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Research article

New classes of graphs with edge $\delta-$ graceful labeling

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Abstract: Graph labeling is a source of valuable mathematical models for an extensive range of applications in technologies (communication networks, cryptography, astronomy, data security, various coding theory problems). An edge $\delta-$ graceful labeling of a graph G with p vertices and q edges, for any positive integer δ, is a bijective f from the set of edge $E(G)$ to the set of positive integers $\{\delta, 2\delta, 3\delta, \cdots, q\delta\}$ such that all the vertex labels $f^*[V(G)]$, given by: $f^*(u) = (\sum_{uv\in E(G)} f(uv))\ mod\ (\delta\ k)$, where $k = max(p, q)$, are pairwise distinct. In this paper, we show the existence of an edge $\delta-$ graceful labeling, for any positive integer δ, for the following graphs: the splitting graphs of the cycle, fan, and crown, the shadow graphs of the path, cycle, and fan graph, the middle graphs and the total graphs of the path, cycle, and crown. Finally, we display the existence of an edge $\delta-$ graceful labeling, for the twig and snail graphs.

Keywords: edge $\delta-$ graceful labeling; splitting graph; shadow graph; middle graph; total graph
Mathematics Subject Classification: 05C78, 05C90

1. Introduction

Labeling graphs have attracted the attention of numerous researchers in different disciplines. The importance of this research line is in fact due to the following:

(1) Finding different coding techniques for securing the communication networks and database managements.

(2) Providing a high-level secrecy to military services which is the most important factor for coding.

(3) Providing confidentiality, and integrity of messages transferred between group members which is a critical networking issue. For more application, see [1, 2].

Coding through special kinds of graphs with different kinds of labeling is structured by many papers. Coding with Fibonacci web graph using super mean labeling was introduced by Uma

Maheswari et al. [3]. Prasad et al. developed a technique of coding secret messages using sun flower graphs SF_n. Furthermore, every labeling graph can be converted to a code by using GMJ coding methods (see, [4] and the references therein)

A graph G is a pair (V, E), where $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The position of the vertices and the length of the edges do not concern us, what is important is the size of a graph (number of vertices) and the pairs of vertices which are connected by an edge. If $e = \{u, v\}$ is an edge of a graph G, then u and v are adjacent while u and e are incident. Let $q = |E(G)|$ be the cardinality of $E(G)$ and $p = |V(G)|$ be that of $V(G)$. For every vertex $u \in V(G)$, the open neighborhood set $N(u)$ is the set of all vertices adjacent to u in G. Graph operations [5] allow us to generate many new graphs from old ones. A fan graph F_n is defined as the join $P_n + K_1$ where P_n is the path graph on n vertices and K_1 is a complete graph on one vertex. The crown graph Cr_n(Sunlet graph) is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n, i.e., the coronas $C_n \odot K_1$.

Definition 1.1. *(i) The splitting graph $S'(G)$ of a connected graph G is the graph obtained by adding new vertex u_i corresponding to each vertex v_i of $V(G)$ such that $N(v_i) = N(u_i)$.*

(ii) The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2 . Join each vertex v_i in G_1 to the neighbors of the corresponding vertex u_i in G_2. The shadow graph $D_2(G)$ can be obtained from the splitting graph $S'(G)$ by adding edge between any two new vertices u_i and u_j if the corresponding original vertices v_i and v_j are adjacent. $V[D_2(G)] = V[S'(G)]$, $E[D_2(G)] = E[S'(G)] \cup \{u_iu_{i+1},\ i = 1, 2, \cdots, n\}$.

(iii) The middle graph $M(G)$ of a connected graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices in $M(G)$ are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

(iv) The total graph $T(G)$ of a connected graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if their corresponding elements are either adjacent or incident in G. The total graph $T(G)$ can be obtained from the middle graph $M(G)$ by adding edge between any two original vertices v_i and v_j if the corresponding new vertices u_i and u_j are adjacent. $V[T(G)] = V[M(G)]$, $E[T(G)] = E[M(G)] \cup E(G)$.

There are many different kinds of graph labeling [6–13], all that kinds of labeling problem will have following three common characteristics. A set of numbers from which vertex or edge labels are chosen, A rule that assigns a value to each edge or vertex, A condition that these values must satisfy. For a comprehensive survey on graph labeling refers to a dynamic survey of graph labeling [14].

Zeen El Deen [15] introduced the edge $\delta-$ graceful labeling of graphs by using the numbers $\{\delta,\ 2\delta,\ 3\delta,\ \cdots,\ q\delta\}$ to label the edges of a graph, for any positive integer δ. He showed edge $\delta-$ graceful labeling for some graphs related to cycles.

Definition 1.2. *An edge $\delta-$ graceful labeling of a graph $G = (V(G),\ E(G))$ with $p = |V(G)|$ vertices and $q = |E(G)|$ edges is a bijective mapping f of the edge set $E(G)$ into the set $\{\delta,\ 2\delta,\ 3\delta,\ \cdots,\ q\delta\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, \delta,\ 2\delta,\ 3\delta,\ \cdots, k\delta - \delta\}$, given by: $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (\delta\, k)$, where $k = max(p, q)$, is an injective function. The graph that admits an edge $\delta-$ graceful labeling is called an edge $\delta-$ graceful graph, if $\delta = 2$ we have the edge even labeling also if $\delta = 3$ we have the edge triple labeling and so on.*

Example 1.1. *In Figure 1 we present an edge $\delta-$ graceful labeling of a tree graph on $p = 11$ vertices and $q = 10$ edges $f : E(G) \to \{\delta, 2\delta, 3\delta, \cdots, 10\delta\}$ and $f^* : V(G) \to \{0, \delta, 2\delta, 3\delta, \cdots, 10\delta\}$, given by: $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (11\delta)$.*

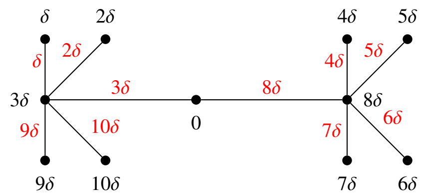

Figure 1. A tree with edge $\delta-$ graceful labeling.

Example 1.2. *Duplication of an edge $e = (xy)$ by a new vertex u in a graph G produces a new graph H such that $N(u) = \{x, y\}$. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set in the path P_n and G be the graph obtained by duplication of each edge $v_i v_{i+1}$ of path P_n by vertex u_i, $(1 \le i < n)$. Then $V(G) = V(P_n) \cup \{u_1, u_2, \cdots, u_{n-1}\}$ and the edge set are $\{v_i v_{i+1}, v_i u_i, u_i v_{i+1}, i = 1, 2, \cdots, n-1\}$, so the graph G has $p = 2n - 1$ vertices and $q = 3n - 3$ edges, $k = max(p, q) = 3n - 3$. An edge $5-$ graceful labeling of a graph obtained duplication of each edge of P_{11} by a vertex is shown in Figure 2.*

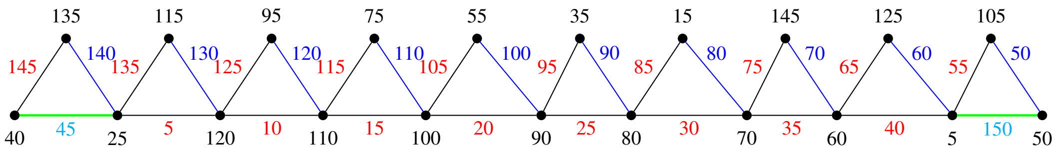

Figure 2. Graph obtained duplication of each edge of P_{11} by a vertex and its edge $5-$ graceful labeling.

2. Edge $\delta-$ graceful labeling of some splitting graphs

2.1. Edge $\delta-$ graceful labeling of the splitting graph $S'(C_n)$

Theorem 2.1. *For any positive integer δ, the splitting graph $S'(C_n)$ of the cycle C_n, $n > 3$ is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of C_n where these vertices are in their natural order module n. To form the splitting graph $S'(C_n)$ we add new vertices $\{u_1, u_2, \cdots, u_n\}$ corresponding to vertices of C_n. The edges set of $S'(C_n)$ are $\{v_i v_{i+1}, v_i u_{i+1}, u_i v_{i+1}, i = 1, 2, \cdots, n\}$, the graph $S'(C_n)$ has $p = 2n$ vertices and $q = 3n$ edges, $k = max(p, q) = 3n$.

Case (1): When $n \equiv 2 \bmod 4$, $n > 3$. We define the labeling function $f : E(S'(C_n)) \longrightarrow \{\delta, 2\delta, \cdots, (3n)\delta\}$ as follows:

$$f(v_i\, u_{i+1}) = \delta\,(i), \qquad for \quad 1 \le i \le n,$$

$$f(v_i\, v_{i+1}) = \begin{cases} \delta\,(n+1), & \text{if} \quad i = 1; \\ \delta\,(2n - 1 + i), & \text{if} \quad 2 \le i \le n. \end{cases}$$

$$f(u_i\, v_{i+1}) = \begin{cases} \delta\,(2n), & \text{if} \quad i = 1; \\ \delta\,(n+i), & \text{if} \quad 2 \le i \le n-1; \\ \delta\,(3n), & \text{if} \quad i = n. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$f^*(v_1) = (n+1)\,\delta,$ $f^*(v_2) = (2n+4)\,\delta,$ $f^*(u_1) = 0,$ $f^*(u_n) = (n-1)\,\delta,$ and
$f^*(v_i) = (2n+4i-4)\,\delta \; mod\,(3n\delta),$ $for \;\; 3 \le i \le n,$
$f^*(u_i) = (n+2i-1)\,\delta \; mod\,(3n\delta),$ $for \;\; 2 \le i \le n-1.$

Hence the vertex labels are all distinct and a multiple of δ.

Case (2): When $n \equiv 0 \; mod\, 4,$ $n = 4$, the graph $S'(C_4)$ is an edge $\delta-$ graceful graph for any positive integer δ define the labeling function $f : E(S'(C_4)\,) \longrightarrow \{\delta,\; 2\,\delta,\; \cdots,\; (12)\delta\}$ as shown in Figure 3.

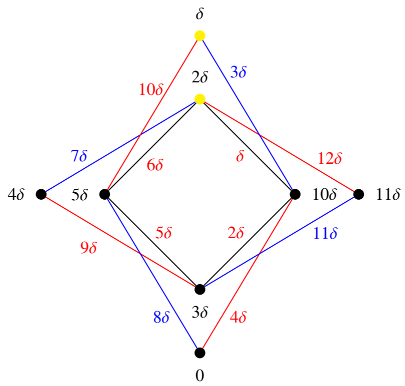

Figure 3. The splitting graphs $S'(C_4)$ with an edge $\delta-$ graceful labeling.

- When $n \equiv 0 \; mod\, 4,$ $n > 4$, we define the labeling function f as follows:

$$f(u_i\, v_{i+1}) = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$

$$f(v_i\, u_{i+1}) = \begin{cases} \delta\,(i+1), & \text{if} \quad 2 \le i \le n-1; \\ \delta, & \text{if} \quad i = n. \end{cases}$$

$$f(v_i\, v_{i+1}) = \begin{cases} \delta\,(2n+i), & \text{if} \quad 1 \le i \le n-3; \\ \delta\,(4n-2-i), & \text{if} \quad n-2 \le i \le n. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$f^*(v_1) = (n+1)\,\delta,$ $f^*(v_{n-2}) = (3n-7)\,\delta,$ $f^*(v_{n-1}) = (3n-3)\,\delta,$ $f^*(v_n) = (2n-3)\,\delta,$
$f^*(v_i) = (2n+4i-1)\,\delta \; mod\,(3n\delta),$ $for \quad 2 \le i \le n-3,$
$f^*(u_1) = (n+2)\,\delta,$ $f^*(u_i) = (n+2i)\,\delta \; mod\,(3n\delta),$ $for \quad 2 \le i \le n.$

Hence the vertex labels are all distinct and a multiple of δ.

Case (3): When n is odd, we define the labeling function f as follows:

$$f(u_i\, v_{i+1}) = \delta\,(3i-2), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\, v_{i+1}) = \delta\,(3i-1), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\, u_{i+1}) = \delta\,(3i), \qquad for \quad 1 \le i \le n.$$

In view of the above labeling pattern then the induced vertex labels are:

$$f^*(v_i) = \ (12i-10)\,\delta \; mod\,(3n\delta), \qquad for \quad 1 \le i \le n, \quad \text{and}$$
$$f^*(u_i) = \ (6i-5)\,\delta \; mod\,(3n\delta), \qquad for \quad 1 \le i \le n.$$

Hence the vertex labels are all distinct and a multiple of δ. Therefore $S'(C_n)$ admits an edge $\delta-$ graceful labeling. □

Illustration: The splitting graphs $S'(C_8)$ with an edge 4− graceful labeling, $S'(C_9)$ with an edge 5− graceful labeling and splitting graph $S'(C_{10})$ with an edge 3− graceful labeling are shown in Figure 4.

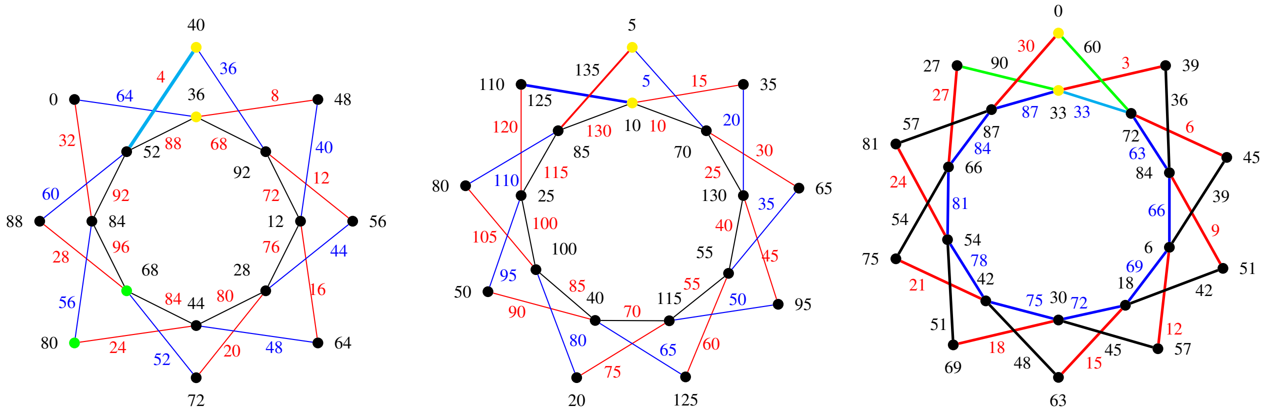

Figure 4. The splitting graphs $S'(C_8)$ with an edge 4− graceful labeling, $S'(C_9)$ with an edge 5− graceful labeling and $D_2(C_{10})$ with an edge 3− graceful labeling.

2.2. Edge $\delta-$ graceful labeling of the splitting graph $S'(F_n)$

Theorem 2.2. *For any positive integer δ, the splitting graph $S'(F_n)$ of the fan graph F_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{\, v_0,\ v_1,\ v_2,\ \cdots,\ v_n\}$ be the vertices of the fan F_n, to form the splitting graph $S'(F_n)$ we add new vertices $\{u_0,\ u_1,\ u_2,\ \cdots,\ u_n\,\}$ corresponding to the vertices of the fan F_n. The edges set of $S'(F_n)$ are $\{v_iu_{i+1},\ v_iv_{i+1},\ u_iv_{i+1},\ \ 1 \le i \le n-1\,\} \cup \{u_iv_0\,,\ v_iu_0,\ v_iv_0\,,\ \ 1 \le i \le n\,\}$, the graph $S'(F_n)$ has $p = \ 2n+2$ vertices and $q = 6n-3$ edges, $k = max(p, q) \ = \ 6n-3$, see Figure 5.

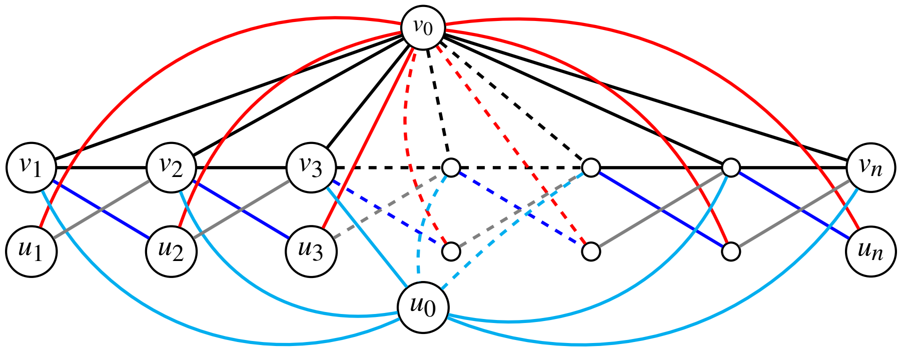

Figure 5. The splitting graph $S'(F_n)$ of the fan graph F_n.

Case (1): When n is even, we define the labeling function $f : E(S'(F_n)) \longrightarrow \{\delta,\ 2\delta,\ \cdots,\ (6n-3)\delta\}$ as follows:

$$f(v_0\, v_i) = \begin{cases} \delta\,(n), & \text{if } i = 1; \\ \delta\,(i), & \text{if } 2 \le i \le n-1; \\ \delta, & \text{if } i = n. \end{cases}$$

$$f(u_i\, v_{i+1}) = \delta\,(n+i), \qquad for \quad 1 \le i \le n-1,$$
$$f(v_i\, u_{i+1}) = \delta\,(2n-1+i), \qquad for \quad 1 \le i \le n-1,$$

$$f(u_0\, v_i) = \begin{cases} \delta\,(3n-2+i), & \text{if } 1 \le i \le n-1; \\ \delta\,(6n-3), & \text{if } i = n. \end{cases}$$

$$f(v_i\, v_{i+1}) = \delta\,(5n-3-i), \qquad for \quad 1 \le i \le n-1,$$
$$f(v_0\, u_i) = \delta\,(6n-3-i), \qquad for \quad 1 \le i \le n.$$

In view of the above labeling pattern then the induced vertex labels are:

$$f^*(v_0) = 0, \qquad f^*(v_1) = (5n-2)\,\delta, \qquad f^*(v_n) = \delta,$$
$$f^*(u_1) = (n)\,\delta, \qquad f^*(u_n) = (2n-2)\,\delta,$$
$$f^*(v_i) = (4n-3+2i)\,\delta \; mod\; [(6n-3)\delta], \qquad for \quad 2 \le i \le n-1,$$
$$f^*(u_i) = (3n-2+i)\,\delta \; mod\; [(6n-3)\delta], \qquad for \quad 2 \le i \le n-1,$$

Finally, $f^*(u_0) = [\sum_{i=1}^{n-1} f(u_0v_i) + f(u_0v_n)]\; mod(6n-3)\,\delta = [\sum_{i=1}^{n-1}(3n-2+i)\,\delta]\; mod(6n-3)\,\delta$

$$= [(\frac{7n^2}{2} - \frac{11n}{2} + 2)\,\delta]\; mod(6n-3)\,\delta.$$

If $n \equiv 2\; mod\; 12 \implies n = 12k+2$

$$f^*(u_0) = [(504k^2 + 102k + 5)\,\delta]\; mod(72k+9)\,\delta = [(39k+5)\,\delta]\; mod(72k+9)\,\delta$$
$$= [\,(\frac{13n-6}{4})\,\delta]\; mod(6n-3)\,\delta.$$

Similarly, If $n \equiv 0\; mod\; 12 \implies n = 12k$ then $f^*(u_0) = [(\frac{9n-4}{4})\,\delta]\; mod(6n-3)\,\delta.$

If $n \equiv 4\; mod\; 12 \implies n = 12k+4$ then $f^*(u_0) = [\,(\frac{17n-8}{4})\,\delta]\; mod(6n-3)\,\delta.$

If $n \equiv 6 \; mod \; 12 \implies n = 12k + 6$, then $f^*(u_0) = [\,(\frac{21n-10}{4})\,\delta]\; mod(6n-3)\,\delta.$

If $n \equiv 8 \; mod \; 12 \implies n = 12k + 8$ then $f^*(u_0) = [\,(\frac{n}{4})\,\delta]\; mod(6n-3)\,\delta.$

If $n \equiv 10 \; mod \; 12 \implies n = 12k + 10$ then $f^*(u_0) = [\,(\frac{5n-2}{4})\,\delta]\; mod(6n-3)\,\delta.$

Case (2): When n is odd, $n > 3$, we define the labeling function f as follows:

$$f(v_i\, u_{i+1}) = \delta\,(i), \qquad for \quad 1 \le i \le n-1,$$
$$f(v_0\, u_i) = \delta\,[\frac{3}{2}(n-1) + i], \qquad for \quad 1 \le i \le n,$$
$$f(v_0\, v_i) = \delta\,(\frac{9n-3}{2} - i), \qquad for \quad 1 \le i \le n.$$

$$f(v_i\, v_{i+1}) = \begin{cases} \delta\,(5n-1), & \text{if } i = 1; \\ \delta\,(\frac{9n-5}{2} + \frac{i}{2}), & \text{if } i = 2, 4, \cdots, n-3, n-1; \\ \delta\,(n + \frac{i-3}{2}), & \text{if } i = 3, 5, \cdots, n-4, n-2. \end{cases}$$

$$f(u_i\, v_{i+1}) = \begin{cases} \delta\,(\frac{3}{2}(n-1)), & \text{if } i = 1; \\ \delta\,(6n-2-i), & \text{if } i = 2, 3, \cdots, n-2; \\ \delta\,(\frac{5n-1}{2}), & \text{if } i = n-1. \end{cases}$$

$$f(v_i\, u_0) = \begin{cases} \delta\,(6n-3), & \text{if } i = 1; \\ \delta\,(\frac{5n-3}{2} + i), & \text{if } i = 2, 3, \cdots, n-1; \\ \delta\,(5n-2), & \text{if } i = n. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$f^*(v_0) = 0, \quad f^*(v_1) = (\frac{7n+1}{2})\,\delta, \quad f^*(v_2) = (4)\,\delta, \quad f^*(v_n) = (4n-1)\,\delta,$

$f^*(u_1) = (3n-2)\,\delta, \quad f^*(u_{n-1}) = (6n-5)\,\delta, \quad f^*(u_n) = (\frac{7n-5}{2})\,\delta,$

$f^*(v_i) = (\frac{37}{2}n - \frac{17}{2} + i)\,\delta \; mod \; [(6n-3)\delta] = (\frac{n+1}{2} + i)\,\delta, \qquad for \quad 3 \le i \le n-1,$

$f^*(u_i) = (\frac{15n-9}{2} + i)\,\delta \; mod \; [(6n-3)\delta] = (\frac{3n-3}{2} + i)\,\delta, \qquad for \quad 2 \le i \le n-2,$

Finally, $f^*(u_0) = \sum_{i=1}^{n} f(u_0 v_i) = [\sum_{i=2}^{n-1} f(u_0 v_i) + f(u_0 v_1) + f(u_0 v_n)]\; mod(6n-3)\,\delta$

$= [\sum_{i=2}^{n-1}(\frac{5}{2}n - \frac{3}{2} + i)\,\delta + (5n-2)\,\delta]\; mod(6n-3)\,\delta = [(3n^2 - 2n)\,\delta]\; mod(6n-3)\,\delta$

$\because \; n \equiv 1 \; mod \; 2 \implies n = 2k+1$

$f^*(u_0) = [(12k^2 + 8k + 1)\,\delta]\; mod(12k+3)\,\delta = [(5k+1)\,\delta]\; mod(12k+3)\,\delta = (\frac{5n-3}{2})\,\delta.$

Hence the vertex labels are all distinct and a multiple of δ. Therefore $S'(F_n)$ admits an edge $\delta-$ graceful labeling. □

Illustration: The splitting graphs $S'(F_8)$ with an edge $5-$ graceful labeling and $S'(F_9)$ with an edge $4-$ graceful labeling are presented in Figure 6.

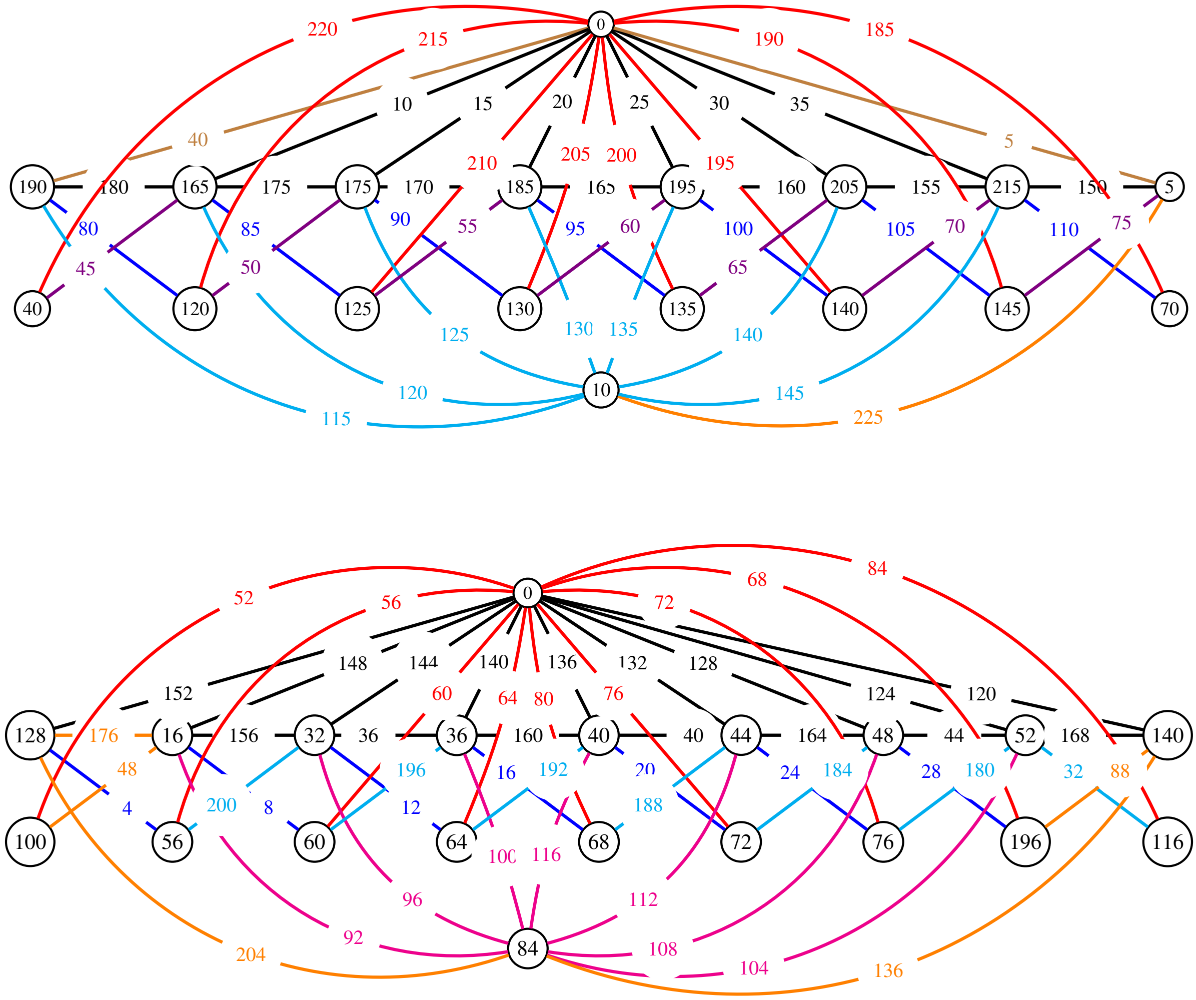

Figure 6. The graphs $S'(F_8)$ with an edge 5− graceful labeling and $S'(F_9)$ with an edge 4−graceful labeling.

2.3. Edge δ−graceful labeling of the splitting graph $S'(Cr_n)$ of the crown graph

Theorem 2.3. *For any positive integer δ, the splitting graph $S'(Cr_n)$ of the crown graph is an edge δ− graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ be the vertices of the crown Cr_n, to form the splitting graph $S'(Cr_n)$ we add newly vertices $\{v'_1, v'_2, \cdots, v'_n\}$ and $\{u'_1, u'_2, \cdots, u'_n\}$ corresponding to the vertices of the crown Cr_n. The edges set of the splitting graph $S'(Cr_n)$ are $\{v_i\ u'_i,\ v'_i\ v_{i+1},\ v_i\ v'_{i+1},\ v_i\ v_{i+1},\ v'_i\ u_i,\ u_i\ v_i,\ 1 \le i \le n\}$, see Figure 7. The graph $S'(Cr_n)$ has $p = 4n$ vertices and $q = 6n$ edges, $k = max(p, q) = 6n$.

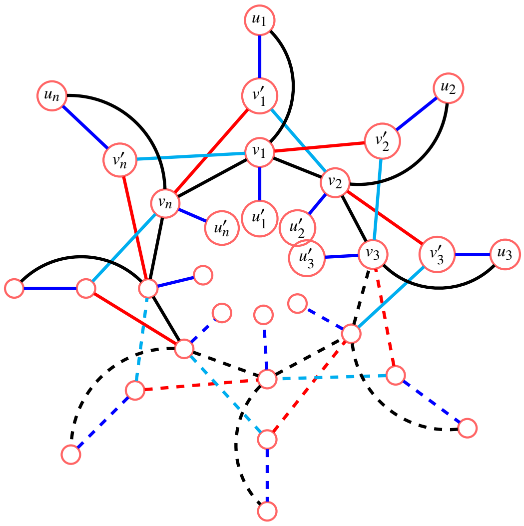

Figure 7. The splitting graph $S'(Cr_n)$ of the crown graph.

Case (1): When n is odd. We define the labeling function $f : E(S'(Cr_n)) \longrightarrow \{\delta, 2\delta, \cdots, (6n)\delta\}$ as follows:

$$f(v'_i v_{i+1}) = \delta(n+i), \quad for \quad 1 \le i \le n,$$
$$f(v'_i u_i) = \delta(5n-i), \quad for \quad 1 \le i \le n,$$
$$f(u_i v_i) = \delta(6n-i), \quad for \quad 1 \le i \le n,$$

$$f(v_i v'_{i+1}) = \delta(3n-i), \quad for \quad 1 \le i \le n-1, \quad \text{and} \quad f(v_n v'_1) = \delta(6n),$$
$$f(v_i v_{i+1}) = \delta(3n+i), \quad for \quad 1 \le i \le n-1, \quad \text{and} \quad f(v_n v_1) = \delta(3n),$$

$$f(v_i u'_i) = \begin{cases} \delta(n), & \text{if } i = 1; \\ \delta(i), & \text{if } for\ 2 \le i \le n-1; \\ \delta, & \text{if } i = n. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$$f^*(u'_1) = n\,\delta, \quad f^*(u'_n) = \delta, \quad f^*(v'_1) = 0, \quad f^*(v_1) = (6n-1)\,\delta, \quad f^*(v_n) = (2n-1)\,\delta,$$
$$f^*(u_i) = (5n-2i)\,\delta, \quad for \quad 1 \le i \le n,$$
$$f^*(u'_i) = (i)\,\delta, \quad for \quad 2 \le i \le n-1,$$
$$f^*(v'_i) = (3n-i+1)\,\delta, \quad for \quad 2 \le i \le n,$$
$$f^*(v_i) = (4n+2i-2)\,\delta, \quad for \quad 2 \le i \le n-1.$$

Case (2): When n is even. The labeling function $f : E(S'(Cr_n)) \longrightarrow \{\delta,\ 2\,\delta,\ \cdots,\ (6n)\delta\}$ defined as follows:

$$f(u_i\, v_i) = \delta\,(i), \qquad for \quad 1 \le i \le n,$$
$$f(v'_i\, v_{i+1}) = \delta\,(3n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\, v'_{i+1}) = \delta\,(5n+1-i), \qquad for \quad 1 \le i \le n,$$

$$f(v_1\, u'_1) = \delta\,(6n), \qquad f(v_i\, u'_i) = \delta\,(2n+1-i), \quad for \quad 2 \le i \le n,$$
$$f(v'_1\, u_1) = \delta\,(6n-2), \qquad f(v'_i\, u_i) = \delta\,(2n+i), \quad for \quad 2 \le i \le n,$$

$$f(v_i\, v_{i+1}) = \begin{cases} \delta\,(5n-1+2i), & \text{if } \ 1 \le i \le \frac{n}{2}; \\ \delta\,(2n+1), & \text{if } \ i = \frac{n}{2}+1; \\ \delta\,(4n+2i-2), & \text{if } \ \frac{n}{2}+2 \le i \le n-1; \\ \delta\,(2n), & \text{if } \ i = n. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$$f^*(v_1) = (4n+2)\,\delta, \quad f^*(v_{\frac{n}{2}+1}) = \delta, \quad f^*(v_{\frac{n}{2}+2}) = (5n+4)\delta, \quad f^*(v_n) = (6n-3)\delta,$$
$$f^*(u_1) = (6n-1)\,\delta, \quad f^*(u'_1) = 0, \quad f^*(v'_1) = (n)\,\delta,$$

$$f^*(u_i) = (2n+2i)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(u'_i) = (2n+1-i)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(v'_i) = (4n+i+2)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(v_i) = (2n+4i-3)\,\delta, \qquad for \quad 2 \le i \le \frac{n}{2},$$
$$f^*(v_i) = (4i-5)\,\delta, \qquad for \quad \frac{n}{2}+3 \le i \le n-1.$$

Hence the vertex labels are all distinct. Therefore $S'(Cr_n)$ admits an edge $\delta-$ graceful labeling. □

Illustration: The splitting graphs $S'(Cr_8)$ of the crown graph with an edge 5− graceful labeling and $S'(Cr_9)$ with an edge 3− graceful labeling are shown in Figure 8.

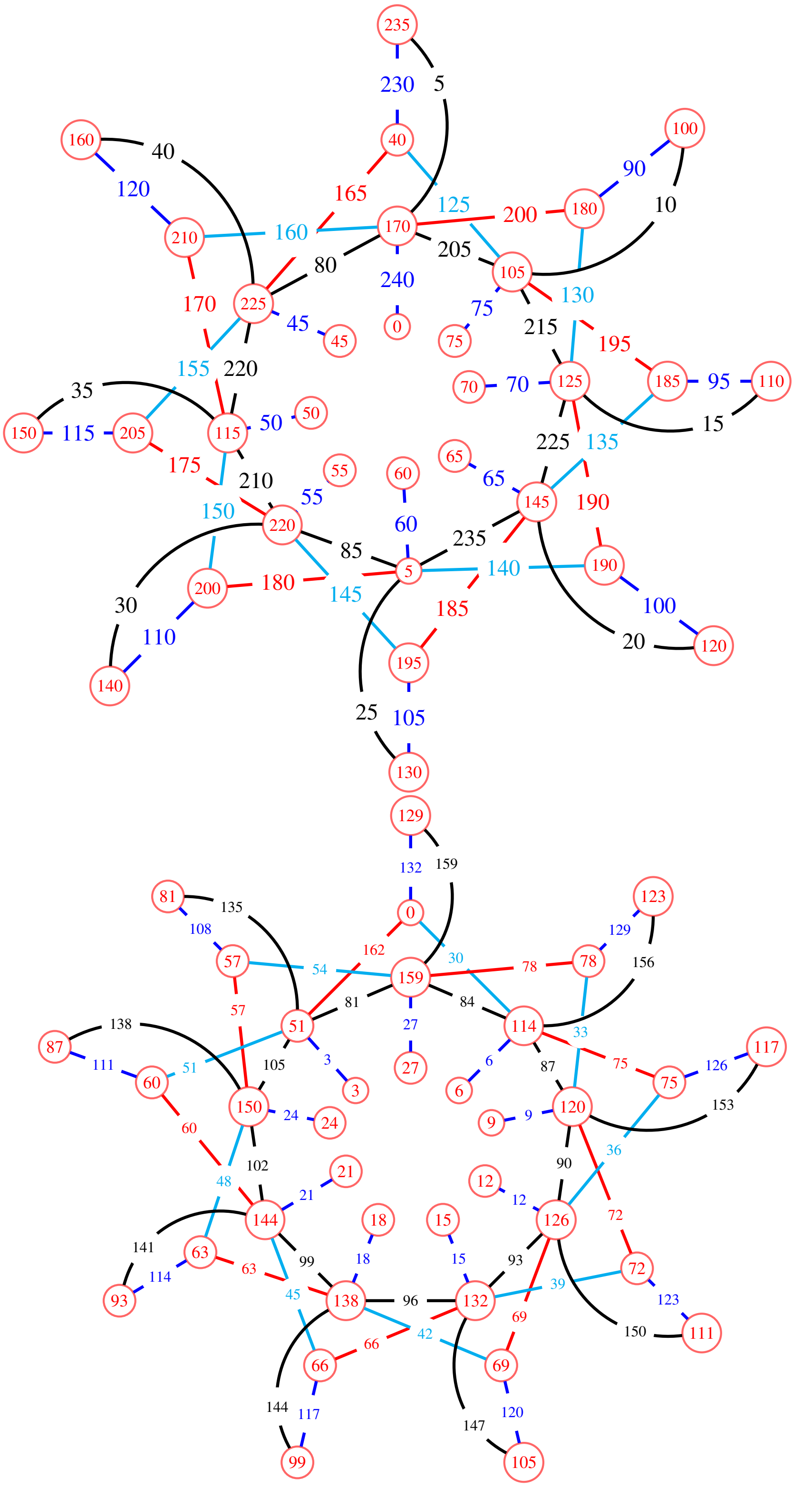

Figure 8. The graphs $S'(Cr_8)$ with an edge 5− graceful labeling and $S'(Cr_9)$ with an edge 3− graceful labeling.

3. Edge $\delta-$ graceful labeling of the shadow graph of some graphs

3.1. Edge $\delta-$ graceful labeling of the shadow graph $D_2(P_n)$

Theorem 3.1. *For any positive integer δ, the shadow graph $D_2(P_n)$, $n > 2$ of the path P_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices in first copy of P_n and $\{u_1, u_2, \cdots, u_n\}$ be that in second copy of P_n. The edges set in the shadow graph $D_2(P_n)$ are $\{v_iv_{i+1}, v_iu_{i+1}, u_iv_{i+1}, u_iu_{i+1}, 1 \leq i \leq n-1\}$. The four vertices v_1, v_n, u_1 and u_n are of degree 2 and the remaining vertices are of degree 4, so the graph $D_2(P_n)$ has $p = 2n$ vertices and $q = 4n - 4$ edges, $k = max(p, q) = 4n - 4$.

- If $n = 2$ the graph $D_2(P_2)$ is not an edge $\delta-$ graceful graph since it isomorphic to C_4 [2].
- If $n = 3$ the graph $D_2(P_3)$ is an edge $\delta-$ graceful graph for any positive integer δ define the labeling function $f : E(D_2(P_3)) \longrightarrow \{\delta, 2\,\delta, \cdots, 8\delta\}$ as shown in Figure 9.

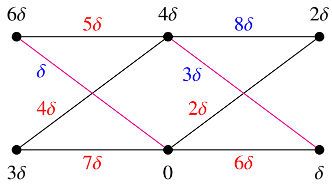

Figure 9. The shadow graph $D_2(P_3)$ with edge $\delta-$ graceful labeling.

- If $n \geq 4$. We define the labeling function $f : E(D_2(P_n)) \longrightarrow \{\delta, 2\,\delta, \cdots, (4n-4)\delta\}$ as follows:

$$\begin{aligned}
f(v_i\,u_{i+1}) &= \delta\, i, && for \quad 1 \leq i \leq n-1,\\
f(v_i\,v_{i+1}) &= \delta\,(4n-4-i), && for \quad 1 \leq i \leq n-1,\\
f(u_i\,v_{i+1}) &= \delta\,(2n-i-1), && for \quad 1 \leq i \leq n-1,
\end{aligned}$$

$$f(u_1\,u_2) = \delta(4n-4), \quad f(u_2\,u_3) = \delta(2n), \quad f(u_3\,u_4) = \delta(2n-1),$$

$$f(u_i\,u_{i+1}) = \delta\,(2n+i-3), \qquad for \quad 4 \leq i \leq n-1.$$

In view of the above labeling pattern then the induced vertex labels are:

$$f^*(v_1) = 0, \quad f^*(v_n) = \delta, \quad f^*(u_1) = (2n-2)\,\delta, \quad f^*(u_n) = (4n-5)\,\delta,$$

$$f^*(u_2) = 2\,\delta, \quad f^*(u_3) = (2n+1)\,\delta, \quad f^*(u_4) = (2n+2)\,\delta,$$

$$\begin{aligned}
f^*(v_i) &= [f(v_iv_{i+1}) + f(v_{i-1}v_i) + f(v_iu_{i-1}) + f(v_iu_{i+1})]\ mod\ [(4n-4)\delta]\\
&= (2n+1-2i)\,\delta, \qquad for \quad 2 \leq i \leq n-1.
\end{aligned}$$

Similarly, $f^*(u_i) = (2n-5+2i)\,\delta,\ for\ 5 \leq i \leq n-1.$

Hence the labels of the vertices are all distinct numbers and a multiple of δ. Thus $D_2(P_n)$ is an edge $\delta-$ graceful graph. □

Illustration: The shadow graph $D_2(P_{13})$ with edge $3-$ graceful labeling is presented in Figure 10.

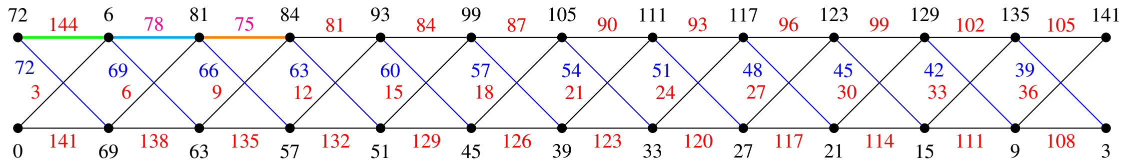

Figure 10. The shadow graph $D_2(P_{13})$ with edge 3− graceful labeling.

3.2. Edge δ− graceful labeling of the shadow graph $D_2(C_n)$ of the cycle C_n

Theorem 3.2. *For any positive integer δ , the shadow graph $D_2(C_n)$, $n \geq 3$ of the cycle C_n is an edge δ− graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices in first copy of C_n and $\{u_1, u_2, \cdots, u_n\}$ be that in second copy of C_n. According to the construction of the shadow graph $D_2(C_n)$ of the cycle C_n, the edges set in the shadow graph $D_2(C_n)$ are $\{v_iv_{i+1}, v_iu_{i+1}, u_iv_{i+1}, u_iu_{i+1}, i = 1, 2, \cdots, n\}$. All the vertices v_i and u_i are of degree 4, so the graph $D_2(C_n)$ has $p = 2n$ vertices and $q = 4n$ edges, $k = max(p, q) = 4n$. There are two cases:

Case (1): When n is odd, we define the labeling $f : E(D_2(C_n)) \longrightarrow \{\delta, 2\delta, 3\delta, \cdots, 4n\,\delta\}$ as follows:

$$\begin{array}{lll} f(v_1\,v_n) = \delta\,n, & f(v_i\,v_{i+1}) = \delta\,i & for \quad 1 \leq i \leq n-1, \\ f(u_1\,u_n) = \delta\,2n, & f(u_i\,u_{i+1}) = \delta\,(n+i) & for \quad 1 \leq i \leq n-1, \\ f(v_1\,u_n) = \delta\,3n, & f(v_i\,u_{i+1}) = \delta\,(4n-i) & for \quad 1 \leq i \leq n-1, \\ f(u_1\,v_n) = \delta\,4n, & f(u_i\,v_{i+1}) = \delta\,(2n+i) & for \quad 1 \leq i \leq n-1. \end{array}$$

In view of the above labeling pattern we have:

$$f^*(v_1) = n\,\delta \quad \text{and} \quad f^*(v_i) = [2\,\delta\,(n+i-1)]\; mod\,(4n)\,\delta, \qquad for \quad 2 \leq i \leq n,$$

$$f^*(u_n) = 3\,n\,\delta \quad \text{and} \quad f^*(u_i) = (2\,\delta\,i)\; mod\,(4n)\,\delta, \qquad for \quad 1 \leq i \leq n-1.$$

Hence the labels of the vertices $v_2, v_3, v_4, \cdots, v_{n-1}, v_n$ are $2\,\delta(n+1)$, $2\,\delta(n+2)$, $2\,\delta(n+3)$, $\cdots$, $2\,\delta(2n-2)$, $2\,\delta(2n-1)$, respectively,and the labels of the vertices $u_1, u_2, u_3, \cdots, u_{n-1}, u_n$ are $2\,\delta, 4\,\delta, 6\,\delta, \cdots, 2\,\delta(n-1), 3\,\delta\,n$, respectively, which are distinct numbers.

Case (2): When n is even, we define the labeling f as follows:

$$\begin{array}{lll} & f(u_i\,v_{i+1}) = i\,\delta, & for \quad 1 \leq i \leq n, \\ f(v_1\,v_n) = \delta(4n-1), & f(v_i\,v_{i+1}) = \delta\,(2n+i), & for \quad 1 \leq i \leq n-1, \\ f(v_n\,u_1) = \delta\,2n, & f(v_i\,u_{i+1}) = \delta\,(4n-1-i), & for \quad 1 \leq i \leq n-1, \\ f(u_n\,u_1) = \delta\,4n, & f(u_i\,u_{i+1}) = \delta\,(n+i), & for \quad 1 \leq i \leq n-1. \end{array}$$

In view of the above labeling pattern we have:

$$f^*(u_1) = (\,3n+2)\,\delta, \qquad f^*(u_n) = (\,2n-1)\,\delta, \quad f^*(v_1) = (3n-2)\,\delta, \qquad f^*(v_n) = (2n-3)\,\delta,$$

$$f^*(u_i) = (2n-1+2i)\,\delta \;\; mod\,(4n)\,\delta, \qquad for \quad 2 \leq i \leq n-1,$$

$$f^*(v_i) = [\,\delta\,(2i-3)]\; mod\,(4n)\,\delta, \qquad for \quad 2 \leq i \leq n-1.$$

Obviously the vertex labels are all distinct. Thus, the graph $D_2(C_n)$ is an edge δ− graceful graph for all n. □

Illustration: The shadow graph $D_2(C_9)$ with an edge 5− graceful labeling and $D_2(C_{10})$ with an edge 3− graceful labeling are shown in Figure 11.

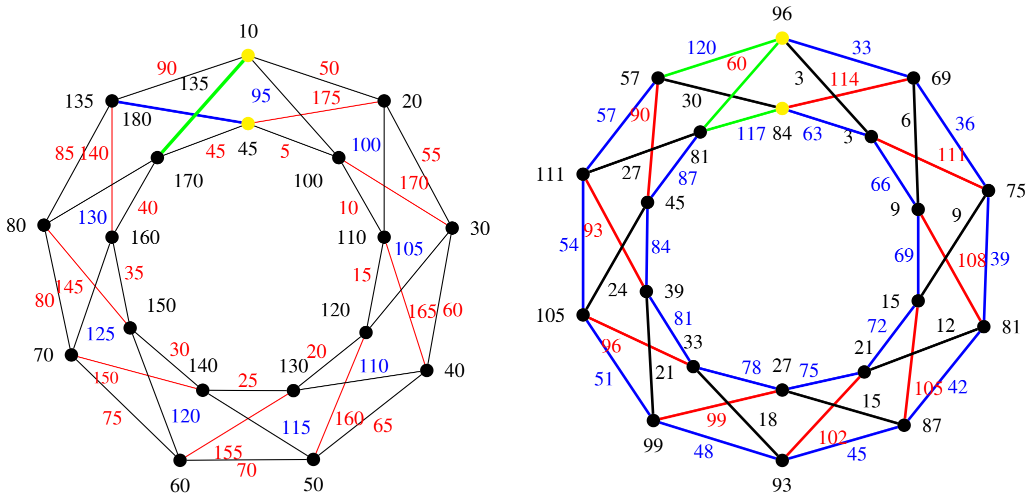

Figure 11. The shadow graph $D_2(C_9)$ with an edge 5− graceful labeling and $D_2(C_{10})$ with an edge 3− graceful labeling.

3.3. Edge δ− graceful labeling of the shadow graph $D_2(F_n)$ of the fan graph F_n

Theorem 3.3. *For any positive integer δ, the shadow graph $D_2(F_n)$, $n \geq 3$ of the fan graph F_n is an edge δ− graceful graph.*

Proof. Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices in first copy of F_n and $\{u_0, u_1, u_2, \cdots, u_n\}$ be that in second copy of F_n. The edges set in the shadow graph $D_2(F_n)$ are $\{v_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1}, u_i u_{i+1}, \ 1 \leq i \leq n-1\} \cup \{u_i v_0, v_i u_0, v_i v_0, u_0 u_i, \ 1 \leq i \leq n\}$, so the graph $D_2(F_n)$ has number of total vertices $p = |V(D_2(F_n)| = 2n + 2$ and edges $q = |E(D_2(F_n)| = 8n - 4$, $n = 2, 3, 4, \cdots$, see Figure 12. There are three cases:

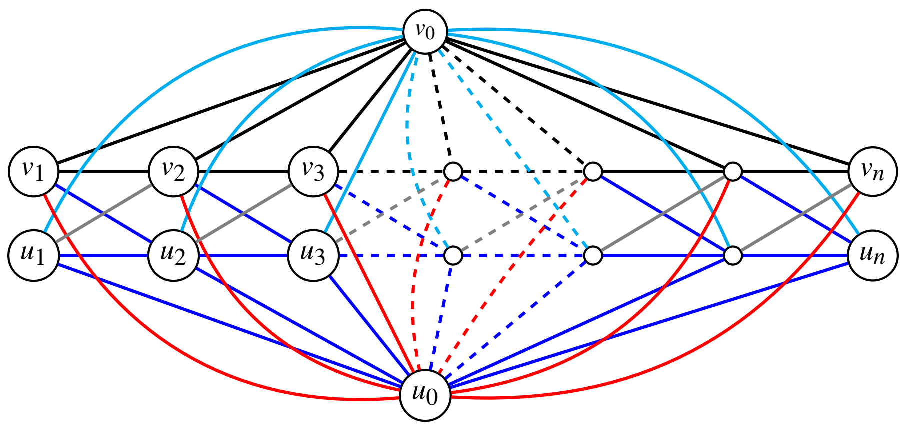

Figure 12. The shadow graph $D_2(F_n)$ of the fan graph F_n.

Case (1): When $n \equiv 0 \; mod \; 4$ and $n \equiv 2 \; mod \; 4$, we define the labeling
$f : E[D_2(F_n)] \longrightarrow \{\delta,\ 2\delta,\ 3\delta,\ \cdots,\ (8n-4)\,\delta\}$ as follows:

$$\begin{aligned}
f(v_0\, v_i) &= \delta\, i, && for \quad 1 \le i \le n,\\
f(v_0\, u_i) &= \delta\,(8n-4-i), && for \quad 1 \le i \le n,\\
f(v_i\, v_{i+1}) &= \delta\,(n+i), && for \quad 1 \le i \le n-1,\\
f(u_i\, u_{i+1}) &= \delta\,(3n-1-i), && for \quad 1 \le i \le n-1,\\
f(v_i\, u_{i+1}) &= \delta\,(6n-3-i), && for \quad 1 \le i \le n-1,\\
f(u_i\, v_{i+1}) &= \delta\,(7n-4-i), && for \quad 1 \le i \le n-1,\\
f(u_0\, u_i) &= \delta\,(4n-1-i), && for \quad 1 \le i \le n,
\end{aligned}$$

$$f(u_0\, v_1) = \delta(\,8n-4), \qquad f\,(u_0\, v_i) = \delta\,(5n-1-i) \qquad for \quad 1 \le i \le n.$$

In view of the above labeling pattern, we can check that

(i) $[f(v_0\, v_i) + f(v_0\, u_i)] \; mod \; [(8n-4)\,\delta\,] = 0, \qquad for \quad 1 \le i \le n.$
(ii) $[f(u_0\, u_{n-i}) + f(u_0\, v_{i+2})] \; mod \; [(8n-4)\,\delta] = 0, \qquad for \quad 0 \le i \le \frac{n}{2}.$

Then, the induced vertex labels are:

$f^*(v_0) = \sum_{i=1}^{n}[\, f(v_0\, v_i) + f(v_0\, u_i)] \; mod \; [(8n-4)\,\delta] = 0,$
$f^*(u_0) = \sum_{i=1}^{n}[\, f(u_0\, v_i) + f(u_0\, u_i)] \; mod \; [(8n-4)\,\delta] = [f(u_0\, v_1) + f(u_0\, u_1)] mod\, (8n-4)\,\delta = (4n-2)\delta,$

$f^*(v_1) = (n-1)\,\delta, \qquad f^*(v_n) = (5n)\,\delta, \qquad f^*(u_1) = (4n-3)\,\delta, \qquad f^*(u_n) = n\,\delta,$
$f^*(v_i) = [f(v_i\, v_{i+1}) + f(v_{i-1}\, v_i) + f(v_i\, u_0) + f(v_i\, v_0) + f(v_i\, u_{i+1}) + f(u_{i-1}\, v_i)] \; mod \; [(8n-4)\,\delta]$

$= [\,\delta\,(8n-4i)] \; mod \; [(8n-4)\,\delta], \qquad for \quad 2 \le i \le n-1.$

$f^*(u_i) = [f(u_i\, u_{i+1}) + f(u_{i-1}\, u_i) + f(u_i\, u_0) + f(u_i\, v_0) + f(u_i\, v_{i+1}) + f(v_{i-1}\, u_i)] \; mod \; (8n-4)\,\delta$
$= [\,\delta\,(3n-2i)] \; mod \; [(8n-4)\,\delta], \qquad for \quad 2 \le i \le n-1.$

Hence the labels of the vertices $v_2,\ v_3,\ v_4,\ \cdots,\ v_{n-2},\ v_{n-1}$ are $\delta(8n-8),\ \delta(8n-12),\ \delta(8n-16),\ \cdots,\ \delta(4n+8),\ \delta(4n+4)$, respectively, and the labels of the vertices $u_2,\ u_3,\ u_4,\ \cdots,\ u_{n-2},\ u_{n-1}$ are $\delta(3n-4),\ \delta(3n-6),\ \delta(3n-8),\ \cdots,\ \delta(n+4),\ \delta(n+2)$, respectively, which are distinct numbers.
Case (2): When $n \equiv 1 \; mod \; 4$, we define the labeling f as follows:

$$\begin{aligned}
f(v_0\, v_i) &= \delta\,(n-1+i), && for \quad 1 \le i \le n,\\
f(v_0\, u_i) &= \delta\,(7n-3-i), && for \quad 1 \le i \le n,\\
f(u_i\, u_{i+1}) &= \delta\,(4n-3-i), && for \quad 1 \le i \le n-1,
\end{aligned}$$

$$\begin{aligned}
f\,(u_0\, v_i) &= \delta(\,n-i), && for \quad 1 \le i \le n-1, && f\,(u_0\, v_n) = \delta\,(4n-3),\\
f\,(u_0\, u_i) &= \delta(\,8n-4-i), && for \quad 1 \le i \le n-1, && f\,(u_0\, u_n) = \delta\,(8n-4),\\
f\,(v_i\, u_{i+1}) &= \delta(\,2n-1+i), && for \quad 1 \le i \le n-2, && f\,(v_{n-1}\, u_n) = \delta\,(4n-2),
\end{aligned}$$

$$f\,(u_1\, v_2) = \delta(\,4n-1), \qquad f\,(u_i\, v_{i+1}) = \delta(\,6n-2-i), \qquad for \quad 2 \le i \le n-1,$$

$$f\,(v_i\, v_{i+1}) = \begin{cases} \delta(\,4n+i), & \text{if} \quad i = 1;\\ \delta(\,4n), & \text{if} \quad i = 2;\\ \delta\,(4n-1+i), & \text{if} \quad 3 \le i \le n-1. \end{cases}$$

In view of the above labeling pattern, the induced vertex labels are:

$f^*(v_0) = 0,$ $\quad f^*(v_1) = 4\,\delta,$ $\quad f^*(v_2) = 8\,\delta,$

$f^*(v_3) = \delta\,(2n+7),$ $\quad f^*(v_{n-1}) = 5n\,\delta,$ $\quad f^*(v_n) = \delta,$

$f^*(u_0) = (4n-3)\,\delta,$ $\quad f^*(u_1) = (7n-6)\,\delta\,,$ $\quad f^*(u_n) = (5n-3)\,\delta,$

$$f^*(v_i) = [\,\delta\,(2n+2+2i)]\ mod\ [(8n-4)\,\delta], \qquad for\ \ 4 \le i \le n-2.$$

Finally, $f^*(u_i) = [\,\delta\,(7n-4-4i)]\ mod\ [(8n-4)\,\delta], \qquad for\ \ 2 \le i \le n-1.$

Hence the labels of the vertices $v_2,\ v_3,\ v_4,\ \cdots,\ v_{n-2}\ ,v_{n-1}$ are $\delta(8n-8),\ \delta(8n-12),\ \delta(8n-16),\ \cdots,\ \delta(4n+8),\ \delta(4n+4)$, respectively, and the labels of the vertices $u_2,\ u_3,\ u_4,\ \cdots,\ u_{n-2},\ u_{n-1}$ are $\delta(7n-12),\ \delta(7n-16),\ \delta(7n-20),\ \cdots,\ \delta(3n+4),\ \delta(3n)$, respectively, which are distinct numbers.

In this labeling, the induced labeling of the vertex u_0 will equal the induced labeling of the vertex

u_i when $i = \dfrac{3n-1}{4}$ i.e., when $n = 4k+3 \ \Rightarrow\ n \equiv 3\ mod\ 4.$

Case (3): When $n \equiv 3\ mod\ 4$, we define the labeling f as in the case $n \equiv 1\ mod\ 4$ but we change the labeling of two edges (u_0v_n) and (u_1v_2) as follows:

$f(u_0v_n) = \delta\,(4n-1)$ and $f(u_1v_2) = \delta\,(4n-3).$

The induced vertex labels are:

$f^*(v_0) = 0,$ $\quad f^*(v_1) = 4\,\delta,$ $\quad f^*(v_2) = 6\,\delta,$

$f^*(v_3) = \delta\,(2n+7),$ $\quad f^*(v_{n-1}) = 5n\,\delta,$ $\quad f^*(v_n) = 3\,\delta,$

$f^*(u_0) = (4n-1)\,\delta,$ $\quad f^*(u_1) = (7n-8)\,\delta\,,$ $\quad f^*(u_n) = (5n-3)\,\delta,$

$$f^*(v_i) = [\,\delta\,(2n+2+2i)]\ mod\ [(8n-4)\,\delta], \qquad for\ \ 4 \le i \le n-2$$

$$f^*(u_i) = [\,\delta\,(7n-4-4i)]\ mod\ [(8n-4)\,\delta], \qquad for\ \ 2 \le i \le n-1.$$

Hence the vertex labels are all distinct and a multiple of δ. Therefore the shadow graph $D_2(F_n)$ admits an edge $\delta-$ graceful labeling. □

Illustration: The shadow graphs $D_2(F_8)$ with edge 4− graceful labeling, $D_2(F_9)$ with edge 3− graceful labeling and $D_2(F_7)$ with an edge 5− graceful labeling. are presented in Figure 13.

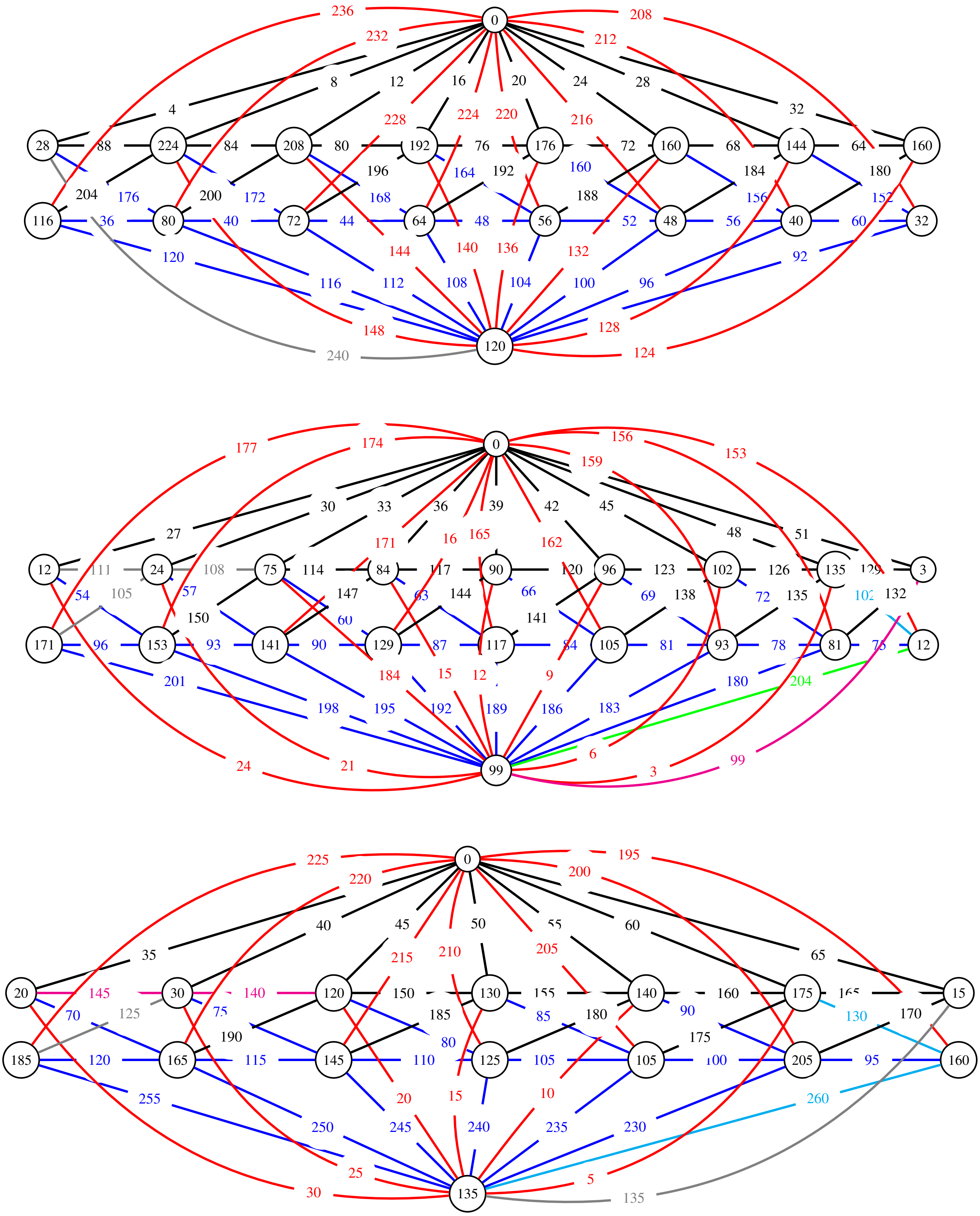

Figure 13. Some shadow graphs $D_2(F_n)$ with distinct edge $\delta-$ graceful labeling.

4. Edge $\delta-$ graceful labeling of the middle graph of some graphs

4.1. Edge $\delta-$ graceful labeling of the middle graph $M(P_n)$

Theorem 4.1. *For any positive integer δ , the middle graph $M(P_n)$ of path P_n is an edge $\delta-$ graceful graph when n is even and $n \equiv 1 \; mod \; 4$.*

Proof. Let $\{v_1, \; v_2, \; \cdots, \; v_n\}$ be the vertices and $\{u_1, \; u_2, \; \cdots, \; u_{n-1}\}$ be the edges of path P_n. Then $V[M(P_n)] = V(P_n) \cup E(P_n)$ and $E[M(P_n)] = \{v_iu_i; \; 1 \le i \le n-1, \; v_iu_{i-1}; \; 2 \le i \le n, \; u_iu_{i+1}; \; 1 \le i \le n-2\}$. Here $p = 2n-1$ vertices and $q = 3n-4$ edges, $k = max(p,q) = 3n-4$.

Case (1): When n is even, We define the labeling function $f : E(M(P_n)) \longrightarrow \{\delta, \; 2\,\delta, \; \cdots, \; (3n-4)\delta\}$ as follows:

$$f\,(u_i\,u_{i+1}) = \begin{cases} \delta\,(n-2i-2), & \text{if} \;\; 1 \le \; i \le \frac{n}{2}-2; \\ \delta\,(n-1), & \text{if} \;\; i = \frac{n}{2}-1; \\ \delta\,(2n-2i-5), & \text{if} \;\; \frac{n}{2} \le i \le n-3; \\ \delta\,(n-2), & \text{if} \;\; i = n-2. \end{cases}$$

$$f\,(u_i\,v_i) = \begin{cases} \delta\,(n-3), & \text{if} \quad i = 1; \\ \delta\,(n+i-2), & \text{if} \;\; 2 \le i \le n-1. \end{cases}$$

$$f(v_i\,u_{i-1}) = \delta(2n-4+i), \qquad for \;\; 2 \le i \le \; n.$$

Then the induced vertex labels are:

$f^*(v_1) = \; \delta(n-3), \qquad f^*(v_n) = 0, \qquad f^*(u_1) = \; (n-5)\,\delta, \qquad f^*(u_{n-1}) = \; (3n-5)\,\delta,$
$f^*(u_{n-2}) = \; (3n-6)\,\delta, \qquad f^*(u_{\frac{n}{2}}) = \; (3n-7)\,\delta, \quad and \quad f^*(u_{\frac{n}{2}-1}) = \; (2n-2)\,\delta,$
$f^*(v_i) = [f(v_iu_{i-1}) + f(v_iu_i)] \; mod \; [(3n-4)\delta] = \; (2i-2)\,\delta, \qquad for \quad 2 \le i \le n-1;$
$f^*(u_i) = [f(u_{i-i}u_i) + f(u_iu_{i+1}) + f(u_iv_i) + f(u_iv_{i+1})] \; mod \; [(3n-4)\delta]$
$\qquad = \; (2n-3-2i)\,\delta, \qquad for \qquad 2 \le \; i \; \le \frac{n}{2}-2$, and
$f^*(u_i) = \; (4n-9-2i)\,\delta, \qquad for \qquad \frac{n}{2}+1 \le i \le \; \cdots, n-3.$

Case (2): When $n \equiv \; 1 \; mod \; 4$, We define the labeling function $f : E(M(P_n)) \longrightarrow \{\delta, \; 2\,\delta, \cdots, \; (3n-4)\delta\}$ as follows:

$$f(v_i\,u_i) = \delta\,i, \qquad for \quad 1 \le \; i \le n-1,$$
$$f(u_i\,u_{i+1}) = \delta\,(n+i), \qquad for \;\; 1 \; \le \; i \le n-2, \quad \text{and}$$

$$f\,(v_i\,u_{i-1}) = \begin{cases} \delta\,(2n+i-3), & \text{if} \;\; 2 \le \; i \le n-1; \\ \delta\,n, & \text{if} \;\; i = n. \end{cases}$$

Then the induced vertex labels are:

$$f^*(v_1) = \ \delta, \qquad f^*(v_n) = n\,\delta, \qquad f^*(u_1) = \ 5\,\delta, \qquad f^*(u_{n-1}) = \ (n+1)\,\delta,$$
$$f^*(v_i) = [f(v_iu_{i-1}) + f(v_iu_i)] \ mod \ [(3n-4)\delta] = \ (2n-3+2i)\ \delta mod\ [(3n-4)\delta] \qquad 2 \le \ i \le \ n-1,$$
$$f^*(u_i) = [f(u_{i-i}u_i) + f(u_iu_{i+1}) + f(u_iv_i) + f(u_iv_{i+1})] \ mod \ [(3n-4)\delta]$$
$$= \ [(n+1+4i)\ \delta] \ mod\ [(3n-4)\delta], \qquad for \quad 2 \ \le i \le n-2.$$

Hence the labels of the vertices $v_2, \ v_3, \ \cdots, v_{n-2}, \ v_{n-1}$ are $(2n+1)\delta, \ (2n+3)\delta, \ \cdots, \ (n-3)\delta, \ (n-1)\delta,$ respectively, and the labels of the vertices $u_2, \ u_3, \ \cdots, u_{n-3}, \ u_{n-2}$ will be $(n+9)\delta, \ (n+13)\delta, \ \cdots, \ (2n-7)\delta, \ (2n-3)\delta,$ respectively. Hence the vertex labels are all distinct and a multiple of δ. Therefore $M(P_n)$ admits an edge δ graceful labeling when n is odd. □

Illustration: The middle graphs $M(P_{14})$ with an edge 2− graceful labeling and $M(P_{13})$ with an edge 3− graceful labeling are shown in Figure 14.

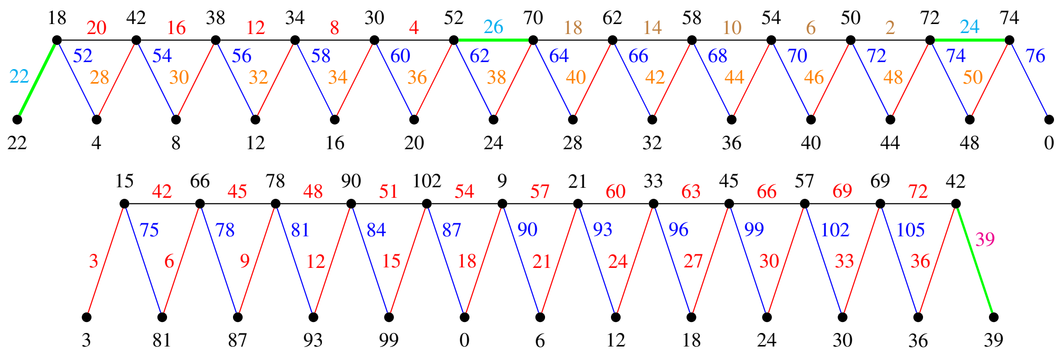

Figure 14. The middle graphs $M(P_{14})$ with an edge 2− graceful labeling and $M(P_{13})$ with an edge 3− graceful labeling.

4.2. The middle graph $M(C_n)$, $n \ge 3$ of the cycle C_n

Theorem 4.2. *For any positive integer δ, the middle graph $M(C_n)$, $n \ge 3$ of the cycle C_n is an edge δ− graceful graph when n is odd number.*

Proof. Let $\{v_1, v_2, \ \cdots, \ v_n\}$ be the vertices of the cycle C_n and $\{u_1, u_2, \ \cdots, \ u_{n-1}\}$ be the edges of the cycle C_n. The $V[M(C_n)] = V(C_n) \cup E(C_n)$ and $E[M(C_n)] = \{u_iu_{i+1}, \ v_iu_i, \ u_iv_{i+1}; \ 1 \le i \le n\}$, so the number of vertices $p = |V[M(C_n]| = 2n$ and edges $q = |E[M(C_n)]| = 3n, \ n = 3, 4, \ \cdots$.

When n is odd, we define the labeling function $f : E(M(C_n)\,) \longrightarrow \{\delta, \ 2\,\delta, \ \cdots, \ (3n)\delta\}$ as follows:

$$f(u_i\ v_{i+1}) = \delta\,(3i-2), \qquad for \quad 1 \le \ i \le \ n,$$
$$f(u_i\ u_{i+1}) = \delta\,(3i-1), \qquad for \quad 1 \le \ i \le \ n,$$
$$f(v_1\ u_1) = \ 3n\,\delta, \qquad f(v_i\ u_i) = 3\delta\,(i-1), \qquad for \quad 2 \le \ i \le \ n.$$

In view of the above labeling pattern,the induced vertex labels are:

$$f^*(v_1) = \delta(3n-2), \qquad f^*(v_i) = \ (6i-8)\,\delta \ mod\ [(3n)\delta], \qquad 2 \le \ i \le \ n,$$
$$f^*(u_1) = \ 2\,\delta, \qquad f^*(u_i) = \ (12i-10)\,\delta \ mod\ [(3n)\delta], \qquad 2 \le \ i \le \ n.$$

Hence the vertex labels are all distinct and a multiple of δ. Therefore $M(C_n)$ admits an edge δ− graceful labeling. □

Illustration: The middle graph $M(C_9)$ of the cycle C_9 with an edge 6− graceful labeling is shown in Figure 15.

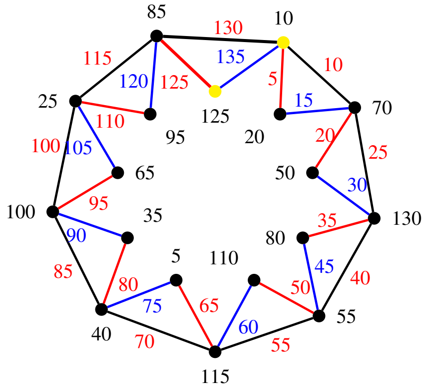

Figure 15. The middle graph $M(C_9)$ with an edge 5− graceful labeling.

4.3. The middle graph $M(Cr_n)$ of the crown graph Cr_n

Theorem 4.3. *For any positive integer δ, the middle graph of the crown graph Cr_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_{n-1}\}$ be the vertices of the crown graph Cr_n and $v'_1, v'_2, \cdots, v'_n$ and $u'_1, u'_2, \cdots, u'_{n-1}$ be the edges of the crown graph Cr_n. Then $V(M(Cr_n)) = V(Cr_n) \cup E(Cr_n)$ and $E(M(Cr_n)) = \{v_i u'_i,\ v'_i v_{i+1},\ v_i v'_i,\ u_i u'_i,\ u'_i v'_i,\ v'_i u'_{i+1},\ v'_i v'_{i+1};\ 1 \le i \le n\}$, so $p = 4n$ vertices and $q = 7n$ edges, $k = max(p, q) = 7n$, see Figure 16.

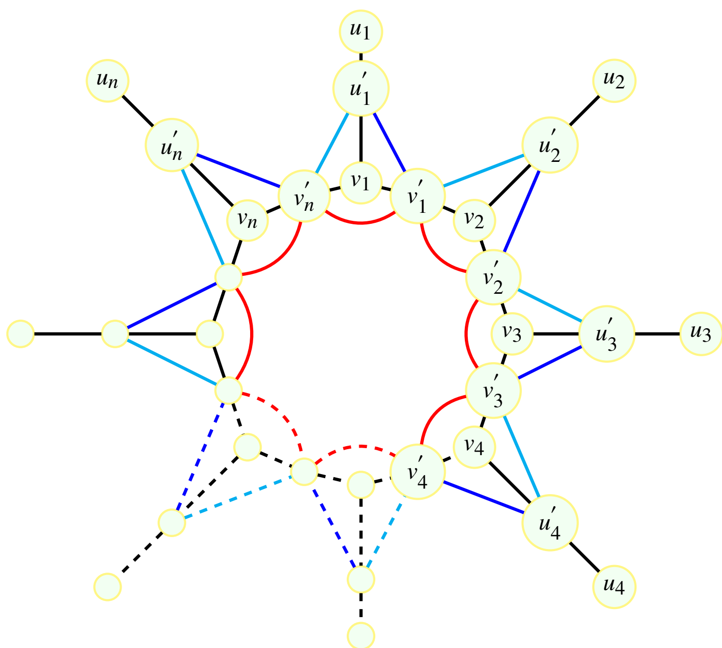

Figure 16. The middle graph $M(Cr_n)$ of the crown graph Cr_n.

Case (1): When n is odd, we define the labeling function $f : E(M(Cr_n)) \longrightarrow \{\delta,\ 2\,\delta,\ \cdots,\ 7n\delta\}$ as follows:

$$f(v_i\, v_i') = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\, v_{i+1}) = \delta\,(3n-i+1), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\, u_{i+1}') = \delta\,(4n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\, u_i') = \delta\,(3n+i), \qquad for \quad 1 \le i \le n,$$

$$f\,(v_1\, u_1') = \delta\,(7n), \qquad f\,(v_i\, u_i') = \delta\, i, \qquad for \quad 2 \le i \le n,$$
$$f\,(u_1\, u_1') = \delta, \qquad f\,(u_i\, u_i') = \delta\,(7n+1-i), \qquad for \quad 2 \le i \le n,$$

$$f\,(v_i'\, v_{i+1}') = \begin{cases} \delta\,(6n-2i), & \text{if} \quad 1 \le i \le \frac{n-1}{2}; \\ \delta\,(6n-1), & \text{if} \quad i = \frac{n+1}{2}; \\ \delta\,(7n-2i), & \text{if} \quad \frac{n+3}{2} \le i \le n-1; \\ \delta\,(6n), & \text{if} \quad i = n. \end{cases}$$

Then the induced vertex labels are:

$$f^*(v_1') = (2n+1)\,\delta, \qquad f^*(v_{\frac{n+1}{2}}') = (2n+2)\,\delta, \qquad f^*(v_{\frac{n+3}{2}}') = (3n)\,\delta, \qquad f^*(v_n') = (3n+3)\,\delta,$$
$$f^*(v_1) = (3n+2)\,\delta, \qquad f^*(u_1) = \delta, \qquad f^*(u_1') = (n+2)\,\delta,$$

$$f^*(v_i) = (4n+i+2)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(u_i) = (7n-i+1)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(u_i') = (2i)\,\delta, \qquad for \quad 2 \le i \le n,$$

$$f\,(v_i') = \begin{cases} \delta\,(2n-2i+3)\ mod\ [(7n)\delta], & \text{if} \quad 2 \le i \le \frac{n-1}{2}; \\ \delta\,(4n-2i+3)\ mod\ [(7n)\delta], & \text{if} \quad \frac{n+5}{2} \le i \le n-1. \end{cases}$$

Hence the labels of the vertices $v_2',\ v_3',\ \cdots,v_{\frac{n-3}{2}}',\ v_{\frac{n-1}{2}}'$ are $(2n-1)\,\delta,\ (2n-3)\,\delta,\ \cdots,\ (n+6)\,\delta,\ (n+4)\,\delta$, respectively, and the labels of the vertices $v_{\frac{n+5}{2}}',\ v_{\frac{n+7}{2}}',\ \cdots,v_{n-2}',\ v_{n-1}'$ are $(3n-2)\,\delta,\ (3n-4)\,\delta,\cdots,\ (2n+7)\,\delta,\ (2n+5)\,\delta$, respectively.

Case (2): When $n \equiv 0\ mod\ 6$, and $n \equiv 2\ mod\ 6$. We define the labeling function $f : E(M(Cr_n)) \longrightarrow \{\delta,\ 2\,\delta,\cdots,\ 7n\delta\}$ as follows:

$$f(v_i\, u_i') = \delta\, i, \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\, v_{i+1}) = \delta\,(3n-i+1), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\, v_i') = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$

$$f\,(u_i\, u_i') = \begin{cases} \delta\,(3n+1), & \text{if} \quad i = 1; \\ \delta\,(7n-i+1), & \text{if} \quad 2 \le i \le n. \end{cases}$$

$$f\,(u_i'\, v_i') = \begin{cases} \delta\,(7n), & \text{if} \quad i = 1; \\ \delta\,(3n+i), & \text{if} \quad 2 \le i \le n. \end{cases}$$

$$f\,(v_i'\, u_{i+1}') = \begin{cases} \delta\,(4n+1+i), & \text{if} \quad 1 \le i \le n-1; \\ \delta\,(4n+1), & \text{if} \quad i = n. \end{cases}$$

$$f\,(v'_i\, v'_{i+1}) = \begin{cases} \delta\,(5n+2i-1), & \text{if } \ 1 \le i \le \frac{n}{2}; \\ \delta\,(4n+2i), & \text{if } \ \frac{n}{2}+1 \le i \le n. \end{cases}$$

Then the induced vertex labels are:

$$\begin{array}{lll} f^*(v_1) = (3n+3)\,\delta, & f^*(v_i) = (4n+i+2)\,\delta, & for \quad 2 \le i \le n, \\ f^*(u_1) = (3n+1)\,\delta, & f^*(u_i) = (7n-i+1)\,\delta, & for \quad 2 \le i \le n, \\ f^*(u'_1) = 3\,\delta, & f^*(u'_i) = (2i+1)\,\delta, & for \quad 2 \le i \le n, \\ f^*(v'_1) = (5n+4)\,\delta, & f^*(v'_{\frac{n}{2}+1}) = (2n+5)\,\delta, & f^*(v'_n) = (3n)\,\delta \quad \text{and} \end{array}$$

$$f\,(v'_i) = \begin{cases} \delta\,(6i-2)\ mod\ [(7n)\delta] & \text{if } \ 2, \le i \le \frac{n}{2}; \\ \delta\,(5n+6i)\ mod\ [(7n)\delta] & \text{if } \ \frac{n}{2}+2 \le i \le n-1. \end{cases}$$

Hence the labels of the vertices $v'_2,\ v'_3,\ \cdots,\ v'_{\frac{n}{2}-1},\ v'_{\frac{n}{2}}$ are $10\delta,\ 16\,\delta,\ \cdots,\ (3n-8)\,\delta,\ (3n-2)\,\delta$, respectively, and the labels of the vertices $v'_{\frac{n}{2}+2},\ v'_{\frac{n}{2}+3},\ \cdots,v'_{n-2},\ v'_{n-1}$ are $(n+12)\,\delta,\ (n+18)\,\delta,\ \cdots,\ (4n-12)\,\delta,\ (4n-6)\,\delta$, respectively.

We notice that, these values are distinct for all n even except when $n \equiv 4\ mod\ 6$, since

$$\begin{array}{ll} v'_{\frac{n}{2}+2} = (n+12)\,\delta = v'_3 = 16\,\delta, & when \quad n = 4 \\ v'_{\frac{n}{2}+2} = (n+12)\,\delta = v'_4 = 22\,\delta, & when \quad n = 10 \\ v'_{\frac{n}{2}+2} = (n+12)\,\delta = v'_5 = 28\,\delta, & when \quad n = 16 \end{array}$$

Case (3): When $n \equiv 4\ mod\ 6$. Define the labeling function $f : E(M(Cr_n)) \longrightarrow \{\delta,\ 2\,\delta,\ \cdots,\ 7n\delta\}$ as in the first case with changes in the following edges

$$f\,(v'_i\, v'_{i+1}) = \begin{cases} \delta\,(5n+2i), & \text{if } \ 1 \le i \le \frac{n}{2}; \\ \delta\,(4n+2i-1), & \text{if } \ \frac{n}{2}+1 \le i \le n. \end{cases}$$

Then the induced vertex labels are:

$$f^*(v'_1) = (5n+4)\,\delta, \qquad f^*(v'_{\frac{n}{2}+1}) = (2n+5)\,\delta, \qquad f^*(v'_n) = (3n-2)\,\delta \quad \text{and}$$

$$f\,(v'_i) = \begin{cases} \delta\,(6i)\ mod\ [(7n)\delta] & \text{if } \ 2 \le i \le \frac{n}{2}; \\ \delta\,(5n+6i-2)\ mod\ [(7n)\delta] & \text{if } \ \frac{n}{2}+2 \le i \le n-1. \end{cases}$$

Hence the labels of the vertices $v'_2,\ v'_3,\ \cdots,\ v'_{\frac{n}{2}-1},\ v'_{\frac{n}{2}}$ are $12\delta,\ 18\,\delta,\ \cdots,\ (3n-6)\,\delta,\ (3n)\,\delta$, respectively, and the labels of the vertices $v'_{\frac{n}{2}+2},\ v'_{\frac{n}{2}+3},\ \cdots,v'_{n-2},\ v'_{n-1}$ are $(n+10)\,\delta,\ (n+16)\,\delta,\ \cdots,\ (4n-14)\,\delta,\ (4n-8)\,\delta$, respectively. Hence there are no repetition in the vertex labels which completes the proof. □

Illustration: The middle graphs $M(Cr_{10})$ with an edge $3-$ graceful labeling and $M(Cr_9)$ with an edge $2-$ graceful labeling are shown in Figure 17.

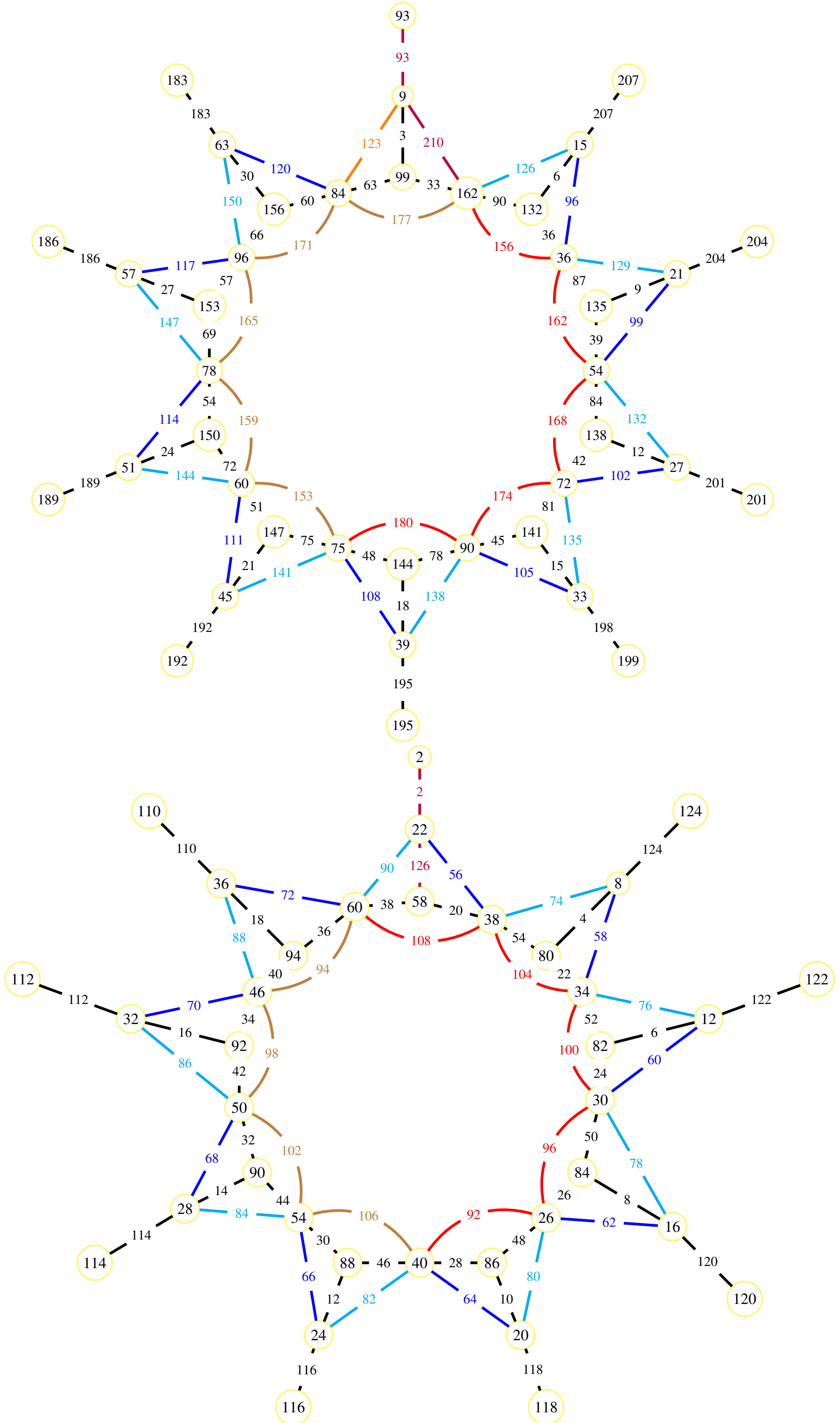

Figure 17. The middle graphs $M(Cr_{10})$ with an edge 3− graceful labeling and $M(Cr_9)$ with an edge 2− graceful labeling.

5. Edge $\delta-$ graceful labeling of the total graph of some graphs

5.1. Edge $\delta-$ graceful labeling of the total graph $T(P_n)$

Theorem 5.1. *For any positive integer δ, the total graph $T(P_n)$ of the path P_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_1, v_1, \cdots, v_n\}$ be the vertices of the path P_n , the total graph $T(P_n)$ of path P_n has vertices set $V(T(P_n)) = \{v_i, \ 1 \le i \le n\} \cup \{u_i, \ 1 \le i \le n-1\}$ and edges set $E(T(P_n)) = \{v_i v_{i+1}, \ v_i u_i, \ u_i v_{i+1}, \ 1 \le i \le n-1\} \cup \{u_i u_{i+1}, \ 1 \le i \le n-2\}$, so the graph $T(P_n)$ has $p = 2n-1$ and $q = 4n-5$, $k = max(p,q) = 4n-5$.

If $n = 2$ the graph $T(P_2)$ is an edge $\delta-$ graceful graph since it isomorphic to C_3 [15]. There are two cases:

Case (1): When n is even, $n \ge 4$. We define the labeling function $f : E(T(P_n)) \longrightarrow \{\delta, \ 2\delta, \ \cdots, \ (4n-5)\delta \}$ as follows:

$$f(v_i\, u_i) = \delta\, i, \qquad for \ \ 1 \le i \le n-1,$$
$$f(v_i\, v_{i+1}) = \delta\, (n-1+i), \qquad for \ \ 1 \le i \le n-1,$$
$$f(u_i\, v_{i+1}) = \delta\, (4n-5-i), \qquad for \ \ 1 \le i \le n-1,$$

$$f(u_i\, u_{i+1}) = \begin{cases} \delta\, (4n-5), & \text{if } \ i = 1; \\ \delta\, (2n-3+i), & \text{if } \ 2 \le i \le n-2. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$f^*(v_1) = (n+1)\,\delta, \quad f^*(v_n) = (n-1)\,\delta, \quad f^*(u_1) = 0, \quad f^*(u_2) = (2n-1)\,\delta, \quad f^*(u_{n-1}) = (3n-5)\,\delta,$
$f^*(v_i) = [(2n-2+2i)\,\delta]\ mod\ [(4n-5)\delta], \qquad for \ \ 2 \le i \le n-1,$
$f^*(u_i) = (2i-2)\,\delta, \qquad for \ \ 3 \le i \le n-2.$

Hence the labels of the vertices are all distinct numbers.

Case (2): When n is odd:

- If $n = 3$ the graph $T(P_3)$ is an edge $\delta-$ graceful graph for any positive integer δ define the labeling function $f : E(T(P_3)) \longrightarrow \{\delta, \ 2\delta, \ \cdots, \ 7\delta\}$ as shown in Figure 18.

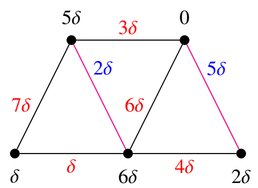

Figure 18. The total graph $T(P_3)$ with edge $\delta-$ graceful labeling.

- If $n \ge 5$. Define the labeling function $f : E(T(P_n)) \longrightarrow \{\delta, \ 2\delta, \ \cdots, \ (4n-5)\delta \}$ as follows:

$$f(v_i\, u_i) = \delta\, i, \qquad for \ \ 1 \le i \le n-1,$$
$$f(u_i\, u_{i+1}) = \delta\, (2n-2+i), \qquad for \ \ 1 \le i \le n-2,$$
$$f(u_i\, v_{i+1}) = \delta\, (2n-1-i), \qquad for \ \ 1 \le i \le n-1, \quad \text{and}$$

$$f(v_i\, v_{i+1}) = \begin{cases} \delta\,(4n-5), & \text{if} \quad i = 1; \\ \delta\,(3n-4+i), & \text{if} \quad 2 \le\ i \le n-2; \\ \delta\,(3n-3), & \text{if} \quad i = n-1. \end{cases}$$

In view of the above labeling pattern then the induced vertex labels are:

$f^*(v_1) = \ \delta, \quad f^*(v_2) = \ (n+3)\delta, \quad f^*(v_{n-1}) = \ (n+1)\,\delta,$
$f^*(v_n) = \ 2\,\delta, \quad f^*(u_1) = \ 3\,\delta, \quad f^*(u_{n-1}) = \ n\,\delta,$
$f^*(v_i) = [(1+2i)\,\delta]\ mod\ [(4n-5)\delta], \qquad for \quad 3 \le\ i \le n-2,$ and
$f^*(u_i) = [\,(2n-1+2i)\,\delta]\ mod\ [(4n-5)\delta], \qquad for \quad 2 \le\ i \le n-2.$

Hence the labels of the vertices $v_3,\ v_4,\ \cdots, v_{n-3},\ v_{n-2}$ will be $7\,\delta,\ 9\,\delta,\ \cdots,\ (2n-5)\,\delta,\ (2n-3)\,\delta$, respectively, and the labels of the vertices $u_2,\ u_3,\ \cdots,\ u_{n-3},\ u_{n-2}$ will be $(2n+3)\,\delta,\ (2n+5)\,\delta,\ \cdots,\ (4n-7)\,\delta,\ 0$, respectively.

In this case we have $f^*(v_{\frac{n-1}{2}})$ will equal $f^*(u_{n-1})$, so we change the labeling of two edges $(v_{\frac{n-5}{2}} v_{\frac{n-3}{2}})$ and $(v_{\frac{n-3}{2}}\ v_{\frac{n-1}{2}}\ )$ as follows $f(v_{\frac{n-5}{2}} v_{\frac{n-3}{2}}) = \dfrac{\delta}{2}(7n-11)$. and $f(v_{\frac{n-3}{2}}\ v_{\frac{n-1}{2}}\ ) = \dfrac{\delta}{2}(7n-13)$.

Then $f^*(v_{\frac{n-5}{2}}) = (n-3)\delta$ and $f^* v_{\frac{n-1}{2}} = (n-1)\,\delta$ Hence the vertex labels are all distinct and a multiple of δ. Therefore $T(P_n)$ admits an edge $\delta-$ graceful labeling. □

Illustration: The total graphs $T(P_{10})$ and $T(P_{11})$ with an edge 6− graceful labeling are shown in Figure 19.

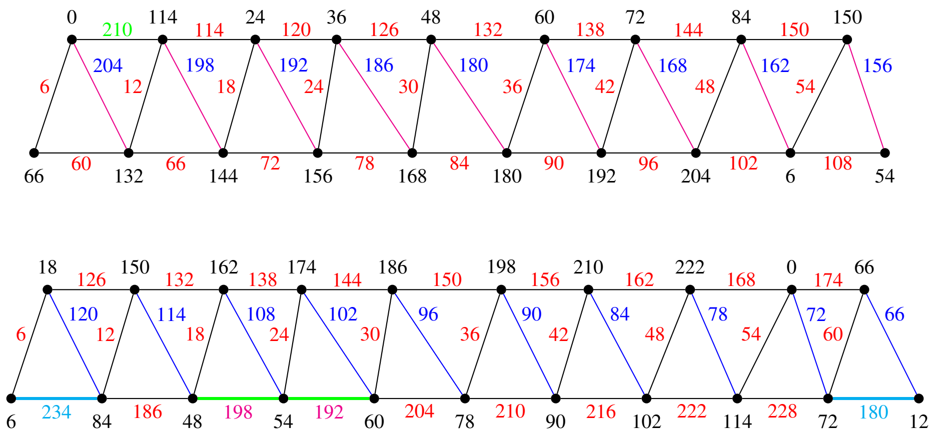

Figure 19. The total graphs $T(P_{10})$ and $T(P_{11})$ with an edge 6− graceful labeling.

5.2. *Edge $\delta-$ graceful labeling of the total graph $T(C_n)$*

Theorem 5.2. *For any positive integer δ, the total graph $T(C_n)$, $n \ge 3$ of the cycle C_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_i,\ \ i = 1, 2,\ \cdots\ ,\ n\}$ be the vertices of the path C_n , the total graph $T(C_n)$ of path C_n has vertices set $V(T(C_n)) = \{v_i,\ u_i\ ,\ i = 1, 2,\ \cdots, n\}$ and $E(T(C_n)) = \{v_iv_{i+1},\ v_iu_i,\ u_iv_{i+1},\ u_iu_{i+1},\ i = 1, 2,\ \cdots\ ,\ n\}$, so the graph $T(C_n)$ has $p = \ 2n$ and $q = 4n$, $k = max(p, q) = 4n$. There are two cases:

Case (1): When n is even, we define the labeling $f : E(T(C_n)) \longrightarrow \{\delta,\ 2\,\delta,\ 3\,\delta,\ \cdots,\ 4n\,\delta\}$ as follows:

$$f(v_i\, v_{i+1}) = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$
$$f(u_i\, v_i) = \delta\,(4n-i), \qquad for \quad 1 \le i \le n,$$
$$f(u_1\, u_n) = \delta\, 4n, \quad f(u_i\, u_{i+1}) = \delta\, i, \qquad for \quad 1 \le i \le n-1,$$
$$f(v_1\, u_n) = \delta\, n, \quad f(u_i\, v_{i+1}) = \delta\,(2n+i), \qquad for \quad 1 \le i \le n-1,$$

In view of the above labeling pattern we have:

$$f^*(v_1) = 0, \quad f^*(u_1) = (2\,n+1)\,\delta, \qquad f^*(u_n) = (\,n-1)\,\delta,$$
$$f^*(v_i) = 2\,\delta\,(i-1), \qquad for \quad 2 \le i \le n,$$
$$f^*(u_i) = \delta\,(2i+2n-1), \qquad for \quad 2 \le i \le n-1.$$

Hence the labels of the vertices $v_2,\ v_3,\ v_4,\ \cdots,\ v_{n-1}\,, v_n$ are $2\,\delta,\ 4\,\delta,\ 6\,\delta,\ \cdots,\ 2\,\delta(n-2),\ 2\,\delta(n-1)$, respectively, and the labels of the vertices $u_2,\ u_3,\ u_4,\ \cdots,\ u_{n-2},\ u_{n-1}$ are $\delta\,(2\,n+3),\ \delta\,(2\,n+5),\ \delta\,(2\,n+7),\ \cdots,\ \delta(4n-5),\ \delta\,(4n-3)$, respectively.

It is easy to see that: $f^*(u_n) = f^*(v_i)$ when $i = \dfrac{n+1}{2}$, but n is even number, so all the labels are distinct numbers.

Case (2): When n is odd, we define the labeling f as in the above case where n is even but we change the labeling of two edges $(u_n v_1)$ and $(v_n v_1)$ as follows $f(u_n v_1) = 2\,n\,\delta$ and $f(v_n v_1) = n\,\delta$.

The induced vertex labels are:

$$f^*(v_1) = [f(v_1\, v_2) + f(v_n\, v_1) + f(u_1\, v_1) + f(u_n\, v_1)]\ mod\ (4n)\,\delta = 0,$$
$$f^*(v_n) = [f(v_n\, v_1) + f(v_{n-1}\, v_n) + f(u_n\, v_n) + f(u_{n-1}\, v_n)]\ mod\ (4n)\,\delta = \delta\ (n-2),$$
$$f^*(u_n) = [f(u_n\, u_1) + f(u_{n-1}\, u_n) + f(u_n\, v_1) + f(u_n\, v_n)]\ mod\ (4n)\,\delta = (\,2n-1)\,\delta.$$

It is easy to see that: $f^*(v_n) = f^*(v_i)$ when $i = \dfrac{n}{2}$, but n is odd number, so all the labels are distinct numbers. □

Illustration: The total graph $T(C_{10})$ of the cycle C_{10} with an edge 5− graceful labeling and the total graph $T(C_9)$ of the cycle C_9 with an edge 6− graceful labeling are shown in Figure 20.

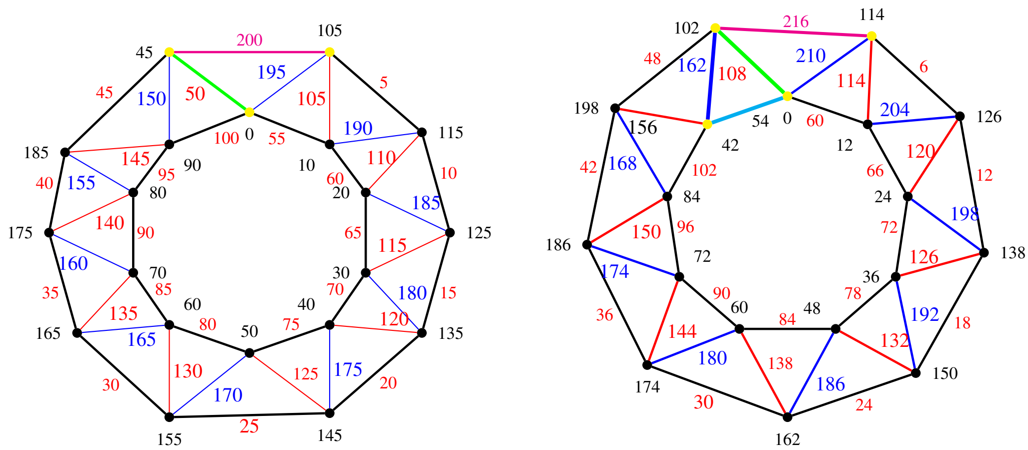

Figure 20. The total graph $T(C_{10})$ with an edge 5− graceful labeling and $T(C_9)$ with an edge 6− graceful labeling.

5.3. Edge $\delta-$ graceful labeling of the total graph $T(Cr_n)$

Theorem 5.3. *For any positive integer δ, the total graph of the crown graph Cr_n is an edge $\delta-$ graceful graph.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ be the vertices of the crown graph Cr_n and $\{v'_1, v'_2, \cdots, v'_n\}$ and $\{u'_1, u'_2, \cdots, u'_n\}$ be the edges of the crown graph Cr_n. Then $V(T(Cr_n)) = V(Cr_n) \cup E(Cr_n)$ and
$E(T(Cr_n)) = \{v_iu'_i,\ v'_iv_{i+1},\ v_iv'_i,\ u_iu'_i,\ u'_iv'_i,\ v'_iu'_{i+1},\ v'_iv'_{i+1},\ v_iu_i,\ v_iv_{i+1};\ 1 \le i \le n\}$, see Figure 21.

Here $p = 4n$ vertices and $q = 9n$ edges, $k = max(p, q) = 9n$.

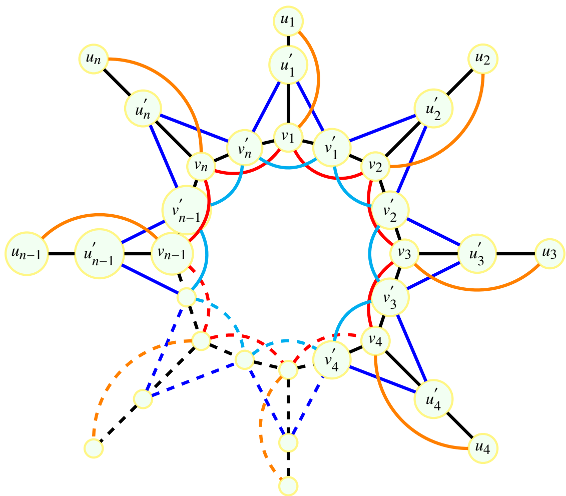

Figure 21. The total graph $T(Cr_n)$ of the crown graph.

Case (1): When n is odd. Define the labeling function $f : E(T(Cr_n)) \longrightarrow \{\delta, 2\delta, \cdots, 9n\delta\}$ as follows:

$$f(v_i u'_i) = \begin{cases} \delta(n+1), & \text{if } i = 1; \\ \delta i, & \text{if } 2 \le i \le n. \end{cases}$$

$$f(u_i u'_i) = \delta(2n + 2 - i), \qquad for \quad 1 \le i \le n,$$

$$f(v'_i u'_i) = \begin{cases} \delta(3n+1), & \text{if } i = 1; \\ \delta(2n+i), & \text{if } 2 \le i \le n. \end{cases}$$

$$f(v_i v'_i) = \begin{cases} \delta, & \text{if } i = 1; \\ \delta(3n+i), & \text{if } 2 \le i \le n. \end{cases}$$

$$f(v'_i v_{i+1}) = \delta(4n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i v_{i+1}) = \delta(5n+i), \qquad for \quad 1 \le i \le n,$$

$$f(v'_i v'_{i+1}) = \begin{cases} \delta(7n-i), & \text{if } 1 \le i \le n-1; \\ \delta(7n), & \text{if } i = n. \end{cases}$$

$$f(v_i'\,u_{i+1}') = \delta\,(7n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\,u_i) = \delta\,(9n+1-i), \qquad for \quad 1 \le i \le n.$$

Then the induced vertex labels are:

$$f^*(v_1) = (8n+3)\,\delta, \qquad f^*(v_i) = (8n+4i-1)\,\delta \ \ mod\,[(9n)\delta], \qquad for \quad 2 \le i \le n,$$
$$f^*(u_1') = (5n+3)\delta, \qquad f^*(u_i') = (2n+2i+1)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(u_i) = (2n+3-2i)\,\delta, \qquad for \quad 1 \le i \le n,$$
$$f^*(v_1') = (n+3)\,\delta, \qquad f^*(v_n') = (6n+1)\,\delta, \qquad f\,(v_i') = (3n+1+2i)\,\delta, \qquad for \quad 2 \le i \le n-1.$$

Case (2): When $n \equiv 2\ mod\ 4$. The labeling function $f : E(T(Cr_n)) \longrightarrow \{\delta,\ 2\,\delta,\ \cdots,\ 9n\,\delta\}$ defined as follows:

$$f(u_i\,u_i') = \delta\,(i), \qquad for \quad 1 \le i \le n,$$
$$f(u_i'\,v_i) = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\,u_i') = \delta\,(2n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\,v_{i+1}) = \delta\,(3n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\,v_{i+1}) = \delta\,(5n+1-i), \qquad for \quad 1 \le i \le n,$$
$$f\,(v_i'\,v_{i+1}') = \delta\,(5n+i+1), \qquad for \quad 1 \le i \le n-1, \qquad f\,(v_n'\,v_1') = \delta\,(5n+1),$$
$$f(v_i\,v_i') = \delta\,(6n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\,u_{i+1}') = \delta\,(7n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\,u_i) = \delta\,(8n+i), \qquad for \quad 1 \le i \le n.$$

Then the induced vertex labels are:

$$f^*(u_i) = (8n+2i)\,\delta\ mod\,[(9n)\delta], \qquad for \quad 1 \le i \le n,$$
$$f^*(u_1') = (2n+3)\delta, \qquad f^*(u_i') = (n+4i-1)\,\delta, \qquad for \quad 2 \le i \le n,$$
$$f^*(v_i) = (8n+4i+1)\,\delta \ \ mod\,[(9n)\delta], \qquad for \quad 1 \le i \le n,$$
$$f^*(v_1') = (3n+6)\,\delta, \qquad f^*(v_n') = (6n+2)\,\delta, \qquad f\,(v_i') = (3n+2+4i)\,\delta, \qquad for \quad 2 \le i \le n-1.$$

These values are distinct for all $n \equiv 2\ mod\ 4$, but

$$f^*(v_i') = (3n+4i+2)\,\delta = f^*(v_n') = (6n+3)\,\delta, \quad \text{when} \quad i = \frac{3n}{4} \ \text{ i.e., when } \ n \equiv 0\ mod\ 4.$$

Case (3): When $n \equiv 0\ mod\ 4$ define the labeling function f as follows:

$$f(u_i\,u_i') = \delta\,(i), \qquad for \quad 1 \le i \le n\,,$$
$$f(u_i'\,v_i) = \delta\,(n+i), \qquad for \quad 1 \le i \le n,$$

$$f\,(v_i\,v_{i+1}) = \begin{cases} \delta\,(3n+i), & \text{if} \quad 1 \le i \le n-1; \\ \delta\,(2n+1), & \text{if} \quad i = n. \end{cases}$$

$$f\,(v_i'\,u_i') = \begin{cases} \delta\,(4n), & \text{if} \quad i = 1; \\ \delta\,(2n+i), & \text{if} \quad 2 \le i \le n; \end{cases}$$

$$f(v_i'\,v_{i+1}) = \delta\,(5n+1-i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\,v_{i+1}') = \delta\,(5n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\,v_i') = \delta\,(6n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i'\,u_{i+1}') = \delta\,(7n+i), \qquad for \quad 1 \le i \le n,$$
$$f(v_i\,u_i) = \delta\,(8n+i), \qquad for \quad 1 \le i \le n.$$

Then the induced vertex labels are:

$f^*(u_i) = (8n+2i)\,\delta\; mod\,[(9n)\delta], \qquad for \quad 1 \le i \le n,$
$f^*(u_1') = (4n+2)\delta, \qquad f^*(u_i') = (n+4i-1)\,\delta, \qquad for \quad 2 \le i \le n,$
$f^*(v_1') = (6n+3)\,\delta, \qquad f\,(v_i') = (3n+4i)\,\delta, \qquad for \quad 2 \le i \le n,$
$f^*(v_1) = (6n+6)\,\delta, \quad f^*(v_n) = (n+2)\,\delta,$
$f^*(v_i) = (8n+4i+1)\,\delta\; mod\,[(9n)\delta], \qquad for \quad 2 \le i \le n-1.$

We notice that, these values are distinct for all $n \equiv 0\; mod\; 4$. It should be notice that:

$f^*(v_i') = (3n+4i)\,\delta = f^*(u_1') = (4n+2)\,\delta,$ when $i = \dfrac{n+2}{4}$ i.e., when $n \equiv 2\; mod\; 4$. Hence there are no repetition in the vertex labels which completes the proof. □

Illustration: The total graphs $T(Cr_8)$ and $T(Cr_9)$ of the crown graph with an edge 3− graceful labeling are shown in Figure 22.

6. Edge $\delta-$ graceful labeling of the twig graph TW_n

Definition 6.1. *The twig graph $TW_n, n \ge 4$ is the graph obtained from path $P_n = \{v_1, v_2, \cdots, v_n\}$ by attaching exactly two pendant edges to each internal vertex of the path P_n.*

Theorem 6.1. *For any positive integer δ, the twig graph TW_n is an edge $\delta-$ graceful graph when n is odd.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the path P_n and the new attaching vertices are $\{u_2, u_3, \cdots, u_{n-1}\}$ and $\{w_2, w_3, \cdots, w_{n-1}\}$. The edges of TW_n are denoted by $\{a_i, b_i, c_i, d_i\}$, where $a_i = \{v_{(\frac{n-4i+3}{2})}\; v_{(\frac{n-4i+5}{2})}\}$, $b_i = \{v_{(\frac{n+4i-3}{2})}\; v_{(\frac{n+4i-1}{2})}\}$, $d_i = \{v_{(\frac{n-4i+1}{2})}\; v_{(\frac{n-4i+3}{2})}\}$ and $c_i = \{v_{(\frac{n+4i-1}{2})}\; v_{(\frac{n+4i+1}{2})}\}$ where,

$$i = \begin{cases} 1, 2, \cdots, \frac{n-1}{4} & \text{if } i \equiv 1\; mod\; 4; \\ 1, 2, \cdots, \frac{n+1}{4} & \text{if } i \equiv 3\; mod\; 4. \end{cases}$$

The graph TW_n has $p = 3n-4$ vertices and $q = 3n-5$ edges, $k = max(p, q) = 3n-4$. see, Figure 23 We define the labeling function $f : E(TW_n) \longrightarrow \{\delta, 2\,\delta, \cdots, (3n-4)\,\delta\}$ as follows:

At the beginning, we determine the middle vertex $v_{\frac{n+1}{2}}$ in the path P_n and we start the labeling from this vertex by using the following algorithm.

First step: Label the edges a_1, b_1, c_1 and d_1 in the following order:

$f(a_1) = \delta$, $f(b_1) = \delta\,(3n-5)$, $f(c_1) = 2\,\delta$ and

$f(d_1) = [f(b_1) - f(a_1)]\; mod\; [\delta\,(3n-4)] = \delta\,(3n-6)$.

Second step: Label the vertices a_i, b_i, c_i and d_i by the following algorithm respectively

(i) $f(a_i) = [f(c_{i-1}) - f(d_{i-1})] mod\; [\delta\,(3n-4)]$, $\quad for \quad i = 2, 3, \cdots, \frac{n-3}{4}$,

(ii) $f(b_i) = (3n-4)\,\delta - f(a_i)$,

(iii) $f(c_i) = [f(a_i) - f(b_i)]\; mod\; [\delta\,(3n-4)]$, $\quad for \quad i = 2, 3, \cdots, \frac{n-7}{4}$,

(iv) $f(d_i) = [f(b_i) - f(a_i)]\; mod\; [\delta\,(3n-4)]$. $\quad for \quad i = 2, 3, \cdots, \frac{n-7}{4}$,

and repeats the second step until we fish the labeling of all edges in the path P_n.

At the end, the edges (v_iu_i) and (v_iw_i), $2 \le i \le n-1$ take any number from the reminder set of the labeling such that $[f(v_iu_i) + f(v_iw_i)]\; mod\; [\delta\,(3n-4)] \equiv 0\; mod\; [\delta\,(3n-4)] \quad for \quad i = 2 \le i \le n-1$.

In view of the above labeling algorithm, the labels of the vertices are,

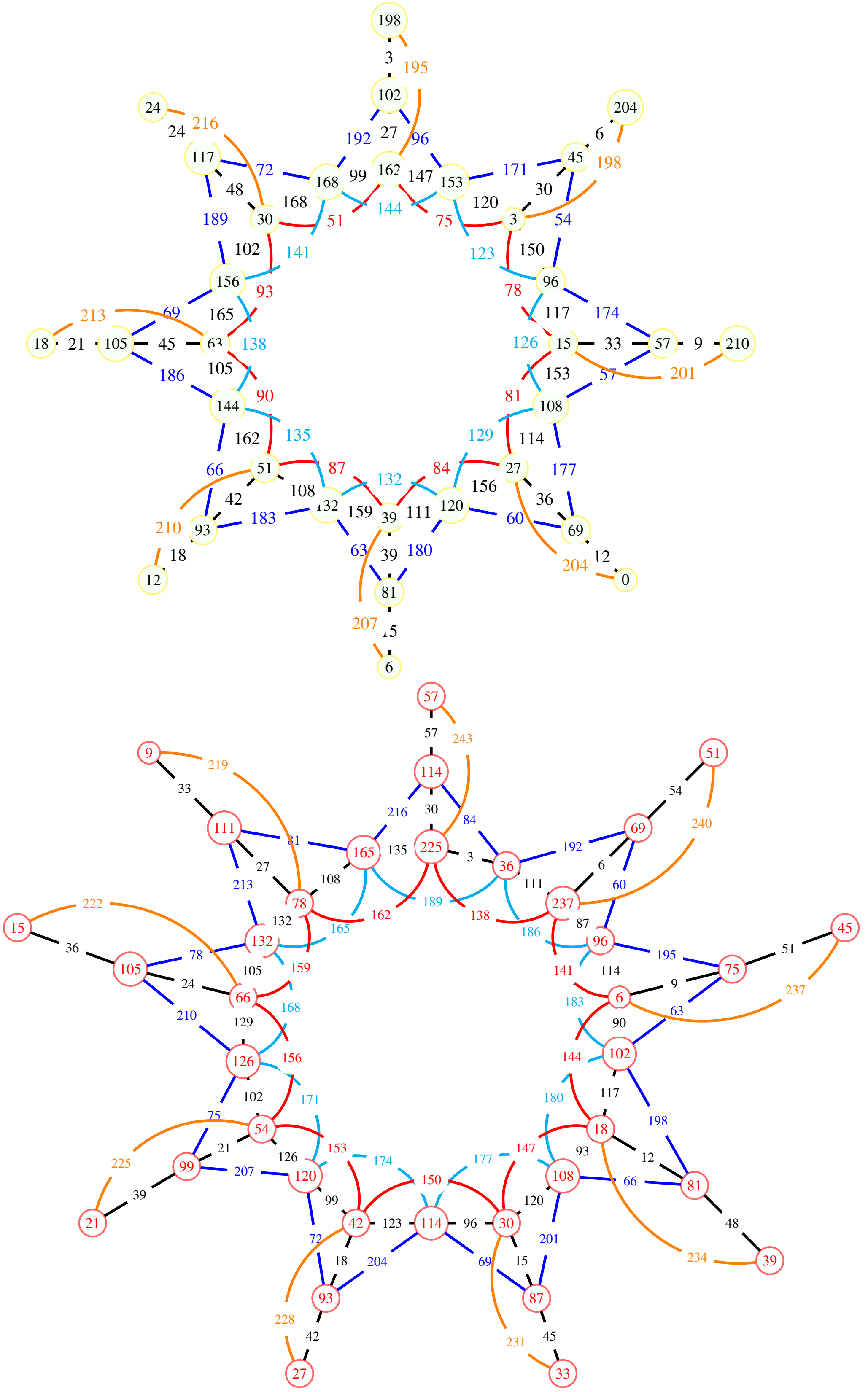

Figure 22. The total graphs $T(Cr_8)$ and $T(Cr_9)$ of the crown graph with an edge 3− graceful labeling.

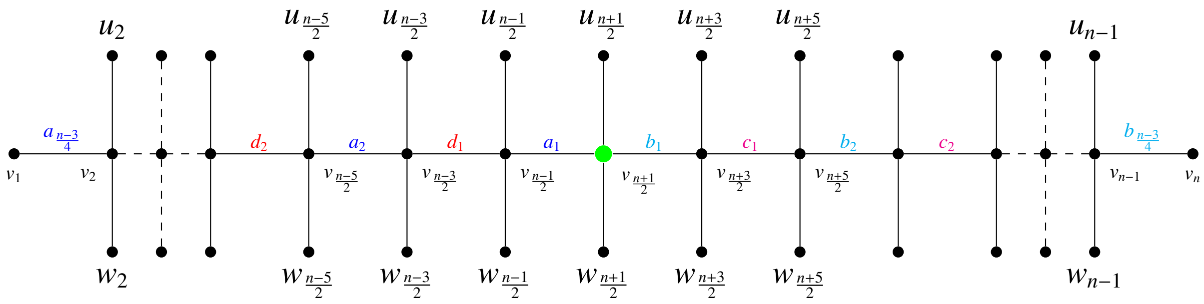

Figure 23. The twig graph TW_n with ordinary labeling.

$$f^*(v_{\frac{n+1}{2}}) = [f(a_1) + f(b_1) + f(u_{\frac{n+1}{2}}) + f(w_{\frac{n+1}{2}})]\ mod\ [\delta\,(3n-4)] = 0,$$

$$f^*(v_{\frac{n-1}{2}}) = f(b_1) = \delta\,(3n-5), \qquad f^*(v_{\frac{n+3}{2}}) = f(a_1) = \delta,$$

$$f^*(v_{\frac{n-3}{2}}) = f(c_1) = 2\delta, \qquad f^*(v_{\frac{n+5}{2}}) = f(d_1) = \delta\,(3n-6),$$

$$f^*(v_{\frac{n-5}{2}}) = f(b_2) = \delta\,(3n-8), \qquad f^*(v_{\frac{n+7}{2}}) = f(a_2) = 4\,\delta,$$

$$f^*(v_{\frac{n+(4i-1)}{2}}) = f(a_i), \quad f^*(v_{\frac{n-(4i-3)}{2}}) = f(b_i), \quad for \quad i = 1, 2, \cdots, \tfrac{n-1}{4},$$

$$f^*(v_{\frac{n+(4i+1)}{2}}) = f(d_i), \quad f^*(v_{\frac{n-(4i-1)}{2}}) = f(c_i), \quad for \quad i = 1, 2, \cdots, \tfrac{n-5}{4}.$$

The pendant vertices v_1 *and* v_n of the path P_n will take the labels of its pendant edges, i.e.,

- If $n \equiv 1\ (mod\ 4)$, then $f^*(v_1) = f(d_{\frac{n-1}{4}})$, *and* $f^*(v_{2n-1}) = f(c_{\frac{n-1}{4}})$.
- If $n \equiv 3\ (mod\ 4)$, then $f^*(v_1) = f(a_{\frac{n+1}{4}})$, *and* $f^*(v_n) = f(b_{\frac{n+1}{4}})$.

It is clear that the labels of the vertices of the path P_n take the labels of the edges of the path and each pendant vertex takes the labels of its incident edge. Then there are no repeated vertex labels, which completes the proof. □

Illustration: The twig graph TW_{13} is presented in Figure 24 with an edge 4− graceful labeling.

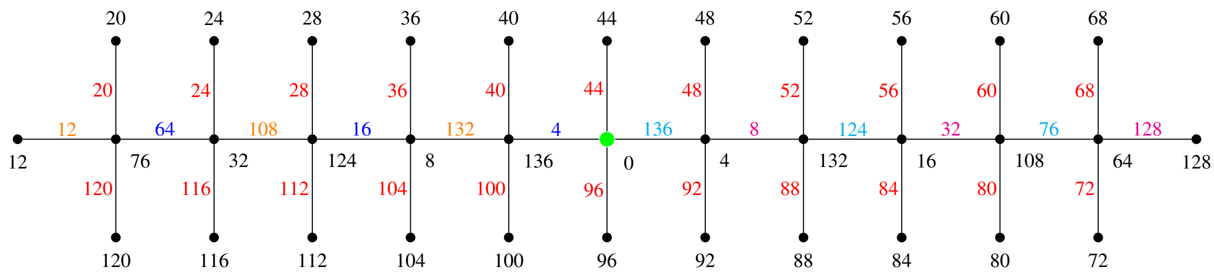

Figure 24. The twig graph TW_{13} with an edge 4- graceful labeling.

7. Edge $\delta-$ graceful labeling of the snail graph S_n

Definition 7.1. *A snail graph* S_n *is obtained from path* $P_n = \{v_1, v_2, \cdots, v_n\}$ *by attaching two parallel edges between* v_i *and* v_{n-i+1} *for* $i = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$.

Theorem 7.1. *For any positive integer* δ*, the snail graph* S_n*,* $n > 2$ *is an edge* $\delta-$ *graceful graph.*

Proof.
Case (1): When n is even, the vertices of the graph S_n are $\{v_0, v_1, v_2, \cdots, v_n\}$ and the edges are $\{e_1, e_1, \cdots, e_{n-1}, a_1, a_2 \cdots, a_{\frac{n}{2}}, b_1, b_2 \cdots, b_{\frac{n}{2}}\}$ where $\{e_i = v_i v_{i+1}, \ i = 1, 2, \cdots, n-1\}$ and $\{a_j = b_j = v_i v_{n-i+1}, \ j = 1, 2, \cdots, \frac{n}{2}\}$, see Figure 25, in this case $p = n$, $q = 2n-1$ and $k = max(p, q) = 2n - 1$.

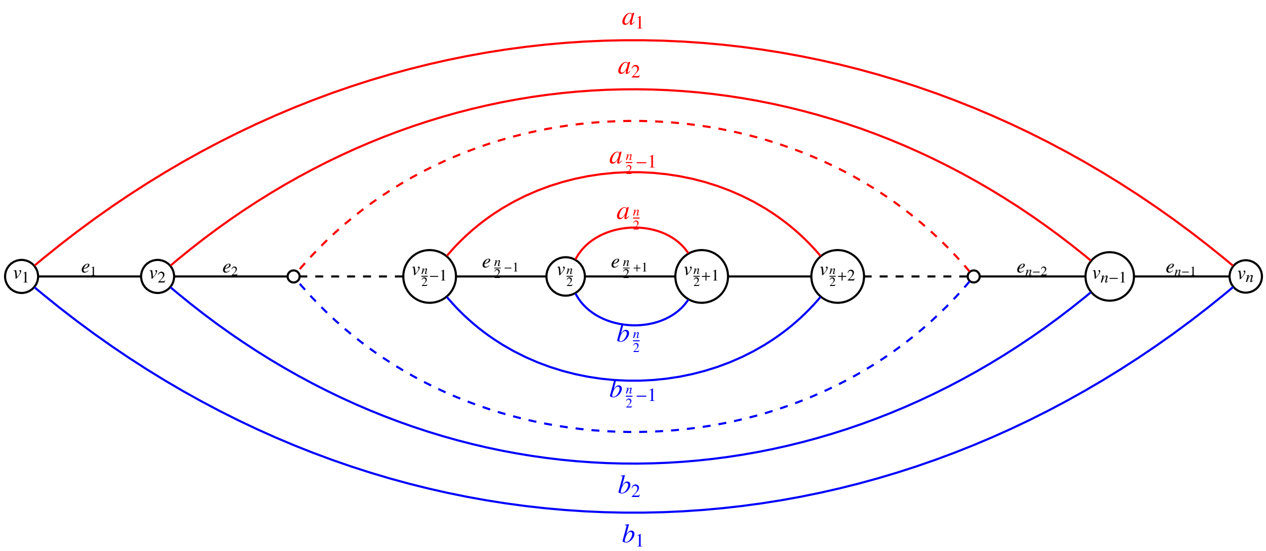

Figure 25. The sinal graph S_n with ordinary labeling when n is even.

We define the labeling function $f : E(S_n)) \longrightarrow \{\delta, 2\delta, \cdots, (2n-1)\delta\}$ as follows:

$$f(a_i) = \delta\, i, \quad for \quad 1 \le i \le \tfrac{n}{2},$$

$$f(b_i) = \delta\,(2n - i - 1), \quad for \quad 1 \le i \le \tfrac{n}{2},$$

$$f(e_i) = \begin{cases} \delta\,(i + \frac{n}{2}), & \text{if } 1 \le i \le \frac{n}{2} - 1; \\ \delta\,(2n-1), & \text{if } i = \frac{n}{2}; \\ \delta\,(\frac{n}{2} + i - 1), & \text{if } \frac{n}{2} + 1 \le i \le n - 1. \end{cases}$$

The induced vertex labels are:

$f^*(v_1) = [\, f(a_1) + f(b_1) + f(e_1)\,]\ mod\ [(2n-1)\,\delta] = \delta[\frac{n}{2} + 1]$,

$f^*(v_n) = [\, f(a_1) + f(b_1) + f(e_{n-1})\,]\ mod\ [(2n-1)\delta] = \delta\,[\frac{3n}{2} - 2]$,

$f^*(v_{\frac{n}{2}}) = \delta\,(n-1)$, $\qquad f^*(v_{\frac{n}{2}+1}) = \delta\, n$.

$$f^*(v_i) = [\ f(a_i)\ + f(b_i)\ + f(e_i)\ + f(e_{i-1})\ ]\ [mod\ (2n-1)\ \delta]$$
$$= \begin{cases} \delta\ (n+2i-1)\ mod\ [(2n-1)\ \delta] & \text{if} \quad 2 \le i \le \frac{n}{2}-1; \\ \delta\ (n+2i-3)\ mod\ [(2n-1)\ \delta] & \text{if} \quad \frac{n}{2}+2 \le i \le n-1. \end{cases}$$

Hence the labels of the vertices $v_2,\ v_3,\ \cdots,\ v_{\frac{n}{2}-1}$ are $\delta\,(n+3),\ \delta\,(n+5),\ \cdots,\ \delta\,(2n-3)$, respectively, and the labels of the vertices $v_{\frac{n}{2}+2},\ v_{\frac{n}{2}+3},\ \cdots,\ v_{n-1}$ are $2\,\delta,\ 4\,\delta,\ \cdots,\ \delta\,(n-4)$, respectively.

Note that S_4 is an edge $\delta-$ graceful graph but not follow this rule, see Figure 26.

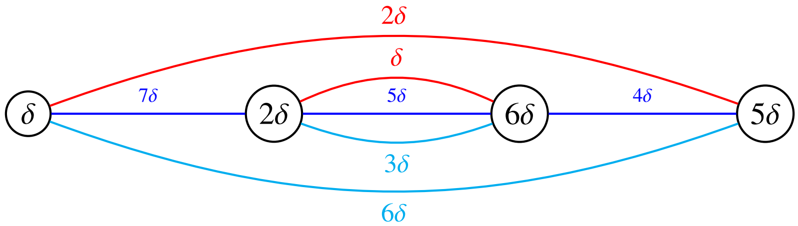

Figure 26. The sinal graph S_4 with an edge $\delta-$ labeling.

Case (2): When n is odd. Let $\{e_1, e_1,\ \cdots,\ e_{n-1}, a_1, a_2,\ \cdots,\ a_{\frac{n-1}{2}}, b_1, b_2\ \cdots,\ b_{\frac{n-1}{2}}\}$ be the edges of S_n which are denoted as in the Figure 27, in this case $p = n,\ q = 2n-2,\ k = max(p, q) = 2n-2.$

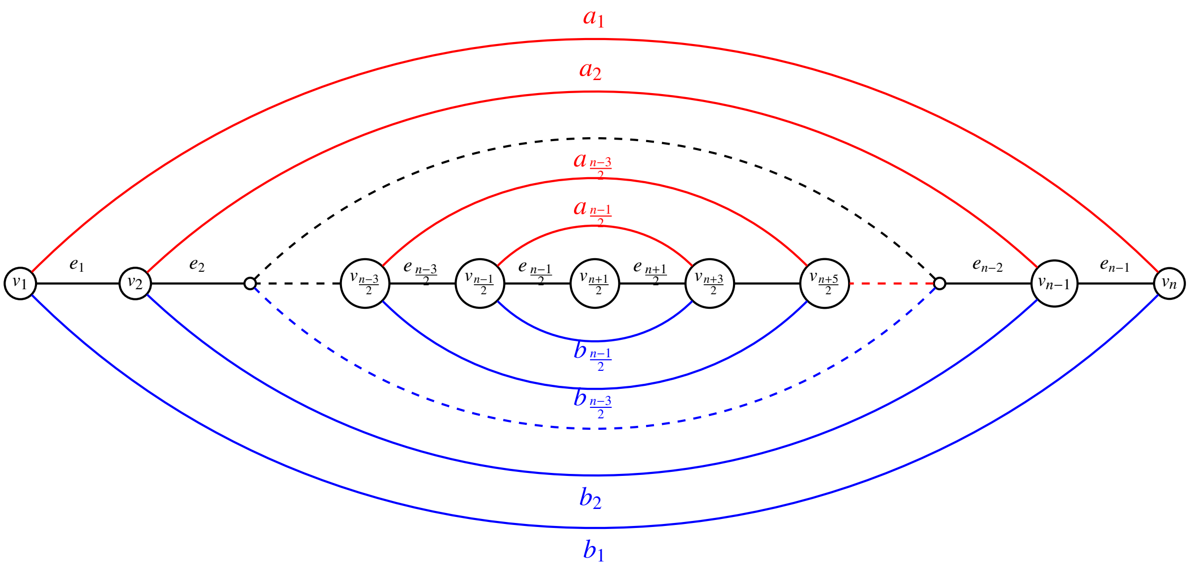

Figure 27. The snail graph S_n with ordinary labeling when n is odd.

Define the labeling function $f : E(S_n)) \longrightarrow \{\delta,\ 2\,\delta, \cdots, (2n-2)\,\delta\}$ as follows:

$$f\ (e_i) = \ i\,\delta, \qquad for \quad 1 \le i \le n-1,$$
$$f(a_i)\ = \delta\ (n+i-1), \qquad for \quad 1 \le i \le \tfrac{n-1}{2},$$
$$f(b_i)\ = \delta\ (2n-i-1), \qquad for \quad 1 \le i \le \tfrac{n-1}{2}.$$

Therefor, the induced vertex labels are

$$f^*(v_1) = \delta\,(n+1)\,, \qquad f^*(v_{\frac{n+1}{2}}) = n\,\delta, \qquad f^*(v_n) = \delta \quad \text{and}$$
$$f^*(v_i) = [\ f(a_i)\ + f(b_i)\ + f(e_i)\ + f(e_{i-1})\ ]\ mod\ [(2n-2)\delta]$$
$$= \delta\,(n+2i-1)\ mod\ [(2n-2)\delta], \quad if \quad i = 2, 3, \ \cdots, \tfrac{n-1}{2}, \tfrac{n+2}{2}, \ \cdots, n-1.$$

Hence the labels of the vertices $v_2, v_3, \ \cdots, \ v_{\frac{n-3}{2}}, v_{\frac{n-1}{2}}$ are $\delta(n+3)$, $\delta\,(n+5)$, $\cdots$, $\delta\,(2n-4)$, 0, respectively, and the labels of the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ \cdots, v_{n-2}, v_{n-1}$ are $4\,\delta$, $6\,\delta$, $\cdots, (n-3)\,\delta$, $(n-1)\,\delta$, respectively. Overall all vertex labels are distinct and a multiple of δ numbers. Hence, the snail graph S_n is an edge $\delta-$ graceful for all n. □

Illustration: In Figure 28, we present a triple graceful labeling of the graph S_{12} and an edge 4 graceful labeling of S_{13}.

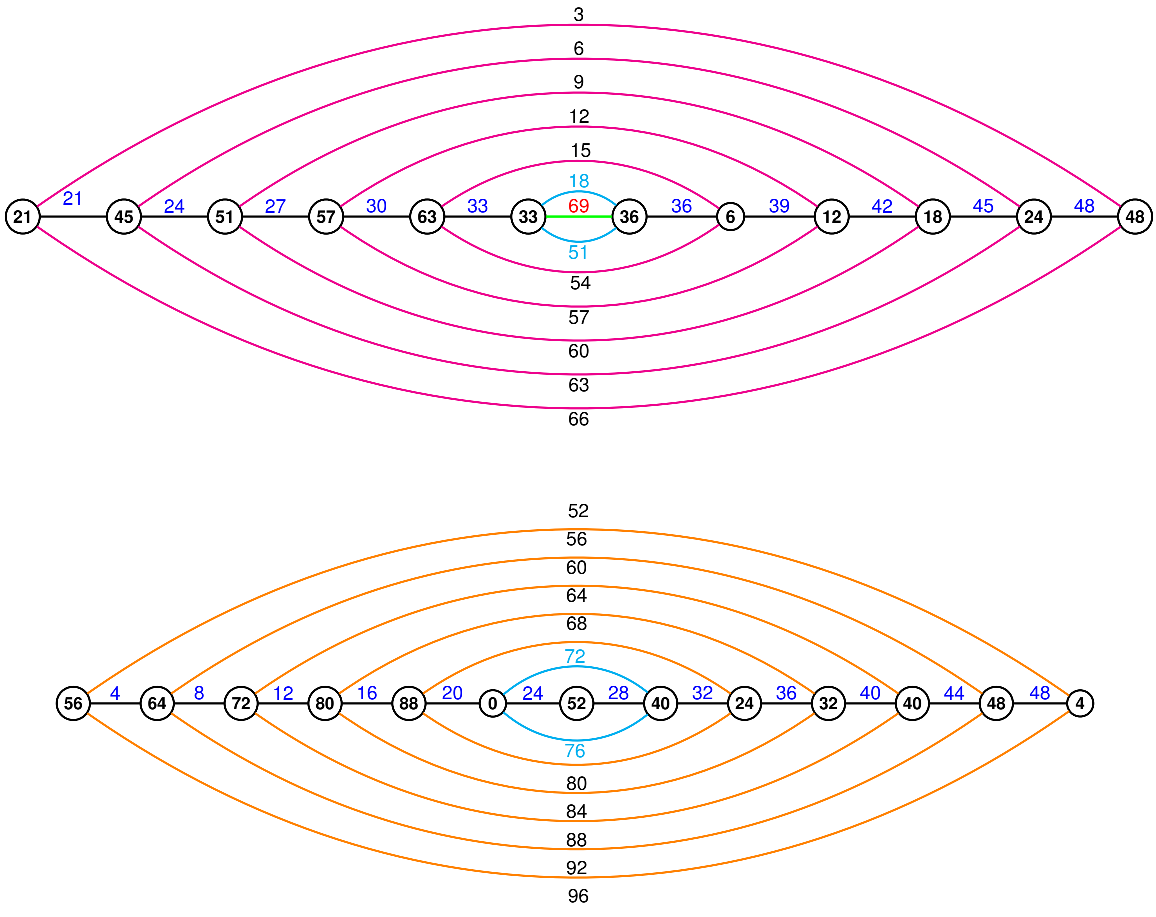

Figure 28. An edge 3− graceful labeling of S_{12} and an edge 4− graceful labeling of S_{13}.

8. Conclusions

Recently, edge graceful labeling of graphs has been studied too much and these objects continue to be an attraction in the field of graph theory and discrete mathematics. A senior number of published papers and results exist. So far, numerous graphs are unknown if it is edge graceful or not.

In this work, we pushed the new type of labeling (edge $\delta-$ graceful labeling), by finding more graphs that have edge $\delta-$ graceful labeling. We prove the existence of an edge $\delta-$ graceful labeling, for any positive integer δ, for the following graphs. The splitting graphs of the cycle, fan, and crown graphs. The shadow graphs of the path, cycle, and fan graphs. The middle graphs and the total graphs of the path, cycle, and crown graphs. Finally, we display the existence of an edge $\delta-$ graceful labeling, for the twig and snail graphs.

Conflict of interest

The authors declare that they have no competing interests.

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