v^I) \le \bar{F}_p^+(v^I) < 0, \bar{F}_p^-(v^J) \le \bar{F}_p^+(v^J) < 0, \bar{F}_p^-(v^K) \le \bar{F}_p^+(v^K) < 0$, for all $u^R \le \bar{\sigma}_p^{(0)}, u^I \le \tilde{\sigma}_p^{(0)}, u^J \le \check{\sigma}_p^{(0)}, u^K \le \hat{\sigma}_p^{(0)}, v^R \ge \bar{\beta}_p^{(\mathcal{K}_p^R)}, v^I \ge \tilde{\beta}_p^{(\mathcal{K}_p^I)}, v^J \ge \check{\beta}_p^{(\mathcal{K}_p^J)}, v^K \ge \hat{\beta}_p^{(\mathcal{K}_p^K)}$.

For any given interval $\mathcal{H} \in \mathbb{R}$, and let $\mathcal{H}^0 = \emptyset, \mathcal{H}^1 = \mathcal{H}$. For convenience, we denote

$$\left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(0)}\right) = \left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(0)}\right)^1 \cup \left[\bar{\beta}_p^{(0)}, \bar{\sigma}_p^{(1)}\right]^0 \cup \ldots \cup \left[\bar{\beta}_p^{(\mathcal{K}_p^R-1)}, \bar{\sigma}_p^{(\mathcal{K}_p^R)}\right]^0 \cup \left(\bar{\sigma}_p^{(\mathcal{K}_p^R)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}\right)^0,$$

$$\left(\bar{\beta}_p^{(\mathcal{K}_p^R-1)}, \bar{\sigma}_p^{(0)}\right) = \left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\bar{\beta}_p^{(r-1)}, \bar{\sigma}_p^{(1)}\right]^1 \cup \left(\bar{\sigma}_p^{(r)}, \bar{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\bar{\sigma}_p^{(\mathcal{K}_p^R)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}\right)^0,$$

$$\left(\bar{\sigma}_p^{(\mathcal{K}_p^R)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}\right) = \left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\bar{\beta}_p^{(r-1)}, \bar{\sigma}_p^{(r)}\right]^0 \cup \left(\bar{\sigma}_p^{(r)}, \bar{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\bar{\sigma}_p^{(\mathcal{K}_p^R)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}\right)^1,$$

$$\left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(0)}\right) = \left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(0)}\right)^1 \cup \left[\tilde{\beta}_p^{(0)}, \tilde{\sigma}_p^{(1)}\right]^0 \cup \ldots \cup \left[\tilde{\beta}_p^{(\mathcal{K}_p^I-1)}, \tilde{\sigma}_p^{(\mathcal{K}_p^I)}\right]^0 \cup \left(\tilde{\sigma}_p^{(\mathcal{K}_p^I)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}\right)^0,$$

$$\left(\tilde{\beta}_p^{(\mathcal{K}_p^I-1)}, \tilde{\sigma}_p^{(0)}\right) = \left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\tilde{\beta}_p^{(r-1)}, \tilde{\sigma}_p^{(1)}\right]^1 \cup \left(\tilde{\sigma}_p^{(r)}, \tilde{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\tilde{\sigma}_p^{(\mathcal{K}_p^I)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}\right)^0,$$

$$\left(\tilde{\sigma}_p^{(\mathcal{K}_p^I)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}\right) = \left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\tilde{\beta}_p^{(r-1)}, \tilde{\sigma}_p^{(r)}\right]^0 \cup \left(\tilde{\sigma}_p^{(r)}, \tilde{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\tilde{\sigma}_p^{(\mathcal{K}_p^I)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}\right)^1,$$

$$\left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(0)}\right) = \left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(0)}\right)^1 \cup \left[\check{\beta}_p^{(0)}, \check{\sigma}_p^{(1)}\right]^0 \cup \ldots \cup \left[\check{\beta}_p^{(\mathcal{K}_p^J-1)}, \check{\sigma}_p^{(\mathcal{K}_p^J)}\right]^0 \cup \left(\check{\sigma}_p^{(\mathcal{K}_p^J)}, \check{\beta}_p^{(\mathcal{K}_p^J)}\right)^0,$$

$$\left(\check{\beta}_p^{(\mathcal{K}_p^J-1)}, \check{\sigma}_p^{(0)}\right) = \left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\check{\beta}_p^{(r-1)}, \check{\sigma}_p^{(1)}\right]^1 \cup \left(\check{\sigma}_p^{(r)}, \check{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\check{\sigma}_p^{(\mathcal{K}_p^J)}, \check{\beta}_p^{(\mathcal{K}_p^J)}\right)^0,$$

$$\left(\check{\sigma}_p^{(\mathcal{K}_p^J)}, \check{\beta}_p^{(\mathcal{K}_p^J)}\right) = \left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\check{\beta}_p^{(r-1)}, \check{\sigma}_p^{(r)}\right]^0 \cup \left(\check{\sigma}_p^{(r)}, \check{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\check{\sigma}_p^{(\mathcal{K}_p^J)}, \check{\beta}_p^{(\mathcal{K}_p^J)}\right)^1,$$

$$\left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(0)}\right) = \left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(0)}\right)^1 \cup \left[\hat{\beta}_p^{(0)}, \hat{\sigma}_p^{(1)}\right]^0 \cup \ldots \cup \left[\hat{\beta}_p^{(\mathcal{K}_p^K-1)}, \hat{\sigma}_p^{(\mathcal{K}_p^K)}\right]^0 \cup \left(\hat{\sigma}_p^{(\mathcal{K}_p^K)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}\right)^0,$$

$$\left(\hat{\beta}_p^{(\mathcal{K}_p^K-1)}, \hat{\sigma}_p^{(0)}\right) = \left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\hat{\beta}_p^{(r-1)}, \hat{\sigma}_p^{(1)}\right]^1 \cup \left(\hat{\sigma}_p^{(r)}, \hat{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\hat{\sigma}_p^{(\mathcal{K}_p^K)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}\right)^0,$$

$$\left(\hat{\sigma}_p^{(\mathcal{K}_p^K)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}\right) = \left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(0)}\right)^0 \cup \ldots \left[\hat{\beta}_p^{(r-1)}, \hat{\sigma}_p^{(r)}\right]^0 \cup \left(\hat{\sigma}_p^{(r)}, \hat{\beta}_p^{(r)}\right)^0 \cup \ldots \cup \left(\hat{\sigma}_p^{(\mathcal{K}_p^K)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}\right)^1,$$

for $r = 1, 2, \ldots, \mathcal{K}_p^\nu, p = 1, 2, \ldots, n$ $(\nu = R, I, J, K)$.

Then the region $\prod_{p=1}^n \left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(\mathcal{K}_p^R)}\right)$, $\prod_{p=1}^n \left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)}\right)$, $\prod_{p=1}^n \left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(\mathcal{K}_p^J)}\right)$ and $\prod_{p=1}^n \left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(\mathcal{K}_p^K)}\right)$ can be divided into $\prod_{p=1}^n (2\mathcal{K}_p^R + 1)$, $\prod_{p=1}^n (2\mathcal{K}_p^I + 1)$, $\prod_{p=1}^n (2\mathcal{K}_p^J + 1)$ and $\prod_{p=1}^n (2\mathcal{K}_p^K + 1)$ subsets respectively. Define the following set for any positive integer $\mathcal{N}$, $\Upsilon(\mathcal{K}_p^\nu + 1) = \left\{\Upsilon_r | r = 1, \ldots \mathcal{K}_p^\nu + 1\right\}$ with $\Upsilon_r = (\xi_1, \ldots, \xi_{\mathcal{N}})^T$ such that $\xi_r = 1, \xi_p = 0$ when $p \neq r$, for $r, p = 1, \ldots, \mathcal{N}$. Then, take $\bar{\Upsilon} = \Upsilon(\mathcal{K}_p^R + 1), \tilde{\Upsilon} = \Upsilon(\mathcal{K}_p^I + 1), \check{\Upsilon} = \Upsilon(\mathcal{K}_p^J + 1), \hat{\Upsilon} = \Upsilon(\mathcal{K}_p^K + 1)$. Therefore, $\mathbb{Q}^n$ can be divided into the following four regions as follows:

$$\bar{\Omega} = \left\{ \prod_{p=1}^n \left(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(0)}\right)^{\theta_1^{(p)}} \cup \left(\cup_{r=1}^{\mathcal{K}_p^R} \left[\bar{\beta}_p^{(r-1)}, \bar{\sigma}_p^{(r)}\right]^{\theta_{2r}^{(p)}} \cup \left(\bar{\sigma}_p^{(r)}, \bar{\beta}_p^{(r)}\right)^{\theta_{2r+1}^{(p)}} \right), \theta = \left(\theta_1^{(p)}, \ldots, \theta_{2\mathcal{K}_p^R+1}^{(p)}\right)^T \in \bar{\Upsilon} \right\},$$

$$\tilde{\Omega} = \left\{ \prod_{p=1}^n \left(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(0)}\right)^{\theta_1^{(p)}} \cup \left(\cup_{r=1}^{\mathcal{K}_p^I} \left[\tilde{\beta}_p^{(r-1)}, \tilde{\sigma}_p^{(r)}\right]^{\theta_{2r}^{(p)}} \cup \left(\tilde{\sigma}_p^{(r)}, \tilde{\beta}_p^{(r)}\right)^{\theta_{2r+1}^{(p)}} \right), \theta = \left(\theta_1^{(p)}, \ldots, \theta_{2\mathcal{K}_p^I+1}^{(p)}\right)^T \in \tilde{\Upsilon} \right\},$$

$$\check{\Omega} = \left\{ \prod_{p=1}^n \left(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(0)}\right)^{\theta_1^{(p)}} \cup \left(\cup_{r=1}^{\mathcal{K}_p^J} \left[\check{\beta}_p^{(r-1)}, \check{\sigma}_p^{(r)}\right]^{\theta_{2r}^{(p)}} \cup \left(\check{\sigma}_p^{(r)}, \check{\beta}_p^{(r)}\right)^{\theta_{2r+1}^{(p)}} \right), \theta = \left(\theta_1^{(p)}, \ldots, \theta_{2\mathcal{K}_p^J+1}^{(p)}\right)^T \in \check{\Upsilon} \right\},$$

$$\hat{\Omega} = \left\{ \prod_{p=1}^{n} \left(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(0)} \right)^{\theta_1^{(p)}} \cup \left(\cup_{r=1}^{\mathcal{K}_p^K} \left[\hat{\beta}_p^{(r-1)}, \hat{\sigma}_p^{(r)} \right]^{\theta_{2r}^{(p)}} \cup \left(\hat{\sigma}_p^{(r)}, \hat{\beta}_p^{(r)} \right)^{\theta_{2r+1}^{(p)}} \right), \theta = \left(\theta_1^{(p)}, \ldots, \theta_{2\mathcal{K}_p^K+1}^{(p)} \right)^T \in \hat{\Upsilon} \right\}.$$

Therefore, it is not difficult to show that $\bar{\Omega}$ has $(2\mathcal{K}_p^R + 1)^n$ members and it can be divided into two subsets with members grouped as $\bar{\Xi}^{(\text{one})} = (\mathcal{K}_p^R + 1)^n$ and $\bar{\Xi}^{(\text{two})} = (2\mathcal{K}_p^R + 1)^n - (\mathcal{K}_p^R + 1)^n$. $\tilde{\Xi}^{(\text{one})}, \check{\Xi}^{(\text{one})}, \hat{\Xi}^{(\text{one})}, \tilde{\Xi}^{(\text{two})}, \check{\Xi}^{(\text{two})}, \hat{\Xi}^{(\text{two})}$ can be defined similarly.

Remark 2.5. *In order to achieve the stability results of the QVNNs, it can be separated into $2n$-dimensional CVNNs with its real and imaginary parts and $4n$-dimensional RVNNs. Then, the regions $\prod_{p=1}^n(\bar{\sigma}_p^{(0)}, \bar{\beta}_p^{(\mathcal{K}_p^R)})$, $\prod_{p=1}^n(\tilde{\sigma}_p^{(0)}, \tilde{\beta}_p^{(\mathcal{K}_p^I)})$, $\prod_{p=1}^n(\check{\sigma}_p^{(0)}, \check{\beta}_p^{(\mathcal{K}_p^J)})$, $\prod_{p=1}^n(\hat{\sigma}_p^{(0)}, \hat{\beta}_p^{(\mathcal{K}_p^K)})$ can be divided into $\prod_{p=1}^n(2\mathcal{K}_p^R+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^I+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^J+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^K+1)$ respectively, and each elements in these subsets have at least one equlibrium point, that is, the neural networks (2.3) has at least $\prod_{p=1}^n(2\mathcal{K}_p^R+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^I+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^J+1)$, $\prod_{p=1}^n(2\mathcal{K}_p^K+1)$ equilibrium points.*

Lemma 2.6. *[7] If*

$$\begin{cases} F_p^{R^-}(\bar{\sigma}_p^{(r)}) > 0, & F_p^{R^+}(\bar{\beta}_p^{(r-1)}) < 0, \\ F_p^{I^-}(\tilde{\sigma}_p^{(r)}) > 0, & F_p^{I^+}(\tilde{\beta}_p^{(r-1)}) < 0, \\ F_p^{J^-}(\check{\sigma}_p^{(r)}) > 0, & F_p^{J^+}(\check{\beta}_p^{(r-1)}) < 0, \\ F_p^{K^-}(\hat{\sigma}_p^{(r)}) > 0, & F_p^{K^+}(\hat{\beta}_p^{(r-1)}) < 0, \end{cases} \tag{2.5}$$

for $p = 1, 2, \ldots, n$ and $r = 1, \ldots \mathcal{K}_p^\nu$, then each subset $\bar{\chi}^\theta \in \bar{\Omega}, \tilde{\chi}^\theta \in \tilde{\Omega}, \check{\chi}^\theta \in \check{\Omega}, \hat{\chi}^\theta \in \hat{\Omega}$ has at least one equilibrium point, and each $\bar{\chi}^{\bar{\theta}} \in \bar{\Xi}^{(\text{one})}, \tilde{\chi}^{\bar{\theta}} \in \tilde{\Xi}^{(\text{one})}, \check{\chi}^{\bar{\theta}} \in \check{\Xi}^{(\text{one})}, \hat{\chi}^{\bar{\theta}} \in \hat{\Xi}^{(\text{one})}$ is positively invariant.

Assumption 2.7. *$d_p, a_{pq}, b_{pq}, \delta_q, \eta_q$ satisfies the following conditions:*

$$\sum_{p=1}^{n} M(A) + \sum_{p=1}^{n} M(B) < D,$$

where,

$$\begin{aligned}
D =& \min(|1 - \max_p(d_p)|, \min_p(d_p)), \\
M(A) =& \max_{q \in S_1}(|a_{pq}^R|\delta_q^{R\nu}) + \max_{q \in S_1}(|a_{pq}^R|\delta_q^{I\nu}) + \max_{q \in S_1}(|a_{pq}^R|\delta_q^{J\nu}) + \max_{q \in S_1}(|a_{pq}^R|\delta_q^{K\nu}) + \max_{q \in S_1}(|a_{pq}^I|\delta_q^{R\nu}) + \max_{q \in S_1}(|a_{pq}^I|\delta_q^{I\nu}) \\
&+ \max_{q \in S_1}(|a_{pq}^I|\delta_q^{J\nu}) + \max_{q \in S_1}(|a_{pq}^I|\delta_q^{K\nu}) + \max_{q \in S_1}(|a_{pq}^J|\delta_q^{R\nu}) + \max_{q \in S_1}(|a_{pq}^J|\delta_q^{I\nu}) + \max_{q \in S_1}(|a_{pq}^J|\delta_q^{J\nu}) + \max_{q \in S_1}(|a_{pq}^J|\delta_q^{K\nu}) \\
&+ \max_{q \in S_1}(|a_{pq}^K|\delta_q^{R\nu}) + \max_{q \in S_1}(|a_{pq}^K|\delta_q^{I\nu}) + \max_{q \in S_1}(|a_{pq}^K|\delta_q^{J\nu}) + \max_{q \in S_1}(|a_{pq}^K|\delta_q^{K\nu}) + \max_{q \in S_2}(|a_{pq}^R|\bar{\delta}_q^{R\nu}) + \max_{q \in S_2}(|a_{pq}^R|\bar{\delta}_q^{I\nu}) \\
&+ \max_{q \in S_2}(|a_{pq}^R|\bar{\delta}_q^{J\nu}) + \max_{q \in S_2}(|a_{pq}^R|\bar{\delta}_q^{K\nu}) + \max_{q \in S_2}(|a_{pq}^I|\bar{\delta}_q^{R\nu}) + \max_{q \in S_2}(|a_{pq}^I|\bar{\delta}_q^{I\nu}) + \max_{q \in S_2}(|a_{pq}^I|\bar{\delta}_q^{J\nu}) + \max_{q \in S_2}(|a_{pq}^I|\bar{\delta}_q^{K\nu}) \\
&+ \max_{q \in S_2}(|a_{pq}^J|\bar{\delta}_q^{R\nu}) + \max_{q \in S_2}(|a_{pq}^J|\bar{\delta}_q^{I\nu}) + \max_{q \in S_2}(|a_{pq}^J|\bar{\delta}_q^{J\nu}) + \max_{q \in S_2}(|a_{pq}^J|\bar{\delta}_q^{K\nu}) + \max_{q \in S_2}(|a_{pq}^K|\bar{\delta}_q^{R\nu}) + \max_{q \in S_2}(|a_{pq}^K|\bar{\delta}_q^{I\nu}) \\
&+ \max_{q \in S_2}(|a_{pq}^K|\bar{\delta}_q^{J\nu}) + \max_{q \in S_2}(|a_{pq}^K|\bar{\delta}_q^{K\nu}), \\
M(B) =& \max_{q \in S_1}(|b_{pq}^R|\eta_q^{R\nu}) + \max_{q \in S_1}(|b_{pq}^R|\eta_q^{I\nu}) + \max_{q \in S_1}(|b_{pq}^R|\eta_q^{J\nu}) + \max_{q \in S_1}(|b_{pq}^R|\eta_q^{K\nu}) + \max_{q \in S_1}(|b_{pq}^I|\eta_q^{R\nu}) + \max_{q \in S_1}(|b_{pq}^I|\eta_q^{I\nu}) \\
&+ \max_{q \in S_1}(|b_{pq}^I|\eta_q^{J\nu}) + \max_{q \in S_1}(|b_{pq}^I|\eta_q^{K\nu}) + \max_{q \in S_1}(|b_{pq}^J|\eta_q^{R\nu}) + \max_{q \in S_1}(|b_{pq}^J|\eta_q^{I\nu}) + \max_{q \in S_1}(|b_{pq}^J|\eta_q^{J\nu}) + \max_{q \in S_1}(|b_{pq}^J|\eta_q^{K\nu})
\end{aligned}$$

$$+\max_{q\in S_1}(|b^K_{pq}|\eta^{R\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{I\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{J\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{K\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{I\nu}_q)$$
$$+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{K\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{I\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{K\nu}_q)$$
$$+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{I\nu}_q)+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{K\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{I\nu}_q)$$
$$+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{K\nu}_q),\ \forall\ \nu=R,I,J,K.$$

3. Main results

Here, define that h^*_p is the equilibrium point of (2.1), then, Eq (2.1) can be written as

$$0=-d_p h^*_p(t)+\sum_{q=1}^{n}a_{pq}f_q(h^*_q(t))+\sum_{q=1}^{n}b_{pq}g_q(h^*_q(t-\tau))+\mathcal{R}_p, \tag{3.1}$$

then making a coordinate transformation $e_p(t)=h_p(t)-h^*_p$, from (3.1) and (2.1) we obtain

$$D^{\upsilon}h_p(t)=-d_p h_p(t)+\sum_{q=1}^{n}a_{pq}f_q(h_q(t))+\sum_{q=1}^{n}b_{pq}g_q(h_q(t-\tau))-(-d_p h^*_p(t)+\sum_{q=1}^{n}a_{pq}f_q(h^*_q(t))$$
$$+\sum_{q=1}^{n}b_{pq}g_q(h^*_q(t-\tau))),$$
$$D^{\upsilon}e_p(t)=-d_p e_p(t)+\sum_{q=1}^{n}a_{pq}f_q+\sum_{q=1}^{n}b_{pq}g_{\tau q}, \tag{3.2}$$

where, $f_q=f_q(h_q(t))-f_q(h^*_q)$, $g_{\tau q}=f_q(h_q(t-\tau))-f_q(h^*_q)$. The initial condition of the given system (3.2) is given by

$$e(s)=\psi^R(s)+i\psi^I(s)+j\psi^J(s)+k\psi^K(s),\ s\in[-\tau,0]. \tag{3.3}$$

Therefore, to find the stability results of (2.1), we can turn to study its equivalent system (3.2). We can write (3.2) as four real valued neural networks similar to (2.3). In order to facilitate the proof of the following theorem, we introduce some notations in Appendix A.

Theorem 3.1. *If Assumptions 2.4, 2.7 and Lemma 1 hold for neural networks (2.3), then there exists at least* $\prod_{p=1}^{n}(2\mathcal{K}^R_p+1)$, $\prod_{p=1}^{n}(2\mathcal{K}^I_p+1)$, $\prod_{p=1}^{n}(2\mathcal{K}^J_p+1)$, $\prod_{p=1}^{n}(2\mathcal{K}^K_p+1)$ *equilibrium points and* $\prod_{p=1}^{n}(\mathcal{K}^R_p+1)$, $\prod_{p=1}^{n}(\mathcal{K}^I_p+1)$, $\prod_{p=1}^{n}(\mathcal{K}^J_p+1)$, $\prod_{p=1}^{n}(\mathcal{K}^K_p+1)$ *of the equilibrium points located in each positively invariant sets* $\bar{\Omega}^{\bar{\theta}}\in\bar{\Xi}^{(\mathrm{one})}$, $\tilde{\Omega}^{\bar{\theta}}\in\tilde{\Xi}^{(\mathrm{one})}$, $\check{\Omega}^{\bar{\theta}}\in\check{\Xi}^{(\mathrm{one})}$, $\hat{\Omega}^{\bar{\theta}}\in\hat{\Xi}^{(\mathrm{one})}$ *respectively, are stable.*

Proof. Based on the properties of Caputo fractional derivatives, we can write the real part of (3.2) as

$$D^{\upsilon}e^R_p=-d_p e^R_p+\sum_{q=1}^{n}\left[a^R_{pq}f^R_q-a^I_{pq}f^I_q-a^J_{pq}f^J_q-a^K_{pq}f^K_q\right]+\sum_{q=1}^{n}\left[b^R_{pq}g^R_{\tau q}-b^I_{pq}g^I_{\tau q}-b^J_{pq}g^J_{\tau q}-b^K_{pq}g^K_{\tau q}\right],$$

$$
\begin{aligned}
e_p^R =&D^{-\upsilon}\left[-d_p e_p^R + \sum_{q=1}^{n}\left[a_{pq}^R f_q^R - a_{pq}^I f_q^I - a_{pq}^J f_q^J\right] + \sum_{q=1}^{n}\left[-a_{pq}^K f_q^K + b_{pq}^R g_{\tau q}^R - b_{pq}^I g_{\tau q}^I - b_{pq}^J g_{\tau q}^J \ -b_{pq}^K g_{\tau q}^K\right]\right],\\
=&\frac{1}{\Gamma(\upsilon)}\int_0^t \frac{1}{(t-s)^{1-\upsilon}}\left[-d_p e_p^R + \sum_{q=1}^{n}\left[a_{pq}^R f_q^R - a_{pq}^I f_q^I - a_{pq}^J f_q^J - a_{pq}^K f_q^K\right]\right.\\
&\left.+\sum_{q=1}^{n}\left[b_{pq}^R g_{\tau q}^R - b_{pq}^I g_{\tau q}^I - b_{pq}^J g_{\tau q}^J - b_{pq}^K g_{\tau q}^K\right]\right] \mathrm{d}s.
\end{aligned}
$$

Multiplying e^{-t} and taking absolute value on both sides, we have

$$
\begin{aligned}
e^{-t}|e_p^R| =&\frac{1}{\Gamma(\upsilon)}\int_0^t \frac{e^{-t}}{(t-s)^{1-\upsilon}}\left[d_p|e_p^R| + \sum_{q=1}^{n}|a_{pq}^R||f_q^R| + \sum_{q=1}^{n}|a_{pq}^I||f_q^I| + \sum_{q=1}^{n}|a_{pq}^J||f_q^J|\right.\\
&\left.-\sum_{q=1}^{n}|a_{pq}^K||f_q^K| + \sum_{q=1}^{n}|b_{pq}^R||g_{\tau q}^R| + \sum_{q=1}^{n}|b_{pq}^I||g_{\tau q}^I| + \sum_{q=1}^{n}|b_{pq}^J||g_{\tau q}^J| + \sum_{q=1}^{n}||b_{pq}^K||g_{\tau q}^K|\right]\mathrm{d}s,\\
\le& d_p\int_0^t \mathcal{I}_1|e_p^R|\mathrm{d}s + + \sum_{q=1}^{n}|a_{pq}^R|\int_0^t \mathcal{I}_1[\delta_q^{RR}|e_q^R| + \delta_q^{RI}|e_q^I| + \delta_q^{RJ}|e_q^J| + \delta_q^{RK}|e_q^K|]\mathrm{d}s + \sum_{q=1}^{n}|a_{pq}^I|\int_0^t \mathcal{I}_1[\delta_q^{IR}|e_q^R|\\
&+\delta_q^{II}|e_q^I| + \delta_q^{IJ}|e_q^J| + \delta_q^{IK}|e_q^K|]\mathrm{d}s + \sum_{q=1}^{n}|a_{pq}^J|\times\int_0^t \mathcal{I}_1[\delta_q^{JR}|e_q^R| + \delta_q^{JI}|e_q^I| + \delta_q^{JJ}|e_q^J| + \delta_q^{JK}|e_q^K|]\mathrm{d}s\\
&+\sum_{q=1}^{n}|a_{pq}^K|\int_0^t \mathcal{I}_1[\delta_q^{KR}|e_q^R| + \delta_q^{KI}|e_q^I| + \delta_q^{KJ}|e_q^J| + \delta_q^{KK}|e_q^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^R|\int_0^t \mathcal{I}_2[\eta_q^{RR}|e_{\tau q}^R| + \eta_q^{RI}|e_{\tau q}^I|\\
&+\eta_q^{RJ}|e_{\tau q}^J| + \eta_q^{RK}|e_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^I|\int_0^t \mathcal{I}_2[\eta_q^{IR}|e_{\tau q}^R| + \eta_q^{II}|e_{\tau q}^I| + \eta_q^{IJ}|e_{\tau q}^J| + \eta_q^{IK}|e_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^J|\\
&\times\int_0^t \mathcal{I}_2[\eta_q^{JR}|e_{\tau q}^R| + \eta_q^{JI}|e_{\tau q}^I| + \eta_q^{JJ}|e_{\tau q}^J| + \eta_q^{JK}|e_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}||b_{pq}^K|\int_0^t \mathcal{I}_2[\eta_q^{KR}|e_{\tau q}^R| + \eta_q^{KI}|e_{\tau q}^I| + \eta_q^{KJ}|e_{\tau q}^J|\\
&+\eta_q^{KK}|e_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^R|\int_0^t \mathcal{I}_3[\eta_q^{RR}|\psi_{\tau q}^R| + \eta_q^{RI}|\psi_{\tau q}^I| + \eta_q^{RJ}|\psi_{\tau q}^J| + \eta_q^{RK}|\psi_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^I|\int_0^t \mathcal{I}_2[\eta_q^{IR}|\psi_{\tau q}^R|\\
&+\eta_q^{II}|\psi_{\tau q}^I| + \eta_q^{IJ}|\psi_{\tau q}^J| + \eta_q^{IK}|\psi_{\tau q}^K|]\mathrm{d}s + \sum_{q=1}^{n}|b_{pq}^J|\times\int_0^t \mathcal{I}_2[\eta_q^{JR}|\psi_{\tau q}^R| + \eta_q^{JI}|\psi_{\tau q}^I| + \eta_q^{JJ}|\psi_{\tau q}^J| + \eta_q^{JK}|\psi_{\tau q}^K|]\mathrm{d}s\\
&+\sum_{q=1}^{n}||b_{pq}^K|\int_0^t \mathcal{I}_2[\eta_q^{KR}|\psi_{\tau q}^R| + \eta_q^{KI}|\psi_{\tau q}^I| + \eta_q^{KJ}|\psi_{\tau q}^J| + \eta_q^{KK}|\psi_{\tau q}^K|]\mathrm{d}s,\\
=&d_p|\tilde{e}_p^{(1)}| + \sum_{l=1}^{4}\left[\bar{A}_p^{(l)}|\tilde{e}_q^{(l)}| + \bar{B}_p^{(l)}|\bar{e}_q^{(l)}| + \bar{B}_p^{(l)}|\bar{\psi}_q^{(l)}|\right],
\end{aligned}
$$

$$
\begin{aligned}
&= d_p|\tilde{e}_p^{(1)}| + \sum_{l=1}^{4}\left[\bar{A}_p^{(l)}|\tilde{e}_q^{(l)}| + \bar{B}_p^{(l)}|\hat{e}_q^{(l)}| + \bar{B}_p^{(l)}|\hat{\psi}_q^{(l)}|\right], \\
&= \max_p(d_p)|s(\check{e}_p^{(1)})| + \sum_{l=1}^{4}\left[\bar{A}_p^{(l)}|s(\check{e}_q^{(l)})| + \bar{B}_p^{(l)}|s(e_q^{(l)})| + \bar{B}_p^{(l)}|s(\psi_q^{(l)})|\right], \\
\|e^R\| &\le \sum_{q=1}^{n}\sup_t(e^{-t}|e_q^R|), \\
&\le \left[\max_p(d_p) + M_1^R\right]\|e^R\| + M_2^R\|e^I\| + M_3^R\|e^J\| + M_4^R\|e^K\| + \bar{M}_1^R\|\psi^R\| + \bar{M}_2^R\|\psi^I\| + \bar{M}_3^R\|\psi^J\| + \bar{M}_4^R\|\psi^K\|.
\end{aligned} \tag{3.4}
$$

From (3.4), one obtains

$$
\|e^R\| \le N_1^R\|e^I\| + N_2^R\|e^J\| + N_3^R\|e^K\| + \bar{N}_1^R\|\psi^R\| + \bar{N}_2^R\|\psi^I\| + \bar{N}_3^R\|\psi^J\| + \bar{N}_4^R\|\psi^K\|. \tag{3.5}
$$

Similar to the proof of $\|e^R\|$, we can estimate $\|e^I\|, \|e^J\|, \|e^K\|$ but are omitted here for space saving. Like $\|e^R\|$, we have

$$
\|e^I\| \le N_1^I\|e^R\| + N_2^I\|e^J\| + N_3^I\|e^K\| + \bar{N}_1^I\|\psi^I\| + \bar{N}_2^I\|\psi^I\| + \bar{N}_3^I\|\psi^J\| + \bar{N}_4^I\|\psi^K\|, \tag{3.6}
$$

$$
\|e^J\| \le N_1^J\|e^R\| + N_2^J\|e^I\| + N_3^J\|e^K\| + \bar{N}_1^J\|\psi^R\| + \bar{N}_2^J\|\psi^I\| + \bar{N}_3^J\|\psi^J\| + \bar{N}_4^J\|\psi^K\|, \tag{3.7}
$$

$$
\|e^K\| \le N_1^K\|e^R\| + N_2^K\|e^I\| + N_3^K\|e^J\| + \bar{N}_1^K\|\psi^R\| + \bar{N}_2^K\|\psi^I\| + \bar{N}_3^K\|\psi^J\| + \bar{N}_4^K\|\psi^K\|. \tag{3.8}
$$

From (3.5)–(3.8), we can easily obtain that

$$
\begin{cases}
\|e^R\| \le \Lambda_2^R\|\psi^R\| + \Lambda_3^R\|\psi^I\| + \Lambda_4^R\|\psi^J\| + \Lambda_5^R\|\psi^K\|, \\
\|e^I\| \le \Lambda_2^I\|\psi^R\| + \Lambda_3^I\|\psi^I\| + \Lambda_4^I\|\psi^J\| + \Lambda_5^I\|\psi^K\|, \\
\|e^J\| \le \Lambda_2^J\|\psi^R\| + \Lambda_3^J\|\psi^I\| + \Lambda_4^J\|\psi^J\| + \Lambda_5^J\|\psi^K\|, \\
\|e^K\| \le \Lambda_2^K\|\psi^R\| + \Lambda_3^K\|\psi^I\| + \Lambda_4^K\|\psi^J\| + \Lambda_5^K\|\psi^K\|,
\end{cases} \tag{3.9}
$$

where,

$$
\begin{aligned}
\Lambda_1^R &= N_1^R T_1^R + N_2^R T_{11}^R + N_3^R T_9^R, \\
\Lambda_1^I &= N_1^I T_{14}^I + N_2^I T_{12}^I + N_3^I T_{10}^I, \\
\Lambda_1^J &= N_1^J T_3^J + N_3^J T_{15}^J + N_2^J T_{17}^J, \\
\Lambda_1^K &= N_1^K T_9^K + N_2^K T_{10}^K + N_3^K T_7^K,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2^R &= \frac{1}{1-\Lambda_1^R}\left[\bar{N}_1^R\bar{D}_1^R + \bar{N}_1^I\bar{D}_2^R + \bar{N}_1^J\bar{D}_3^R + \bar{N}_1^K\bar{D}_4^R\right], \\
\Lambda_3^R &= \frac{1}{1-\Lambda_1^R}\left[\bar{N}_2^R\bar{D}_1^R + \bar{N}_2^I\bar{D}_2^R + \bar{N}_2^J\bar{D}_3^R + \bar{N}_2^K\bar{D}_4^R\right], \\
\Lambda_4^R &= \frac{1}{1-\Lambda_1^R}\left[\bar{N}_3^R\bar{D}_1^R + \bar{N}_3^I\bar{D}_2^R + \bar{N}_3^J\bar{D}_3^R + \bar{N}_3^K\bar{D}_4^R\right],
\end{aligned}
$$

$$\Lambda_5^R = \frac{1}{1-\Lambda_1^R}\left[\bar{N}_4^R\bar{D}_1^R + \bar{N}_4^I\bar{D}_2^R + \bar{N}_4^J\bar{D}_3^R + \bar{N}_4^K\bar{D}_4^R\right],$$

$\Lambda_v^I, \Lambda_v^J, \Lambda_v^K\ \forall v = 2,3,4,5$ can be defined similarly and $\bar{D}_l^R, \tilde{D}_l^I, \check{D}_l^J, \hat{D}_l^K, (l = 1,2,3,4)$ are given in Appendix B.

If we choose $\|\psi^R\| \leq \epsilon_1/4\Lambda_2^R,\ \|\psi^I\| \leq \epsilon_1/4\Lambda_3^R,\ \|\psi^J\| \leq \epsilon_1/4\Lambda_4^R,\ \|\psi^K\| \leq \epsilon_1/4\Lambda_5^R$. From (3.9), $\|e^R\|$ becomes

$$\|e^R\| \leq \epsilon_1. \tag{3.10}$$

Similarly, we can find from (3.9)

$$\|e^I\| \leq \epsilon_2,\ \|e^J\| \leq \epsilon_3,\ \|e^K\| \leq \epsilon_4, \tag{3.11}$$

where,
$\|\psi^R\| \leq \epsilon_2/4\Lambda_2^I, \|\psi^I\| \leq \epsilon_2/4\Lambda_3^I, \|\psi^J\| \leq \epsilon_2/4\Lambda_4^I, \|\psi^K\| \leq \epsilon_2/4\Lambda_5^I, \|\psi^R\| \leq \epsilon_3/4\Lambda_2^J, \|\psi^I\| \leq \epsilon_3/4\Lambda_3^J, \|\psi^J\| \leq \epsilon_3/4\Lambda_4^J, \|\psi^K\| \leq \epsilon_3/4\Lambda_5^J, \|\psi^R\| \leq \epsilon_4/4\Lambda_2^K, \|\psi^I\| \leq \epsilon_4/4\Lambda_3^K, \|\psi^J\| \leq \epsilon_4/4\Lambda_4^K, \|\psi^K\| \leq \epsilon_4/4\Lambda_5^K$.

From (3.10) and (3.11), we take for all $\epsilon = \max\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\} > 0$, there exists a $\omega = \epsilon/\max\{\omega_1,\omega_2,\omega_3,\omega_4\} > 0$, $\omega_1 = \max\left\{\Lambda_2^R,\Lambda_2^I,\Lambda_2^J,\Lambda_2^K\right\}, \omega_2 = \max\left\{\Lambda_3^R,\Lambda_3^I,\Lambda_3^J,\Lambda_3^K\right\}, \omega_3 = \max\left\{\Lambda_4^R,\Lambda_4^I,\Lambda_4^J,\Lambda_4^K\right\}, \omega_4 = \max\left\{\Lambda_5^R,\Lambda_5^I,\Lambda_5^J,\Lambda_5^K\right\}$, such that $\|e(t)\| < \epsilon$ when $\|\psi(t)\| < \omega$. Thus, the equilibrium point located in $\bar{\chi}^{\bar{\theta}}, \tilde{\chi}^{\bar{\theta}}, \check{\chi}^{\bar{\theta}}, \hat{\chi}^{\bar{\theta}}$ for the neural networks (2.3) is stable, and its uniqueness can also be guaranteed in the invariant sets $\bar{\chi}^{\bar{\theta}}, \tilde{\chi}^{\bar{\theta}}, \check{\chi}^{\bar{\theta}}, \hat{\chi}^{\bar{\theta}}$. This completes the proof. □

Remark 3.2. *In recent years, among the applications of fractional-order neural networks, multi-stability problems has become a much more interesting topic in the literature. Previously, stability of fractional-order RVNNs or CVNNs has been the attraction of many researchers [50–53]. In [7], by using the geometrical properties of non-monotonic activation functions, the authors present the problem of multi-stability of recurrent neural networks. Several sufficient conditions were derived to ensure the existence of* $\prod_{i=1}^n(2K_i+1)$ *equilibrium points with* $(K_i \geq 0)$ *in which* $\prod_{i=1}^n(K_i+1)$ *of them are stable. The multi-stability for classical integer-order QVNNs with time delay was considered in [32], by separating the proposed QVNNs into four RVNNs. In this paper, we made an attempt to study several sufficient conditions and are derived to achieve the existence of* $\prod_{p=1}^n(2\mathcal{K}_p^R+1), \prod_{p=1}^n(2\mathcal{K}_p^I+1), \prod_{p=1}^n(2\mathcal{K}_p^J+1), \prod_{p=1}^n(2\mathcal{K}_p^K+1)$ *equilibrium points, in which* $\prod_{p=1}^n(\mathcal{K}_p^R+1), \prod_{p=1}^n(\mathcal{K}_p^I+1), \prod_{p=1}^n(\mathcal{K}_p^J+1), \prod_{p=1}^n(\mathcal{K}_p^K+1)$ *of them are stable and others are unstable. It is obvious that the number of equilibrium points of QVNNs is much more than that of CVNNs as well as RVNNs and hence it is a generalized one.*

Remark 3.3. *According to Assumption 1 and the geometrical properties of activation functions, if* $\mathcal{K}_p^R = \mathcal{K}_p^I = \mathcal{K}_p^J = \mathcal{K}_p^K = 0$, $\bar{\sigma}_p^{(0)} = \tilde{\sigma}_p^{(0)} = \check{\sigma}_p^{(0)} = \hat{\sigma}_p^{(0)} = -\infty$, $\bar{\beta}_p^{(\mathcal{K}_p)} = \tilde{\beta}_p^{(\mathcal{K}_p)} = \check{\beta}_p^{(\mathcal{K}_p)} = \hat{\beta}_p^{(\mathcal{K}_p)} = +\infty$, *then we obtain only one equilibrium point for the neural networks (2.1) and is globally stable, which is similar to the previous works done.*

4. Numerical simulations

In this section, we present two examples to illustrate the theoretical results of this paper.

Example 4.1. *Consider the following two-dimensional fractional-order QVNNs:*

$$D^{\upsilon}h(t) = -\mathcal{D}h(t) + \mathcal{A}f(t) + \mathcal{B}g(t-\tau) + \mathcal{R}, \tag{4.1}$$

where the order $\upsilon = 0.98$*,* $h(t) = (h_1^T(t), h_2^T(t))^T$*,* $\mathcal{D} = \mathrm{diag}\,(1,1)$ *and all the other parameters are* $\mathcal{A} = \mathcal{A}^R + i\mathcal{A}^I + j\mathcal{A}^J + \mathcal{A}^K, \mathcal{B} = \mathcal{B}^R + i\mathcal{B}^I + j\mathcal{B}^J + \mathcal{B}^K, \mathcal{R} = (0,0)^T$*, where*

$$\begin{aligned}
\mathcal{A}^R &= \begin{pmatrix} 3 & 0.2 \\ 0.6 & 6 \end{pmatrix}, \mathcal{A}^I = \begin{pmatrix} 0.1 & -0.3 \\ 0.02 & 0.2 \end{pmatrix}, \\
\mathcal{A}^J &= \begin{pmatrix} 0.3 & 0.5 \\ 1.7 & 0.8 \end{pmatrix}, \mathcal{A}^K = \begin{pmatrix} 0.6 & -1.5 \\ 0.7 & 0.2 \end{pmatrix}, \\
\mathcal{B}^R &= \begin{pmatrix} -0.002 & 0.04 \\ -0.03 & -0.1 \end{pmatrix}, \mathcal{B}^I = \begin{pmatrix} 0.02 & 0.05 \\ 0.04 & -0.1 \end{pmatrix}, \\
\mathcal{B}^J &= \begin{pmatrix} -0.2 & 0.3 \\ -0.06 & 0.06 \end{pmatrix}, \mathcal{B}^K = \begin{pmatrix} 0.01 & 0.01 \\ 0.001 & -0.12 \end{pmatrix},
\end{aligned}$$

$f(t) = f^R + if^I + jf^J + kf^K, g(t-\tau) = g^R + ig^I + jg^J + kg^K$ *where* $f^R = f^I = f^J = f^K = \tanh(t)$ *and* $g^R = g^I = g^J = g^K = \tanh(t-\tau), \tau = 0.6$*. From Assumption 1, there exists* $m_p^R = m_p^I = m_p^J = m_p^K = -0.1, M_p^R = M_p^I = M_p^J = M_p^K = 0.1, p = 1,2$*. Based on the given activation functions and from Assumption 1 we choose* $\bar{\sigma}_p^{(0)} = \tilde{\sigma}_p^{(0)} = \check{\sigma}_p^{(0)} = \hat{\sigma}_p^{(0)} = +\infty, \bar{\beta}_p(0) = \tilde{\beta}_p(0) = \check{\beta}_p(0) = \hat{\beta}_p(0) = -2, \bar{\sigma}_p^{(1)} = \tilde{\sigma}_p^{(1)} = \check{\sigma}_p^{(1)} = \hat{\sigma}_p^{(1)} = 2, \bar{\beta}_p^{(1)} = \tilde{\beta}_p^{(1)} = \check{\beta}_p^{(1)} = \hat{\beta}_p^{(1)} = +\infty$ *for* $p = 1,2$*. It is easy to verify that the condition (2.5) in Lemma 1 are satisfied for* $\mathcal{K}_1^{\gamma} = \mathcal{K}_2^{\gamma} = 1$*. Hence, Theorem 1 satisfies the conditions with above parameters and the neural network (4.1) has* $\prod_{p=1}^{2}(2\mathcal{K}_p^R + 1), \prod_{p=1}^{2}(2\mathcal{K}_p^I + 1), \prod_{p=1}^{2}(2\mathcal{K}_p^J + 1), \prod_{p=1}^{2}(2\mathcal{K}_p^K + 1)$ *equilibrium points in which* $\prod_{p=1}^{2}(\mathcal{K}_p^R + 1), \prod_{p=1}^{2}(\mathcal{K}_p^I + 1), \prod_{p=1}^{2}(\mathcal{K}_p^J + 1), \prod_{p=1}^{2}(\mathcal{K}_p^K + 1)$ *of them are stable. The simulation results are given in Figures 1–12 with 100 random initial conditions.*

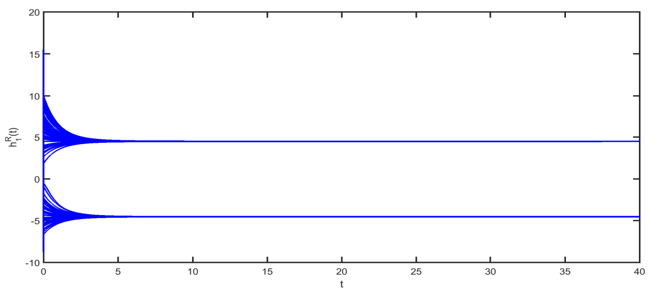

Figure 1. This figure depicts the trajectories of state variables $h_1^R(t)$ in Example 1.

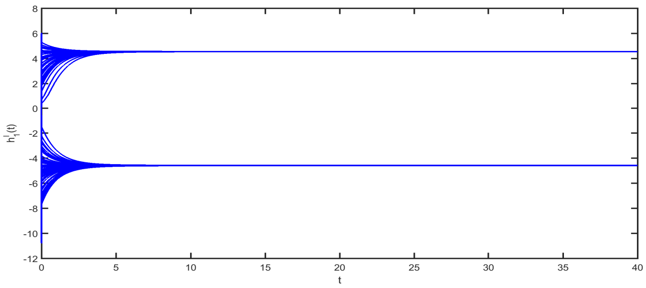

Figure 2. This figure depicts the trajectories of state variables $h_1^I(t)$ in Example 1.

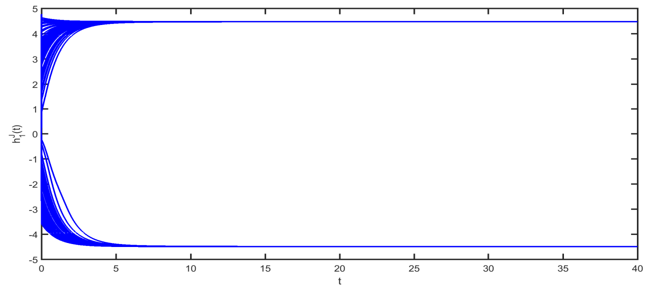

Figure 3. This figure depicts the trajectories of state variables $h_1^J(t)$ in Example 1.

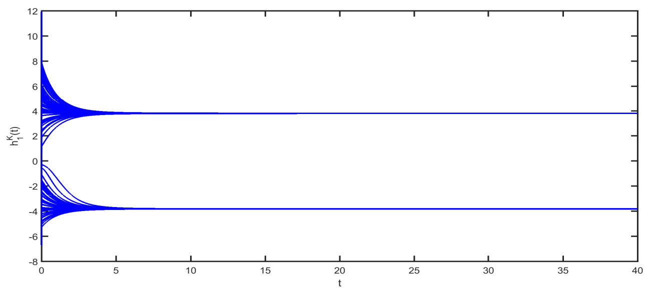

Figure 4. This figure depicts the trajectories of state variables $h_1^K(t)$ in Example 1.

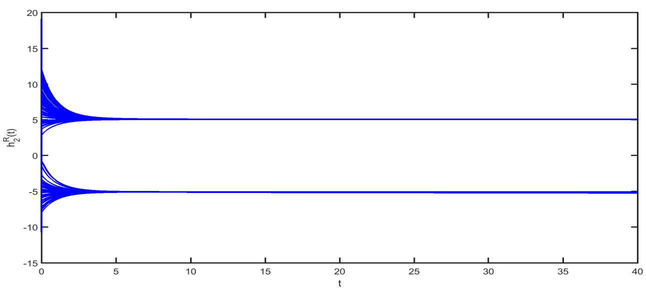

Figure 5. This figure depicts the trajectories of state variables $h_2^R(t)$ in Example 1.

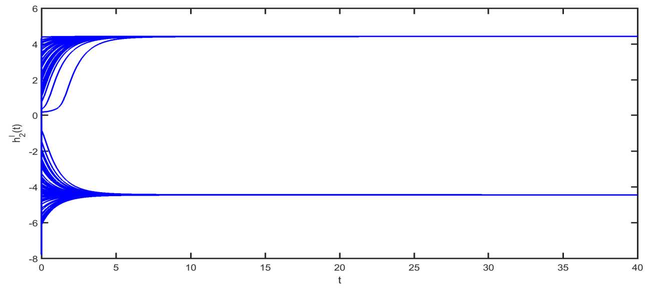

Figure 6. This figure depicts the trajectories of state variables $h_2^I(t)$ in Example 1.

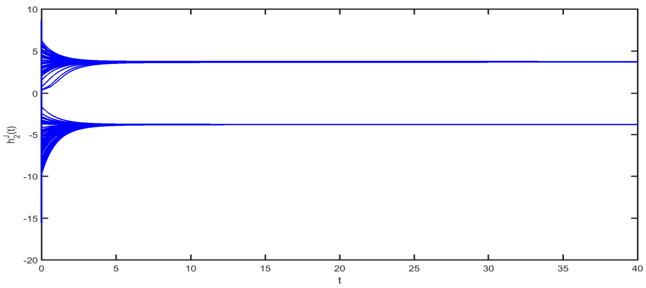

Figure 7. This figure depicts the trajectories of state variables $h_2^J(t)$ in Example 1.

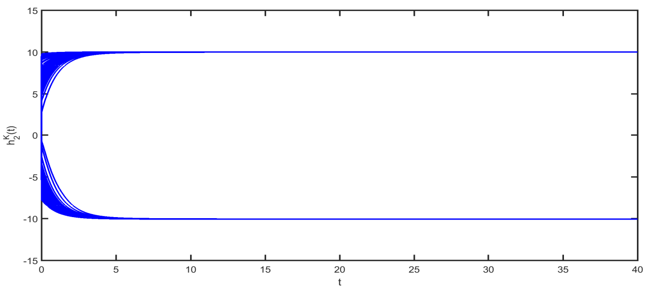

Figure 8. This figure depicts the trajectories of state variables $h_2^K(t)$ in Example 1.

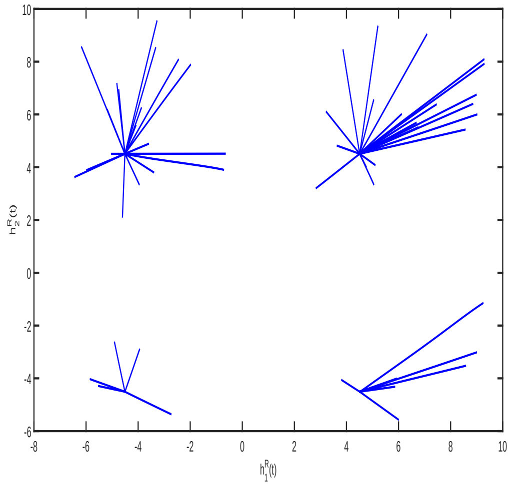

Figure 9. This figure depicts the phase graph of the state variables $h_1^R(t)$ and $h_2^R(t)$ in Example 1.

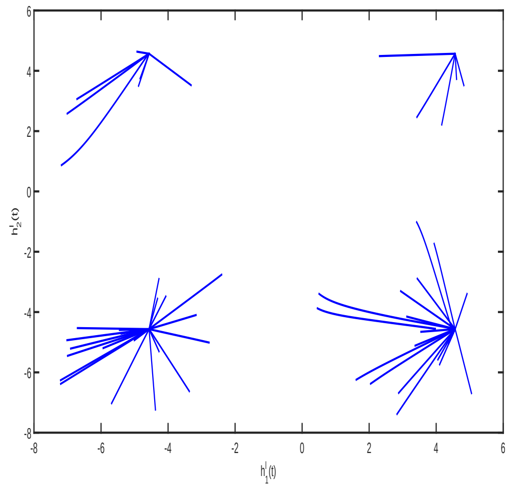

Figure 10. This figure depicts the phase graph of the state variables $h_1^I(t)$ and $h_2^I(t)$ in Example 1.

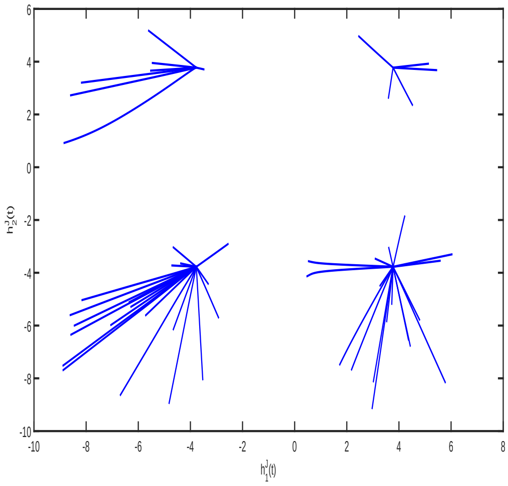

Figure 11. This figure depicts the phase graph of the state variables $h_1^J(t)$ and $h_2^J(t)$ in Example 1.

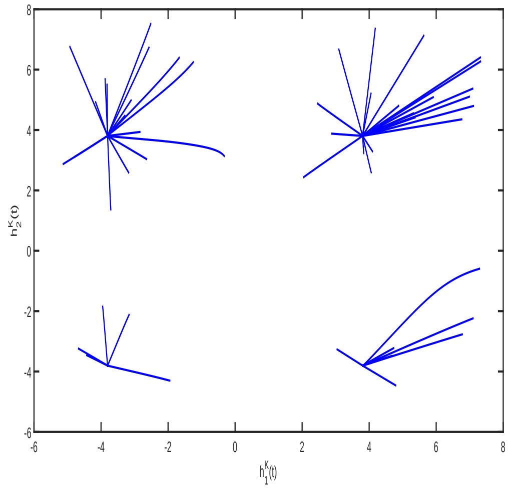

Figure 12. This figure depicts the phase graph of the state variables $h_1^K(t)$ and $h_2^K(t)$ in Example 1.

Example 4.2. *Consider the fractional-order* ($\upsilon = 0.9$) *QVNNs (4.1) with* $\mathcal{D} = \mathrm{diag}\,(1, 1)$

$$A^R = \begin{pmatrix} 4 & 0.3 \\ 0.7 & 6.2 \end{pmatrix}, B^R = \begin{pmatrix} 0.46 & 0.04 \\ -0.3 & -0.1 \end{pmatrix},$$
$$A^I = \begin{pmatrix} 0.35 & -0.62 \\ 0.2 & 0.86 \end{pmatrix}, B^I = \begin{pmatrix} 0.2 & 0.05 \\ 0.04 & -0.1 \end{pmatrix},$$
$$A^J = \begin{pmatrix} 0.5 & 0.7 \\ 0.7 & 1.8 \end{pmatrix}, B^J = \begin{pmatrix} -0.3 & 0.3 \\ -0.06 & 0.1 \end{pmatrix},$$

$$A^K = \begin{pmatrix} 1.8 & -1.4 \\ 1.9 & 2.2 \end{pmatrix}, B^K = \begin{pmatrix} 0.15 & 0.1 \\ 0.3 & -0.12 \end{pmatrix},$$

the activation functions are $f_p^R = f_p^I = f_p^J = f_p^K = \exp(-u^2)$. $\tau = 0.6$. *Hence* $m_p^R = m_p^I = m_p^J = m_p^K = -0.2, M_p^R = M_p^I = M_p^J = M_p^K = 0.2$, *for* $p = 1, 2$. *From the activation functions, the chosen points are* $\bar{\sigma}_p^{(0)} = \tilde{\sigma}_p^{(0)} = \check{\sigma}_p^{(0)} = \hat{\sigma}_p^{(0)} = -\infty, \bar{\beta}_p^{(0)} = \tilde{\beta}_p^{(0)} = \check{\beta}_p^{(0)} = \hat{\beta}_p^{(0)} = -1.5, \bar{\sigma}_p^{(1)} = \tilde{\sigma}_p^{(1)} = \check{\sigma}_p^{(1)} = \hat{\sigma}_p^{(1)} = 1.5$, $\bar{\beta}_p^{(1)} = \tilde{\beta}_p^{(1)} = \check{\beta}_p^{(1)} = \hat{\beta}_p^{(1)} = +\infty, p = 1, 2$. *It is easy to verify that the conditions (2.5) in Lemma 1 are satisfied for* $\mathcal{K}_1^\gamma = \mathcal{K}_2^\gamma = 1$. *Hence, Theorem 1 satisfies the conditions with above parameters and so the considered neural network has* $\prod_{p=1}^2(2\mathcal{K}_p^R + 1), \prod_{p=1}^2(2\mathcal{K}_p^I + 1), \prod_{p=1}^2(2\mathcal{K}_p^J + 1), \prod_{p=1}^2(2\mathcal{K}_p^K + 1)$ *equilibrium points in which* $\prod_{p=1}^2(\mathcal{K}_p^R + 1), \prod_{p=1}^2(\mathcal{K}_p^I + 1), \prod_{p=1}^2(\mathcal{K}_p^J + 1), \prod_{p=1}^2(\mathcal{K}_p^K + 1)$ *of them are stable. The simulation results are given in Figures 13–20 with 100 random initial conditions.*

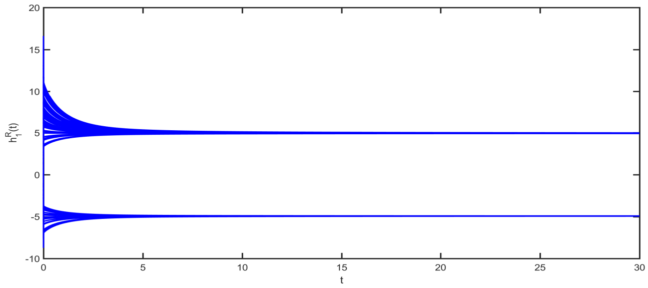

Figure 13. This figure depicts the trajectories of state variables $h_1^R(t)$ in Example 2.

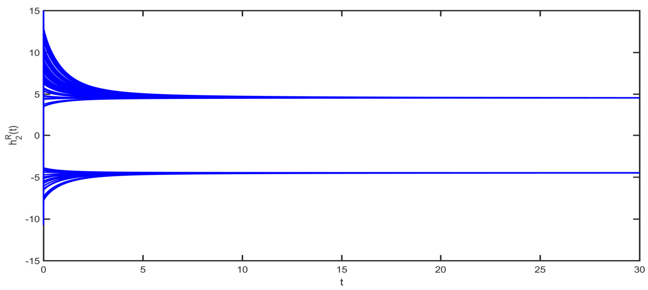

Figure 14. This figure depicts the trajectories of state variables $h_2^R(t)$ in Example 2.

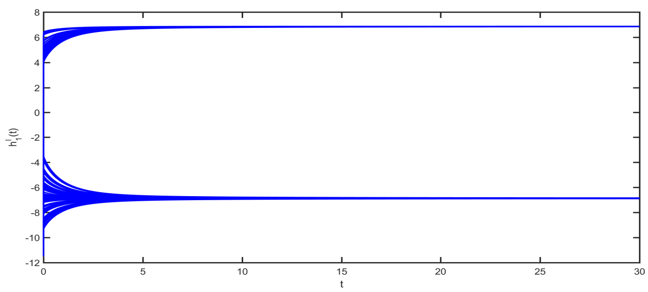

Figure 15. This figure depicts the trajectories of state variables $h_1^I(t)$ in Example 2.

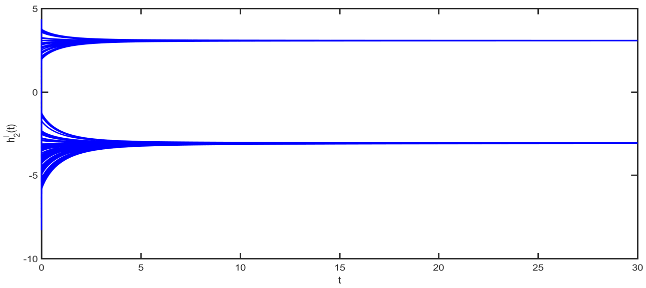

Figure 16. This figure depicts the trajectories of state variables $h_2^I(t)$ in Example 2.

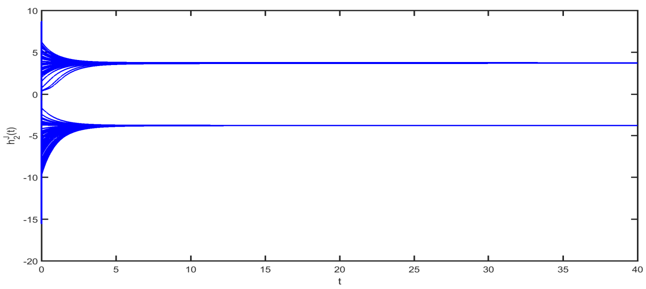

Figure 17. This figure depicts the trajectories of state variables $h_1^J(t)$ in Example 2.

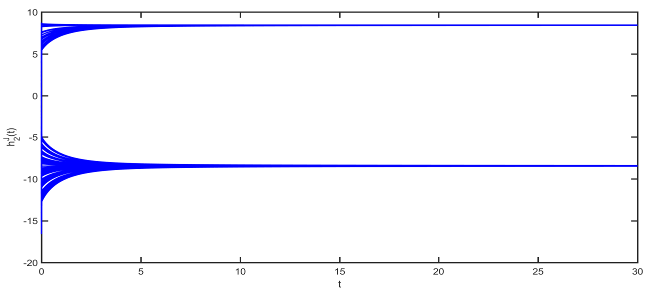

Figure 18. This figure depicts the trajectories of state variables $h_2^J(t)$ in Example 2.

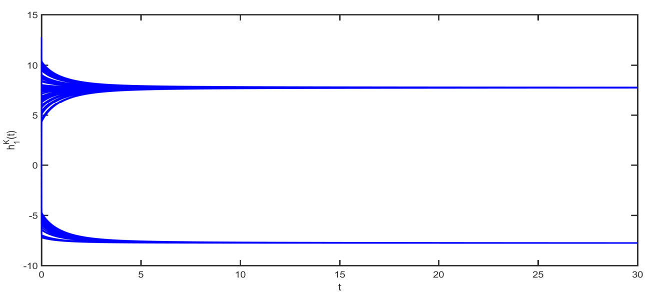

Figure 19. This figure depicts the trajectories of state variables $h_1^K(t)$ in Example 2.

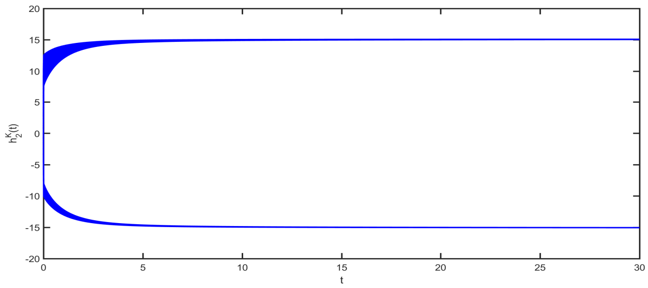

Figure 20. This figure depicts the trajectories of state variables $h_2^K(t)$ in Example 2.

5. Conclusions

For a given set of parameters, multistability refers to the existence of multiple final states that are stable. The initial conditions are critical in determining the convergence of the system to the final state. In this paper, we have investigated the multi-stability analysis for fractional-order QVNNs with delay. By employing the non-commutativity of quaternion multiplications, the QVNNs can be converted into four RVNNs. According to the definition of activation functions, the space of states can be divided into $\prod_{p=1}^{n}(2\mathcal{K}_p^R+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^I+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^J+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^K+1)$ subsets. Some sufficient conditions are derived to assure the existence and stability of multiple equilibrium points for the QVNNs. Under these conditions, QVNNs have $\prod_{p=1}^{n}(2\mathcal{K}_p^R+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^I+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^J+1)$, $\prod_{p=1}^{n}(2\mathcal{K}_p^K+1)$ equilibrium points, of which $\prod_{p=1}^{n}(\mathcal{K}_p^R+1)$, $\prod_{p=1}^{n}(\mathcal{K}_p^I+1)$, $\prod_{p=1}^{n}(\mathcal{K}_p^J+1)$, $\prod_{p=1}^{n}(\mathcal{K}_p^K+1)$ are stable while the other equilibrium points are unstable. Numerical simulation results have been presented to show efficiency of our theoretical results. In comparision with the previous works of real-valued neural networks and complex-valued neural networks, the extension of neural networks in Quaternion form is a more broader extension and dealing multi-stability with fractional-order with time delays makes it different from the existing works in the literature.

Future researches can concentrate in multi-stability analysis of fractional-order multiple Quaternion-valued neural networks with time delay. Also, the researchers can also concentrate in developing varied discussion on stability and synchronization analysis of multiple Quaternion-valued neural network.

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Conflict of interest

All the authors declare no conflict of interest.

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Appendix

In order to facilitate the proof of Theorem 1, we introduce some notations when $(l=1, \nu=R), (l=2, \nu=I), (l=3, \nu=J), (l=4, \nu=K)$.

$$|s(\psi_q^{(l)})| = \sum_{q=1}^{n} \sup_t (e^{-t}|\psi_q^{\nu}|)e^{-\tau}, |s(e_q^{(l)})| = \sum_{q=1}^{n} \sup_t (e^{-t}|e_q^{\nu}|)e^{-\tau}, |s(\check{e}_p^{(1)})| = \sum_{p=1}^{n} \sup_t (e^{-t}|e_p^{R}|), |s(\check{e}_q^{(l)})| = \sum_{q=1}^{n} \sup_t (e^{-t}|e_q^{\nu}|),$$

$$\begin{aligned}
\bar{A}_p^{(l)} =& \max_{q\in S_1}(|a_{pq}^R|\delta_q^{R\nu}) + \max_{q\in S_1}(|a_{pq}^I|\delta_q^{I\nu}) + \max_{q\in S_1}(|a_{pq}^J|\delta_q^{J\nu}) + \max_{q\in S_1}(|a_{pq}^K|\delta_q^{K\nu}) + \max_{q\in S_2}(|a_{pq}^R|\bar{\delta}_q^{R\nu}) + \max_{q\in S_2}(|a_{pq}^I|\bar{\delta}_q^{I\nu}) \\
&+ \max_{q\in S_2}(|a_{pq}^J|\bar{\delta}_q^{J\nu}) + \max_{q\in S_2}(|a_{pq}^K|\bar{\delta}_q^{K\nu}), \\
\tilde{A}_p^{(l)} =& \max_{q\in S_1}(|a_{pq}^R|\delta_q^{I\nu}) + \max_{q\in S_1}(|a_{pq}^I|\delta_q^{R\nu}) + \max_{q\in S_1}(|a_{pq}^J|\delta_q^{K\nu}) + \max_{q\in S_1}(|a_{pq}^K|\delta_q^{J\nu}) + \max_{q\in S_2}(|a_{pq}^R|\bar{\delta}_q^{I\nu}) + \max_{q\in S_2}(|a_{pq}^I|\bar{\delta}_q^{R\nu})
\end{aligned}$$

$$
\begin{aligned}
&+\max_{q\in S_2}(|a^J_{pq}|\bar{\delta}^{K\nu}_q)+\max_{q\in S_2}(|a^K_{pq}|\bar{\delta}^{J\nu}_q),\\
\check{A}^{(l)}_p=&\max_{q\in S_1}(|a^R_{pq}|\delta^{J\nu}_q)+\max_{q\in S_1}(|a^I_{pq}|\delta^{K\nu}_q)+\max_{q\in S_1}(|a^J_{pq}|\delta^{R\nu}_q)+\max_{q\in S_1}(|a^K_{pq}|\delta^{I\nu}_q)+\max_{q\in S_2}(|a^R_{pq}|\bar{\delta}^{J\nu}_q)+\max_{q\in S_2}(|a^I_{pq}|\bar{\delta}^{K\nu}_q)\\
&+\max_{q\in S_2}(|a^J_{pq}|\bar{\delta}^{R\nu}_q)+\max_{q\in S_2}(|a^K_{pq}|\bar{\delta}^{I\nu}_q),\\
\hat{A}^{(l)}_p=&\max_{q\in S_1}(|a^R_{pq}|\delta^{K\nu}_q)+\max_{q\in S_1}(|a^I_{pq}|\delta^{J\nu}_q)+\max_{q\in S_1}(|a^J_{pq}|\delta^{I\nu}_q)+\max_{q\in S_1}(|a^K_{pq}|\delta^{R\nu}_q)+\max_{q\in S_2}(|a^R_{pq}|\bar{\delta}^{K\nu}_q)+\max_{q\in S_2}(|a^I_{pq}|\bar{\delta}^{J\nu}_q)\\
&+\max_{q\in S_2}(|a^J_{pq}|\bar{\delta}^{I\nu}_q)+\max_{q\in S_2}(|a^K_{pq}|\bar{\delta}^{R\nu}_q),\\
\bar{B}^{(l)}_p=&\max_{q\in S_1}(|b^R_{pq}|\eta^{R\nu}_q)+\max_{q\in S_1}(|b^I_{pq}|\eta^{I\nu}_q)+\max_{q\in S_1}(|b^J_{pq}|\eta^{J\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{K\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{I\nu}_q)\\
&+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{K\nu}_q),\\
\tilde{B}^{(l)}_p=&\max_{q\in S_1}(|b^R_{pq}|\eta^{I\nu}_q)+\max_{q\in S_1}(|b^I_{pq}|\eta^{R\nu}_q)+\max_{q\in S_1}(|b^J_{pq}|\eta^{K\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{J\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{I\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{R\nu}_q)\\
&+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{K\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{J\nu}_q),\\
\check{B}^{(l)}_p=&\max_{q\in S_1}(|b^R_{pq}|\eta^{J\nu}_q)+\max_{q\in S_1}(|b^I_{pq}|\eta^{K\nu}_q)+\max_{q\in S_1}(|b^J_{pq}|\eta^{R\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{I\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{J\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{K\nu}_q)\\
&+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{R\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{I\nu}_q),\\
\hat{B}^{(l)}_p=&\max_{q\in S_1}(|b^R_{pq}|\eta^{K\nu}_q)+\max_{q\in S_1}(|b^I_{pq}|\eta^{J\nu}_q)+\max_{q\in S_1}(|b^J_{pq}|\eta^{I\nu}_q)+\max_{q\in S_1}(|b^K_{pq}|\eta^{R\nu}_q)+\max_{q\in S_2}(|b^R_{pq}|\bar{\eta}^{K\nu}_q)+\max_{q\in S_2}(|b^I_{pq}|\bar{\eta}^{J\nu}_q)\\
&+\max_{q\in S_2}(|b^J_{pq}|\bar{\eta}^{I\nu}_q)+\max_{q\in S_2}(|b^K_{pq}|\bar{\eta}^{R\nu}_q),\\
\mathcal{I}_1=&\frac{e^{-(t-s)}e^{-s}}{(t-s)^{1-\upsilon}},\mathcal{I}_2=\frac{e^{-(t-s+\tau)}e^{-(s-\tau)}}{(t-s)^{1-\upsilon}},|\tilde{e}^{(1)}_p|=\sup_t(e^{-t}|e^R_p|)\frac{1}{\Gamma(\upsilon)}\int_0^t u^{(\upsilon-1)}e^{-u}\mathrm{d}u,\\
|\tilde{e}^{(l)}_q|=&\sum_{q=1}^n\sup_t(e^{-t}|e^\nu_q|)\frac{1}{\Gamma(\upsilon)}\int_0^t u^{(\upsilon-1)}e^{-u}\mathrm{d}u,|\bar{\psi}^{(l)}_q|=\frac{1}{\Gamma(\upsilon)}\int_{-\tau}^0(t-\varrho-\tau)^{(\upsilon-1)}e^{-(t-\varrho)}e^{-\varrho}|\psi^\nu_q|\mathrm{d}u,\\
|\bar{e}^{(l)}_q|=&\frac{1}{\Gamma(\upsilon)}\int_0^{t-\tau}(t-\varrho-\tau)^{(\upsilon-1)}e^{-(t-\varrho)}e^{-\varrho}|e^\nu_q|\mathrm{d}u,|\hat{e}^{(l)}_q|=|s(e^{(l)}_q)|*\frac{1}{\Gamma(\upsilon)}\int_{t-\tau}^t\xi^{(\upsilon-1)}e^{-\xi}\mathrm{d}\xi,\\
|\hat{\psi}^{(l)}_q|=&|s(\psi^{(l)}_q)|*\frac{1}{\Gamma(\upsilon)}\int_{t-\tau}^t\xi^{(\upsilon-1)}e^{-\xi}\mathrm{d}\xi,M^R_l=\sum_{p=1}^n(\bar{A}^{(l)}_p+\bar{B}^{(l)}_p),\bar{M}^R_l=\sum_{p=1}^n\bar{B}^{(l)}_p,\\
M^I_l=&\sum_{p=1}^n(\tilde{A}^{(l)}_p+\tilde{B}^{(l)}_p),\bar{M}^I_l=\sum_{p=1}^n\tilde{B}^{(l)}_p,M^J_l=\sum_{p=1}^n(\check{A}^{(l)}_p+\check{B}^{(l)}_p),\bar{M}^J_l=\sum_{p=1}^n\check{B}^{(l)}_p,\\
M^K_l=&\sum_{p=1}^n(\hat{A}^{(l)}_p+\hat{B}^{(l)}_p),\bar{M}^K_l=\sum_{p=1}^n\hat{B}^{(l)}_p,M^R=1-\max_p(d_p)+M^R_1,M^I=1-\max_p(d_p)+M^I_2,\\
M^J=&1-\max_p(d_p)+M^J_3,M^K=1-\max_p(d_p)+M^K_4,\\
N^R_1=&\frac{M^R_2}{M^R},N^R_2=\frac{M^R_3}{M^R},N^R_3=\frac{M^R_4}{M^R},N^I_1=\frac{M^I_1}{M^I},N^I_2=\frac{M^I_3}{M^I},N^I_3=\frac{M^I_4}{M^I},
\end{aligned}
$$

$$
\begin{aligned}
N_1^J =& \frac{M_1^J}{M^J}, N_2^J = \frac{M_2^J}{M^J}, N_3^J = \frac{M_4^J}{M^J}, N_1^K = \frac{M_1^K}{M^K}, N_2^K = \frac{M_2^K}{M^K}, N_3^K = \frac{M_3^K}{M^K}, \bar{N}_l^{\nu} = \frac{\bar{M}_l^{\nu}}{M^{\nu}},\\
T_1^R =& \frac{N_1^J + N_3^J N_1^K}{1 - N_3^J N_3^K}, T_2^R = \frac{N_2^J + N_3^J N_2^K}{1 - N_3^J N_3^K}, T_3^R = \frac{N_3^J N^K + N^J}{1 - N_3^J N_3^K}, T_4^R = (N_1^K + N_3^K T_1),\\
T_5^R =& N_2^K + N_3^K T_2, T_6^R = N_3^K T_3 + N^K, T_7^R = \frac{N_1^I + N_2^I T_1 + N_3^I T_4}{1 - N_2^I T_2 + N_3^I T_5}, T_8^R = \frac{N_2^I T_3 + N_3^I T_6 + N^I}{1 - N_2^I T_2 + N_3^I T_5},\\
T_9^R =& T_4 + T_5 T_7, T_{10}^R = T_5 T_8 + T_6, T_{11}^R = T_1 + T_2 T_7, T_{12}^R = T_2 T_8 + T_3,\\
T_1^I =& \frac{N_1^R + N_2^R N_2^J}{1 - N_2^R N_1^J}, T_2^I = \frac{N_2^R N_3^J}{1 - N_2^R N_1^J}, T_3^I = \frac{N^J N_2^R + N^R}{1 - N_2^R N_1^J},\\
T_4^I =& N_1^J T_1^I + N_2^J, T_5^I = N_1^J T_2^I + N_3^J, T_6^I = N_1^J T_3^I + N^J, T_7^I = N_1^K T_1^I + N_2^K + N_3^K T_4^I, T_8^I = N_1^K T_2^I + N_3^K T_5^I,\\
T_9^I =& N_1^K T_3^I + N_3^K T_6^I + N^K, T_{10}^I = \frac{T_1^I}{1 - T_8^I}, T_{11}^I = \frac{T_9^I}{1 - T_8^I},\\
T_{12}^I =& T_4^I + T_5^I T_{10}^I, T_{13}^I = T_5^I T_{11}^I + T_6^I, T_{14}^I = T_1^I + T_2^I T_{10}^I,\\
T_{15}^I =& T_2^I T_{11}^I + T_3^I,\\
T_1^J =& \frac{N_2^I + N_1^I N_2^R}{1 - N_1^I N_1^R}, T_2^J = \frac{N_1^I N_3^R + N_3^I}{1 - N_1^I N_1^R}, T_3^J = \frac{N_1^I N^R + N^I}{1 - N_1^I N_1^R},\\
T_4^J =& \frac{N_1^K}{1 - N_2^K T_2^J}, T_5^J = \frac{N_2^K T_1^J + N_3^K}{1 - N_2^K T_2^J}, T_6^J = \frac{T_3^J N_2^K + N^K}{1 - N_2^K T_2^J},\\
T_7^J =& T_2^J T_4^J, T_8^J = T_1^J + T_2^J T_5^J, T_9^J = T_2^J T_6^J + T_3^J,\\
T_{10}^J =& N_1^R T_7^J + N_3^R T_4^J, T_{11}^J = N_1^R T_8^J + N_3^R T_5^J + N_2^R, T_{12}^J = N_1^R T_9^J + N_3^R T_6^J, T_{13}^J = \frac{T_{11}^J}{1 - T_{10}^J}, T_{14}^J = \frac{T_{12}^J}{1 - T_{10}^J},\\
T_{15}^J =& T_4^J T_{13}^J + T_5^J, T_{16}^J = T_4^J T_{14}^J + T_6^J, T_{17}^J = T_1^J + T_2^J T_{15}^J, T_{18}^J = T_2^J T_{16}^J + T_3^J, T_1^K = \frac{N_1^R N_2^I + N_2^R}{1 - N_1^R N_1^I},\\
T_2^K =& \frac{N_3^R + N_1^R N_3^I}{1 - N_1^R N_1^I}, T_3^K = \frac{N_1^R N^I + N^R}{1 - N_1^R N_1^I}, T_4^K = N_1^I T_1^K + N_1^I, T_5^K = N_1^I T_2^K + N_3^I, T_6^K = N_1^I T_3^K + N^I,\\
T_7^K =& \frac{N_1^J T_2^K + N_2^J T_5^K + N_3^J}{1 - N_1^J T_1^K + N_2^J T_4^K}, T_8^K = \frac{N_1^J T_3^K + N_2^J T_6^K + N^J}{1 - N_1^J T_1^K + N_2^J T_4^K}.\\
\bar{D}_1^R =& 1 + N_2^K N_3^I - N_2^I N_2^J + N_3^K N_3^I N_2^J - 2N_3^K N_3^J - N_2^K N_2^I N_3^J - N_2^K N_3^K N_3^I N_3^J + N_3^K N_2^I N_2^J N_3^J - N_3^{K^2} N_3^I N_2^J N_3^J\\
&+ N_3^{K^2} N_3^{J^2} + N_2^K N_3^K N_2^I N_3^{J^2},\\
\tilde{D}_1^I =& N_1^I + N_1^K N_3^I + N_2^I N_1^J + N_3^K N_3^I N_1^J - N_3^K N_1^I N_3^J + N_1^K N_2^I N_3^J,\\
\check{D}_1^J =& - N_1^R N_1^I N_1^J - N_2^K N_3^R N_1^I N_1^J + N_1^K N_1^R N_3^R N_1^I N_1^J + N_1^K N_2^K N_3^{R^2} N_1^I N_1^J + N_1^{R^2} N_1^{I^2} N_1^J\\
&+ 2N_2^K N_1^R N_3^R N_1^{I^2} N_1^J + N_2^{K^2} N_3^{R^2} N_1^{I^2} N_1^J + N_2^K N_1^R N_1^I N_3^I N_1^J + N_1^K N_1^{R^2} N_1^I N_3^I N_1^J\\
&+ N_2^{K^2} N_3^R N_1^I N_3^I N_1^J + N_1^K N_2^K N_1^R N_3^R N_1^I N_3^I N_1^J - N_1^I N_2^J + N_1^K N_3^R N_1^I N_2^J + N_1^R N_1^{I^2} N_2^J\\
&+ N_2^K N_3^R N_1^{I^2} N_2^J - N_1^K N_1^R N_3^R N_1^{I^2} N_2^J - N_1^K N_2^K N_3^{R^2} N_1^{I^2} N_2^J + N_2^K N_1^I N_3^I N_2^J\\
&- N_1^K N_2^K N_3^R N_1^I N_3^I N_2^J - N_2^K N_1^I N_3^J - N_1^K N_1^R N_1^I N_3^J + N_2^K N_1^R N_1^{I^2} N_3^J
\end{aligned}
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\begin{aligned}
&+ N_1^K N_1^{R^2} N_1^{I^2} N_3^J + N_2^{K^2} N_3^R N_1^{I^2} N_3^J + N_1^K N_2^K N_1^R N_3^R N_1^{I^2} N_3^J + N_2^{K^2} N_1^I N_3^I N_3^J \\
&+ N_1^K N_2^K N_1^R N_1^I N_3^I N_3^J, \\
\hat{D}_1^K =&1 + N_1^I - N_1^R N_1^I - N_1^R N_1^{I^2} + N_3^K N_1^J - N_2^R N_1^J + N_1^K N_2^R N_1^J - N_3^K N_1^R N_1^I N_1^J - N_2^R N_1^I N_1^J \\
&+ N_2^K N_2^R N_1^I N_1^J + N_2^K N_2^I N_1^J - N_1^R N_2^I N_1^J + N_1^K N_1^R N_2^I N_1^J - N_1^R N_1^I N_2^I N_1^J + N_3^K N_1^I N_2^J \\
&+ N_2^R N_1^I N_2^J + N_1^K N_2^R N_1^I N_2^J - N_3^K N_1^R N_1^{I^2} N_2^J + N_2^R N_1^{I^2} N_2^J + N_2^K N_2^R N_1^{I^2} N_2^J + N_2^I N_2^J \\
&+ N_1^I N_2^I N_2^J + N_2^K N_1^I N_2^I N_2^J + N_1^K N_1^R N_1^I N_2^I N_2^J, \\
\bar{D}_2^R =&N_1^R + N_2^K N_3^R + N_2^R N_2^J + N_3^K N_3^R N_2^J - 2N_3^K N_1^R N_3^J + N_2^K N_2^R N_3^J - N_2^K N_3^K N_3^R N_3^J - N_3^K N_2^R N_2^J N_3^J \\
&- N_3^{K^2} N_3^R N_2^J N_3^J + N_3^{K^2} N_1^R N_3^{J^2} - N_2^K N_3^K N_2^R N_3^{J^2}, \\
\tilde{D}_2^I =&1 - N_2^R N_1^J - N_3^K N_3^J - N_1^K N_2^R N_3^J, \\
\check{D}_2^J =& - N_1^R N_1^J - N_2^K N_3^R N_1^J + N_1^K N_1^R N_3^R N_1^J + N_1^K N_2^K N_3^{R^2} N_1^J + N_1^{R^2} N_1^I N_1^J + 2N_2^K N_1^R N_3^R N_1^I N_1^J + N_2^{K^2} N_3^{R^2} N_1^I N_1^J \\
&+ N_2^K N_1^R N_3^I N_1^J + N_1^K N_1^{R^2} N_3^I N_1^J + N_2^{K^2} N_3^R N_3^I N_1^J + N_1^K N_2^K N_1^R N_3^R N_3^I N_1^J - N_2^J + N_1^K N_3^R N_2^J + N_1^R N_1^I N_2^J \\
&+ N_2^K N_3^R N_1^I N_2^J - N_1^K N_1^R N_3^R N_1^I N_2^J - N_1^K N_2^K N_3^{R^2} N_1^I N_2^J \\
&+ N_2^K N_3^I N_2^J - N_1^K N_2^K N_3^R N_3^I N_2^J - N_2^K N_3^J - N_1^K N_1^R N_3^J + N_2^K N_1^R N_1^I N_3^J + N_1^K N_1^{R^2} N_1^I N_3^J \\
&+ N_2^{K^2} N_3^R N_1^I N_3^J + N_1^K N_2^K N_1^R N_3^R N_1^I N_3^J + N_2^{K^2} N_3^I N_3^J + N_1^K N_2^K N_1^R N_3^I N_3^J, \\
\hat{D}_2^K =&1 + N_1^R - N_1^{R^2} N_1^I - N_1^{R^2} N_1^{I^2} + N_3^K N_1^R N_1^J - N_2^R N_1^J - N_1^R N_2^R N_1^J + N_1^K N_1^R N_2^R N_1^J - N_3^K N_1^{R^2} N_1^I N_1^J \\
&- N_1^R N_2^R N_1^I N_1^J + N_2^K N_1^R N_2^R N_1^I N_1^J - N_1^R N_2^I N_1^J + N_2^K N_1^R N_2^I N_1^J - N_1^{R^2} N_2^I N_1^J + N_1^K N_1^{R^2} N_2^I N_1^J \\
&- N_1^{R^2} N_1^I N_2^I N_1^J + N_3^K N_2^J + N_1^K N_2^R N_2^J + N_2^R N_1^I N_2^J + N_2^K N_2^R N_1^I N_2^J + N_1^R N_2^R N_1^I N_2^J + N_1^K N_1^R N_2^R N_1^I N_2^J \\
&- N_3^K N_1^{R^2} N_1^{I^2} N_2^J + N_1^R N_2^R N_1^{I^2} N_2^J + N_2^K N_1^R N_2^R N_1^{I^2} N_2^J + N_2^I N_2^J + N_2^K N_2^I N_2^J + N_1^R N_2^I N_2^J + N_1^K N_1^R N_2^I N_2^J \\
&+ N_1^R N_1^I N_2^I N_2^J + N_2^K N_1^R N_1^I N_2^I N_2^J + N_1^K N_1^{R^2} N_1^I N_2^I N_2^J, \\
\bar{D}_3^R =&N_2^R + N_3^K N_3^R + N_1^R N_2^I + N_2^K N_3^R N_2^I + N_3^K N_1^R N_3^I + N_2^K N_2^R N_3^I + 2N_2^K N_3^K N_3^R N_3^I + 2N_3^K N_2^R N_3^I N_2^J \\
&+ 2N_3^{K^2} N_3^R N_3^I N_2^J - N_3^K N_2^R N_3^J - N_3^{K^2} N_3^R N_3^J - N_3^K N_1^R N_2^I N_3^J - N_2^K N_3^K N_3^R N_2^I N_3^J \\
&- N_3^{K^2} N_1^R N_3^I N_3^J + N_2^K N_3^K N_2^R N_3^I N_3^J, \\
\tilde{D}_3^I =&N_2^R N_1^I + N_2^I + N_3^K N_3^I + N_1^K N_2^R N_3^I, \\
\check{D}_3^J =& - 1 + N_1^K N_3^R + 2N_1^R N_1^I + 2N_2^K N_3^R N_1^I - N_1^K N_1^R N_3^R N_1^I - N_1^K N_2^K N_3^{R^2} N_1^I - N_1^{R^2} N_1^{I^2} \\
&- 2N_2^K N_1^R N_3^R N_1^{I^2} - N_2^{K^2} N_3^{R^2} N_1^{I^2} + 2N_2^K N_3^I + N_1^K N_1^R N_3^I - N_1^K N_2^K N_3^R N_3^I - 2N_2^K N_1^R N_1^I N_3^I \\
&- N_1^K N_1^{R^2} N_1^I N_3^I - 2N_2^{K^2} N_3^R N_1^I N_3^I - N_1^K N_2^K N_1^R N_3^R N_1^I N_3^I - N_2^{K^2} N_3^{I^2} - N_1^K N_2^K N_1^R N_3^{I^2}, \\
\hat{D}_3^K =&N_3^K + N_1^K N_2^R - N_3^K N_1^R N_1^I + N_2^K N_2^R N_1^I + N_2^K N_2^I + N_1^K N_1^R N_2^I, \\
\bar{D}_4^R =&N_3^R + N_1^R N_3^I + 2N_2^K N_3^R N_3^I - N_3^R N_2^I N_2^J + N_2^R N_3^I N_2^J + 2N_3^K N_3^R N_3^I N_2^J + N_2^R N_3^J - N_3^K N_3^R N_3^J + N_1^R N_2^I N_3^J \\
&- N_3^K N_1^R N_3^I N_3^J + 2N_2^K N_2^R N_3^I N_3^J + N_3^K N_3^R N_2^I N_2^J N_3^J + N_3^K N_2^R N_3^I N_2^J N_3^J - N_3^K N_2^R N_3^{J^2} - N_3^K N_1^R N_2^I N_3^{J^2}, \\
\tilde{D}_4^I =&N_3^I - N_2^R N_3^I N_1^J + N_2^R N_1^I N_3^J + N_2^I N_3^J, \\
\check{D}_4^J =& - N_3^R N_1^J + N_1^K N_3^{R^2} N_1^J + N_1^R N_3^R N_1^I N_1^J + N_2^K N_3^{R^2} N_1^I N_1^J - N_1^R N_3^I N_1^J + N_2^K N_3^R N_3^I N_1^J \\
&+ 2N_1^K N_1^R N_3^R N_3^I N_1^J + N_1^{R^2} N_1^I N_3^I N_1^J + N_2^K N_1^R N_3^R N_1^I N_3^I N_1^J + N_2^K N_1^R N_3^{I^2} N_1^J \\
&+ N_1^K N_1^{R^2} N_3^{I^2} N_1^J - N_3^R N_1^I N_2^J + N_1^R N_3^R N_1^{I^2} N_2^J + N_2^K N_3^{R^2} N_1^{I^2} N_2^J - N_3^I N_2^J + N_1^R N_1^I N_3^I N_2^J
\end{aligned}
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$$+ 2N_2^K N_3^R N_1^I N_3^I N_2^J + N_2^K N_3^{I^2} N_2^J - N_3^J + 2N_1^R N_1^I N_3^J + N_2^K N_3^R N_1^I N_3^J - N_1^{R^2} N_1^{I^2} N_3^J$$
$$- N_2^K N_1^R N_3^R N_1^{I^2} N_3^J + N_2^K N_3^I N_3^J - N_2^K N_1^R N_1^I N_3^I N_3^J,$$
$$\hat{D}_4^K = 1 - N_1^R N_1^I - N_2^R N_1^J - N_1^R N_2^I N_1^J + N_2^R N_1^I N_2^J + N_2^I N_2^J.$$