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Research article

Novel higher order iterative schemes based on the $q-$Calculus for solving nonlinear equations

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Abstract: The conventional infinitesimal calculus that concentrates on the idea of navigating the $q-$symmetrical outcomes free from the limits is known as Quantum calculus (or $q-$calculus). It focuses on the logical rationalization of differentiation and integration operations. Quantum calculus arouses interest in the modern era due to its broad range of applications in diversified disciplines of the mathematical sciences. In this paper, we instigate the analysis of Quantum calculus on the iterative methods for solving one-variable nonlinear equations. We introduce the new iterative methods called $q-$iterative methods by employing the $q-$analogue of Taylor's series together with the inclusion of an auxiliary function. We also investigate the convergence order of our newly suggested methods. Multiple numerical examples are utilized to demonstrate the performance of new methods with an acceptable accuracy. In addition, approximate solutions obtained are comparable to the analogous solutions in the classical calculus when the quantum parameter q tends to one. Furthermore, a potential correlation is established by uniting the $q-$iterative methods and traditional iterative methods.

Keywords: Daftardar-Gejji Jafari decomposition technique; iterative schemes; convergence order; Taylor's series in the $q-$calculus
Mathematics Subject Classification: 65Yxx, 65Hxx, 65Dxx

1. Introduction

It is widely acknowledged that the study of the nonlinear equations characterized as $f(\tilde{u}) = 0$ can be used to investigate a broader range of problems that occur in physical sciences. Because of its significance, many scientists have investigated numerous different order multistep methods to explore the solutions of the nonlinear equations using diversified approaches; such as variational iterative methods, homotopy perturbation method, homotopy analysis method, and the decomposition techniques, for details see [1–9]. These developed approaches are of varying order of convergence. Firstly, Traub [16] has initiated the study of repetitious schemes for solving nonlinear equations who developed a central quadratic convergent Newton iterative method which has much importance in literature.

$$\tilde{u}_{k+1} = \tilde{u}_k - \frac{f(\tilde{u}_k)}{f'(\tilde{u}_k)}, \qquad k = 1, 2, 3, \ldots, \ f'(\tilde{u}_k) \neq 0. \tag{1.1}$$

Later on, to increase the practical usefulness and efficiency index of Newton's method, its various rectifications have been presented by many researchers (see [17–20]). Daftardar-Gejji and Jafari [21] have proposed a straightforward approach that does not necessitate the derivative evaluation of the Adomian polynomial by making different modifications in the Adomian decomposition method [1]. Moreover, this technique helps us to write the nonlinear equation as a combination of both linear and nonlinear components, and plays a remarkable role in developing different iterative schemes to estimate the solution of the nonlinear equations. Saqib and Iqbal [22] have determined the fourth and fifth-order convergent iterative methods for computing roots of the nonlinear equations by using a modified decomposition approach and presented some test examples to check the efficacy and performance of the newly established iterative methods. Ali et al. [23] have established a new class of the iterative methods by implementing the technique [21] and testified the validity of these schemes by considering some mathematical models. Alharbi et al. [2] have introduced some new and efficient iterative methods and implemented a decomposition technique along with an auxiliary function. Variational iteration method (VIM) is another effective tool that is employed to develop effective iterative methods for getting approximate, converging solutions of the nonlinear equations. Based on VIM, Naseem et al. [17] have investigated a new class of iterative methods that are superior in convergence and efficient as compared to other methods. They also elaborated the behavior and dynamical aspects of the suggested iterative schemes by using polynomiographs. The $q-$calculus is materialized as the composition of Physics and Mathematics in the last twenty-five years of the XX century (see [24–27, 43]). Many researchers have been designated considerable thought due to its diversified choice of the utilization in mathematical spheres; such as mechanics, theory of relativity, basic hypergeometric function, quantum, and number theory. Firstly, the $q-$analogue of derivative and the $q-$Taylor's formula were introduced by Jackson [29]. Then, by using the differentiation technique in the $q-$calculus, Jing and Fan [30] have presented some modifications in Taylor's formula together with its remainder in the sphere of the $q-$calculus. They compared the q-Taylor's formula and the ordinary Taylor's formula and found signified results on the $q-$remainder.

Ernst [31] has investigated some novel modifications in the $q-$Taylor's formula with the help of $q-$ integration by parts and developed its varient formulations. Firstly, some novel recursive schemes under the $q-$analysis were suggested and analyzed by Singh et al. [32] and introduced some varied forms of the $q-$iterative schemes by opting several values of the $q-$parameter. The stability and reliability of the $q-$iterative methods are checked by presenting comparative analysis of several nonlinear algebraic equations with some classical methods. The $q-$difference equation plays a vital role in the realm of the $q-$calculus. For solving the partial differential equations in the $q-$calculus, Jafari et al. [33] have applied an iterative method called the Daftardar-Jafari decomposition method. It is demonstrated that the proposed procedure's computational outcome converges to the true solution of the $q-$difference equations subject to specific constraints. The study of $q-$integro differential equation with three criteria was investigated by Abdeljawad and Samei [12] and checked its solution existence by applying the $q-$calculus. Sadik and Orie [11] have introduced a convenient and efficient method based on $q-$calculus known as q-differential transform method for solving partial q-differential equations. The solution obtained by this method is expressed in terms of convergent power series and the validity of this method is checked by computing several examples. Liang and Samei [13] have determined the existence of solutions for non-linear problems regular and singular fractional $q-$differential equation subject to certain constraints. They have presented some results with the support of numerical examples and by applying definitions of the fractional q-derivative of Riemann–Liouville & Caputo type. Many real-life problems can be modeled in the form of $q-$fuzzy differential equations. Noeiaghdam et al. [15] have introduced two fuzzy numerical methods based on the generalized Hukuhara $q-$differentiability named as the fuzzy $q-$Euler's and the local $q-$Taylor's expansion method for solving $q-$fuzzy initial value differential equations. In an attempt of transformation of the classical results towards the $q-$calculus considered by Srivastava et al. [45], the two subclasses of normalized analytic functions are investigated by using various operators of $q-$calculus and fractional $q-$calculus in the complex $z-$plane. Sana et al. [10] have transformed the classical iterative methods over the $q-$iterative methods and presented a comparative analysis of these methods with the classical methods. They also presented the generalized formulation of new methods and test their reliability, effectiveness & convergence speed via various numerical examples.

Motivated and inspired by the research going on in this direction, we have restructured some new multistep iterative methods for computing zeros of the nonlinear equations in the context of the $q-$calculus. First, we find some new $q-$analogues of the iterative methods initiated and advanced by Shah and Noor [9]. Then, to obtain the needed results, we rephrase the supposed nonlinear equation accompanied by an auxiliary function and apply the $q-$Taylor's formula. For the best implementation of the results and the derivations of recursive schemes, we utilize the decomposition methodology [21] under the $q-$paradigm. It is essential to mention that the new suggested algorithms can reduce the number of computing costs compared to conventional iterative methods while good numerical accuracy is maintained by appropriately choosing the parameter $q \in [0, 1]$. Now, we recollect some of the basic ideas in the $q-$analysis [34] that are prerequisites and reinforce the construction of our novel $q-$iterative schemes for computing solutions of the nonlinear equations. Let the $q-$integer, for $q \in(0,1)$ is described such as:

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1} = \frac{1-q^m}{1-q} \qquad \text{for } m = 1, 2, \dots , \tag{1.2}$$

$$[m]_q = m \quad \text{for} \quad q = 1. \tag{1.3}$$

For $0 \leq p \leq m$, the $q-$factorial and the $q-$binomials are defined as:

$$[m]_q! = [m]_q[m-1]_q \cdots [1]_q \qquad [0]_q! = 1, \qquad \begin{bmatrix} m \\ p \end{bmatrix}_q = \frac{[m]_q!}{[p]_q![m-p]_q!}. \tag{1.4}$$

Definition 1 (see [34])**.** *A $q-$analogue of classical exponential function $e_q^{\tilde{u}}$ is defined as*

$$\sum_{k=0}^{\infty} \frac{\tilde{u}^k}{[k]!}. \tag{1.5}$$

The derivative of the classical exponential function remains unchanged under differentiation. The $q-$analogue of exponential function also remains the same in the $q-$calculus such as:

$$D_q e_q^{\tilde{u}} = \sum_{k=0}^{\infty} \frac{D_q \tilde{u}^k}{[k]!} = \sum_{k=1}^{\infty} \frac{[k]\tilde{u}^{k-1}}{[k]!} = \sum_{k=1}^{\infty} \frac{\tilde{u}^{k-1}}{[k-1]!} = \sum_{k=0}^{\infty} \frac{\tilde{u}^k}{[k]!} = e_q^{\tilde{u}}. \tag{1.6}$$

Definition 2 (see [34])**.** *Let $f(\tilde{u})$ is a real valued continuous function and its $q-$derivative is prescribed as follows:*

$$D_q f(\tilde{u}) = \left(\frac{d}{d\tilde{u}}\right)_q f(u) = \frac{f(q\tilde{u}) - f(\tilde{u})}{q\tilde{u} - \tilde{u}}, \qquad q \neq 1. \tag{1.7}$$

where $(D_q f)(\tilde{u})$ represents $q-$derivative is known as Jackson derivative. It reduces to the standard derivative when q approaches to one. The $q-$derivative with higher-order for the function $f(\tilde{u})$ is prescribed as:

$$D_q^0 f = f, \qquad D_q^m f = D_q(D_q^{m-1})f, \qquad \text{for } m = 1, 2, 3 \ldots . \tag{1.8}$$

Definition 3 (see [34])**.** *The $q-$derivative of product and quotient of function $f(\tilde{u})$ and $g(\tilde{u})$ is defined as follows:*

$$D_q(f(\tilde{u})g(\tilde{u})) = g(\tilde{u})D_q f(\tilde{u}) + f(q\tilde{u})D_q g(\tilde{u}) = g(q\tilde{u})D_q f(\tilde{u}) + f(\tilde{u})D_q g(\tilde{u}),$$
$$D_q\left(\frac{f(\tilde{u})}{g(\tilde{u})}\right) = \frac{g(\tilde{u})D_q h(\tilde{u}) - f(\tilde{u})D_q g(\tilde{u})}{g(q\tilde{u})g(\tilde{u})} \quad \text{such that } g(\tilde{u})g(q\tilde{u}) \neq 0.$$

Definition 4 (see [29–31])**.** *Let $f(\tilde{u})$ is a continuous function defined on an interval (k, l) and $c \in [k, l]$. Then, the $q-$Taylor's formula for the function $f(\tilde{u})$ instigated by Jackson is explained as:*

$$f(\tilde{u}) = \sum_{m=1}^{\infty} \frac{D_q^m (\tilde{u} - c)^k}{[k]!} \quad (\forall\ \tilde{u} \in (k, l)), \tag{1.9}$$

where

$$(\tilde{u} - c)^0 = 1, \qquad (\tilde{u} - c)^k = \prod_{i=0}^{m-1} (\tilde{u} - cq)^i, \qquad m \in N, \tag{1.10}$$

$D_q, D_q^2, \ldots$ are all $q-$derivatives, where $0 < q < 1$.

2. Evolvement of the iterative schemes

This section deals with the construction of some novel iterative schemes by employing Taylor's formula and Daftardar-Jafari decomposition technique [21] in the paradigm of the quantum calculus.

2.1. Main results

We consider the nonlinear algebraic equation of the general form:

$$f(\tilde{u}) = 0, \qquad \tilde{u} \in R. \tag{2.1}$$

Let $g(\tilde{u})$ be an auxiliary function. Suppose κ is an initial guess in the neighbourhood of β which is the simple root of nonlinear equation (2.1).

$$f(\tilde{u})g(\tilde{u}) = 0, \qquad \tilde{u} \in R. \tag{2.2}$$

Using $q-$Taylor's series about κ and the technique of He [28], we rewrite the nonlinear equation (2.2) as a parallel coupled system of the equation :

$$f(\kappa)g(\kappa) + (\tilde{u} - \kappa)\Big\{D_q f(\kappa)g(\kappa) + D_q g(v)f(q\kappa)\Big\} + G(\tilde{u}) = 0, \tag{2.3}$$

$$G(\tilde{u}) = f(\tilde{u})g(\kappa) - f(\kappa)g(\kappa) - (\tilde{u} - \kappa)\Big\{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)\Big\}. \tag{2.4}$$

Eq (2.4) can be rewritten as:

$$\tilde{u} = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} - \frac{G(\tilde{u})}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}, \tag{2.5}$$

$$\tilde{u} = c + N_q(\tilde{u}), \tag{2.6}$$

where

$$c := \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}, \tag{2.7}$$

and

$$N_q(\tilde{u}) := -\frac{G(\tilde{u})}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}. \tag{2.8}$$

The term $N_q(\tilde{u})$ is treated as nonlinear and c as a constant.

Let $\tilde{u}_0$ be an initial guess then from relation (2.4), we can easily compute a key equation that is helpful in the development of new $q-$iterative methods :

$$G(\tilde{u}_0) = f(\tilde{u}_0)g(\kappa). \tag{2.9}$$

We now carry out a decomposition technique primarily due to Daftardar-Gejji and Jafari [21], known as the Daftardar-Jafari decomposition technique, to set up arrangements of higher-order iterative methods. The central idea behind using this methodology is to seek the solution of the $q-$basic equation (2.6) in the series form.

$$\tilde{u} = \sum_{k=0}^{\infty} \tilde{u}_k. \tag{2.10}$$

Now, we deteriorate the nonlinear operator $N_q(\tilde{u})$ which is defined in (2.8) such as:

$$N_q(\tilde{u}) = N_q(\tilde{u}_0) + \sum_{k=1}^{\infty}\left\{N_q\left(\sum_{l=0}^{k}\tilde{u}_l\right) - N_q\left(\sum_{l=0}^{k-1}\tilde{u}_l\right)\right\}. \tag{2.11}$$

From the equations (2.6), (2.10) and (2.11), we have

$$\sum_{k=0}^{\infty}\tilde{u}_k = c + N_q(\tilde{u}_0) + \sum_{k=1}^{\infty}\left\{N_q\left(\sum_{l=0}^{k}\tilde{u}_l\right) - N_q\left(\sum_{l=0}^{k-1}\tilde{u}_l\right)\right\}, \tag{2.12}$$

finally, we obtain the following iterative procedure:

$$\begin{cases} \tilde{u}_0 & = c, \\ \tilde{u}_1 & = N_q(\tilde{u}_0), \\ \tilde{u}_2 & = N_q(\tilde{u}_0 + \tilde{u}_1) - N_q(\tilde{u}_0), \\ & \vdots \\ u_{n+1} & = N_q\left(\sum_{l=0}^{n}\tilde{u}_l\right) - N_q\left(\sum_{l=0}^{n-1}\tilde{u}_l\right), \qquad n = 1, 2, \dots. \end{cases} \tag{2.13}$$

It follows that

$$\tilde{u}_1 + \tilde{u}_2 + \cdots + \tilde{u}_{n+1} = N_q(\tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \cdots + \tilde{u}_n),$$

and

$$\tilde{u} = c + \sum_{k=1}^{\infty}\tilde{u}_k. \tag{2.14}$$

Note that $\tilde{u}$ is approximated by

$$\tilde{u}_n = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \cdots + \tilde{u}_n,$$

and thus $\lim\limits_{x\to\infty}\tilde{u}_n = \tilde{u}$.

Theorem 2.1 (see [33])**.** *If N_q is a contraction, then the series specified in* (2.10) *is absolutely convergent.*

Proof. Let N_q is a contraction mapping, then by definition we can write:

$$\left\|N_q(\tilde{u}) - N_q(v)\right\| \le \beta\|\tilde{u} - v\| \qquad 0 < \beta < 1, \tag{2.15}$$

then in view of (2.13), we have

$$\|u_{n+1}\| = \left\|N_q(\tilde{u}_1 + \tilde{u}_2 + ... + \tilde{u}_n) - N_q(\tilde{u}_1 + \tilde{u}_2 + ... + \tilde{u}_{n-1})\right\| \le \beta\|\tilde{u}_n\| \le \beta^n\|\tilde{u}_0\|$$
$$n = 0, 1, 2, \dots,$$

then the series $\tilde{u}=\sum_{k=0}^{\infty}\tilde{u}_k$ is uniformly and absolutely convergent to an answer of the equation (2.6) (see [36]).
This completes the proof. □

Now, we construct the following iterative schemes to find the solution of the nonlinear algebraic equation (2.1)

Algorithm A: From (2.13), we have for $n = 0$:

$$x \approx \tilde{U}_0 = \tilde{u}_0 = c = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}. \tag{2.16}$$

This composition permits us to put forward the subsequent recursive approach for solving the nonlinear equation (2.1), and the iterative schema computes the approximate solution $\tilde{u}_{n+1}$ for a given starting guess $\tilde{u}_0$:

$$u_{n+1} = \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n)f(q\tilde{u}_n)}, \qquad n = 0, 1, \dots. \tag{2.17}$$

This represents main $q-$analogue of iterative scheme which is prospected by He [28] and Shah [9]. This main iterative scheme helps us to generate different $q-$algorithms for solving the nonlinear equation (2.1). Now, with the help of (2.4) and (2.13), we get:

$$\tilde{u}_1 = N_q(\tilde{u}_0) = -\frac{G(\tilde{u}_0)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} = -\frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}. \tag{2.18}$$

Algorithm B: From (2.13), we have for $n = 1$:

$$\tilde{u} \approx \tilde{U}_1 = \tilde{u}_0 + \tilde{u}_1 = \tilde{u}_0 + N_q(\tilde{u}_0). \tag{2.19}$$

By using (2.16) and (2.18), we have

$$\tilde{u} = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} - \frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}. \tag{2.20}$$

This composition permits us to put forward the subsequent iterative approach for solving the nonlinear equation (2.1), and the iterative schema computes the approximate solution $\tilde{u}_{n+1}$ for a given starting guess $\tilde{u}_0$:

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n)f(q\tilde{u}_n)}, \tag{2.21}$$

$$\tilde{u}_{n+1} = \tilde{v}_n - \frac{f(\tilde{v}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n)f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \dots. \tag{2.22}$$

This is $q-$analogue of *Algorithm 2.2* which is investigated by [9] and the error equation of Algorithm B is determined in Theorem 3.1.
By using (2.4), (2.13) and (2.18), we can obtain

$$\tilde{u}_1 + \tilde{u}_2 = N_q(\tilde{u}_0 + \tilde{u}_1) = -\frac{G(\tilde{u}_0 + \tilde{u}_1)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}, \tag{2.23}$$

$$= -\frac{f(\tilde{u}_0 + \tilde{u}_1)g(\kappa) - f(\kappa)g(\kappa) - (\tilde{u}_0 + \tilde{u}_1 - \kappa)\Big\{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)\Big\}}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},$$

$$= -\frac{f(\tilde{u}_0 + \tilde{u}_1)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} - \frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}. \tag{2.24}$$

Algorithm C: Now, considering from (2.13), we have for $n = 2$:

$$\begin{aligned} x \approx \tilde{U}_2 = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 &= \tilde{u}_0 + N_q(\tilde{u}_0) + N_q(\tilde{u}_0 + \tilde{u}_1) - N_q(\tilde{u}_0) \\ &= \tilde{u}_o + N_q(\tilde{u}_0 + \tilde{u}_1). \end{aligned} \tag{2.25}$$

By using (2.16) and (2.23), we get

$$\tilde{u} = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa) f(q\kappa)} - \frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa) f(q\kappa)} - \frac{f(\tilde{u}_0 + \tilde{u}_1)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa) f(q\kappa)}. \tag{2.26}$$

This composition permits us to put forward the subsequent iterative approach for solving the nonlinear equation (2.1), and the iterative schema computes the approximate solution $\tilde{u}_{n+1}$ for a given starting guess $\tilde{u}_0$:

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n) f(q\tilde{u}_n)}, \tag{2.27}$$

$$\tilde{w}_n = \tilde{v}_n - \frac{f(\tilde{v}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n) f(q\tilde{u}_n)}, \tag{2.28}$$

$$\tilde{u}_{n+1} = \tilde{w}_n - \frac{f(\tilde{w}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n) f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \ldots. \tag{2.29}$$

This is $q-$analogue of *Algorithm 2.3* which is investigated by [9] and the error equation of Algorithm C is determined in Theorem 3.1.

Algorithm A, **Algorithm B**, **Algorithm C** are the main and general iterative schemes that are used to generate some new algorithms by considering different choices of auxiliary functions that are the main attractiveness of modification of this technique. To convey the idea, we consider the following auxiliary functions.

Case : Let $g(\tilde{u}_n) = e^{-\beta\tilde{u}_n}$ and $D_q g(\tilde{u}_n) = -\beta e^{-\beta\tilde{u}_n}$. Using these values we obtain the following iterative methods for the solving nonlinear equations.

Algorithm D: For a given $\tilde{u}_0$ (initial guess), approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$u_{n+1} = \tilde{u}_n - \frac{f(\tilde{u}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \qquad n = 0, 1, \ldots . \tag{2.30}$$

Algorithm E: For a given $\tilde{u}_0$ (initial guess), approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \tag{2.31}$$

$$\tilde{u}_{n+1} = \tilde{v}_n - \frac{f(\tilde{v}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \ldots . \tag{2.32}$$

Algorithm F: For a given $\tilde{u}_0$ (initial guess), approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \tag{2.33}$$

$$\tilde{w}_n = \tilde{v}_n - \frac{f(\tilde{v}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \tag{2.34}$$

$$\tilde{u}_{n+1} = \tilde{w}_n - \frac{f(\tilde{w}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n) - \beta f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \dots . \tag{2.35}$$

To the best of our knowledge, the new schemes Algorithm D, Algorithm E, Algorithm F appear to be new ones.

3. Convergence analysis

In this part, the order of convergence of the primary $q-$iterative methods made out by **Algorithm A**, **Algorithm B**, and **Algorithm C** is investigated. In the same approach, the rest of the iterative procedures can be established.

Theorem 3.1. *Let $f : E \subset R \to R$ be a differentiable function, where E is an open interval in R. If $\tilde{u}_0$ is sufficiently close to $\beta \in E$ which is the root of $f(\tilde{u}) = 0$ then the iterative methods Algorithm A, Algorithm B and Algorithm C are convergent algorithms of order at least 2, 3, 4 respectively and we format it as follows:* $[2; q]$, $[3; q]$ *and* $[4; q]$, *where parameter q corresponds to the quantum calculus. Error equations for these newly established algorithms are given as:*

$$e_{n+1} = \left\{\frac{qD_q g(\beta)}{g(\beta)} + c_2\right\} e_n^2 + O(e_n^3),$$

$$e_{n+1} = \left\{\frac{3qc_2 D_q g(\beta)}{g(\beta)} + \frac{q^2 D_q g(\beta)^2}{g(\beta)^2} + 2c_2^2\right\} e_n^3 + O(e_n^4),$$

$$e_{n+1} = \left\{\frac{5q^2 c_2 D_q g(\beta)^2}{g(\beta)^2} + \frac{8qc_2^2 D_q g(\beta)}{g(\beta)} + \frac{q^3 (D_q g(\beta))^3}{g(\beta)^3} + 4c_2^3\right\} e_n^4 + O(e_n^5).$$

Proof. Let f is adequately differentiable function and β is root of $f(\tilde{u})$. Now, expanding $f(\tilde{u}_n)$ and $D_q f(\tilde{u}_n)$ in the $q-$Taylor's series about β we obtain

$$f(\tilde{u}_n) = D_q f(\beta) e_n + \frac{1}{[2]!} D_q^2 f(\beta) e_n^2 + \frac{1}{[3]!} D_q^3 f(\beta) e_n^3 + \cdots , \tag{3.1}$$

$$f(\tilde{u}_n) = D_q f(\beta)\Big\{e_n + b_2 e_n^2 + b_3 e_n^3 + b_4 e_n^4 + \cdots\Big\}, \tag{3.2}$$

$$D_q f(\tilde{u}_n) = D_q f(\beta)\Big\{1 + 2b_2 e_n + 3b_3 e_n^2 + 4b_4 e_n^3 + \cdots\Big\}, \tag{3.3}$$

By expanding $g(\tilde{u}_n)$, $D_q g(\tilde{u}_n)$ in the $q-$Taylor's series, we obtain

$$g(\tilde{u}_n) = g(\beta) + D_q g(\beta) e_n + \frac{e_n^2}{[2]!} D_q^2 g(\beta) + \frac{e_n^3}{[3]!} D_q^3 g(\beta) + \dots, \tag{3.4}$$

$$D_q g(\tilde{u}_n) = D_q g(\beta) + D_q^2 g(\beta) e_n + \frac{e_n^2}{[2]!} D_q^3 g(\beta) + \frac{e_n^3}{[3]!} D_q^{(iv)} g(\beta) + ..., \tag{3.5}$$

where

$$b_k = \frac{D_q^k f(\beta)}{[k]! D_q f(\beta)}, \quad \text{for } k = 2, 3, \ldots \quad \text{and} \quad e_n = \tilde{u}_n - \beta \tag{3.6}$$

By expanding $f(\tilde{u}_n)g(\tilde{u}_n)$, $f(\tilde{u}_n)D_q g(\tilde{u}_n)$, $D_q f(\tilde{u}_n)g(\tilde{u}_n)$, in the $q-$Taylors series about β, we obtain

$$f(\tilde{u}_n)g(\tilde{u}_n) = D_q f(\beta)\Big\{g(\beta)e_n + \Big(D_q g(\beta) + c_2 g(\beta)\Big)e_n^2 + \left(\frac{D_q^2 g(\beta)}{q+1} + c_2 D_q g(\beta) + c_3 g(\beta)\right)e_n^3 + O(e_n^4)\Big\}, \tag{3.7}$$

$$D_q f(\tilde{u}_n)g(\tilde{u}_n) = D_q f(\beta)\Big\{g(\beta) + \Big(D_q g(\beta) + 2c_2 g(\beta)\Big)e_n + \left(\frac{D_q^2 g(\beta) + 2c_2 D_q g(\beta) + 3c_3 g(\beta)}{q+1}\right)e_n^2 + O(e_n^3)\Big\}. \tag{3.8}$$

$$f(q\tilde{u}_n)D_q g(\tilde{u}_n) = D_q f(\beta)\Big\{qD_q g(\beta)e_n + \Big(qD_q^2 g(\beta) + qc_2 D_q g(\beta)\Big)e_n^2 + \left(\frac{qD_q^3 g(\beta)}{q+1} + qc_2 D_q^2 g(\beta) + qc_3 D_q g(\beta)\right)e_n^3 + O(e_n^4)\Big\}, \tag{3.9}$$

From (3.7), (3.9), (3.8), we get

$$\frac{f(\tilde{u}_n)g(\tilde{u}_n)}{D_q f(\tilde{u}_n)g(\tilde{u}_n) + D_q g(\tilde{u}_n)f(q\tilde{u}_n)} = e_n - \Big(\frac{qD_q g(\beta)}{g(\beta)} + c_2\Big)e_n^2 + \Big\{\frac{2c_2 D_q g(\beta)}{g(\beta)} + c_3 - \frac{qD_q^2 g(\beta)}{g(\beta)} + \frac{2qc_2 D_q g(\beta)}{g(\beta)} - \frac{2qc_2 D_q g(\beta)}{(q+1)g(\beta)} - \frac{2c_2 D_q g(\beta)}{(q+1)g(\beta)} + 2c_2^2 - \frac{qD_q g(\beta)^2}{g(\beta)^2} + \frac{q^2 D_q g(\beta)^2}{g(\beta)^2} - 3c_3\Big\}e_n^3 + O(e_n^4). \tag{3.10}$$

Now, using (3.10) into (2.17), we get the error term of the *Algorithm A:*

$$\tilde{u}_{n+1} = \beta + \Big(\frac{qD_q g(\beta)}{g(\beta)} + c_2\Big)e_n^2 - \Big\{\frac{2c_2 D_q g(\beta)}{g(\beta)} - c_3 + \frac{qD_q^2 g(\beta)}{g(\beta)} - \frac{2qc_2 D_q g(\beta)}{g(\beta)} + \frac{2qc_2 D_q g(\beta)}{(q+1)g(\beta)} + \frac{2c_2 D_q g(\beta)}{(q+1)g(\beta)} - 2c_2^2 - \frac{qD_q g(\beta)^2}{g(\beta)^2} - \frac{q^2 D_q g(\beta)^2}{g(\beta)^2} + 3c_3\Big\}e_n^3 + O(e_n^4), \tag{3.11}$$

$$\tilde{e}_{n+1} = \Big(\frac{qD_q g(\beta)}{g(\beta)} + c_2\Big)e_n^2 - \Big\{\frac{2c_2 D_q g(\beta)}{g(\beta)} - c_3 + \frac{qD_q^2 g(\beta)}{g(\beta)} - \frac{2qc_2 D_q g(\beta)}{g(\beta)} + \frac{2qc_2 D_q g(\beta)}{(q+1)g(\beta)} + \frac{2c_2 D_q g(\beta)}{(q+1)g(\beta)} - 2c_2^2 - \frac{qD_q g(\beta)^2}{g(\beta)^2} - \frac{q^2 D_q g(\beta)^2}{g(\beta)^2} + 3c_3\Big\}e_n^3 + O(e_n^4). \tag{3.12}$$

Choosing (3.12), we have

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{D_q g(\tilde{u}_n)f(q\tilde{u}_n) + D_q f(\tilde{u}_n)g(\tilde{u}_n)}, \tag{3.13}$$

$$=\beta + \Big(\frac{qD_qg(\beta)}{g(\beta)} + c_2\Big)e_n^2 - \left\{\frac{2c_2D_qg(\beta)}{g(\beta)} - c_3 + \frac{qD_q^2g(\beta)}{g(\beta)} - \frac{2qc_2D_qg(\beta)}{g(\beta)} + \frac{2qc_2D_qg(\beta)}{(q+1)g(\beta)} + \frac{2c_2D_qg(\beta)}{(q+1)g(\beta)} - 2c_2^2 - \frac{qD_qg(\beta)^2}{g(\beta)^2} - \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 3c_3\right\}e_n^3 + O(e_n^4). \tag{3.14}$$

By expanding $f(\tilde{v}_n)$ in the $q-$Taylor's series about β and using (3.14), we have

$$f(\tilde{v}_n) =\Big(\frac{qD_qg(\beta)}{g(\beta)} + c_2\Big)e_n^2 + \left\{\frac{-2c_2D_qg(\beta)}{g(\beta)} - c_3 + \frac{qD_q^2g(\beta)}{g(\beta)} - \frac{2qc_2D_qg(\beta)}{g(\beta)} + \frac{2qc_2D_qg(\beta)}{(q+1)g(\beta)} + \frac{2c_2D_qg(\beta)}{(q+1)g(\beta)} - 2c_2^2 - \frac{qD_qg(\beta)^2}{g(\beta)^2} - \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 3c_3\right\}e_n^3 + O(e_n^4). \tag{3.15}$$

From (3.4), (3.9), (3.8) and (3.15) we have

$$\frac{f(\tilde{v}_n)g(\tilde{u}_n)}{D_qf(\tilde{u}_n)g(\tilde{u}_n) + D_qg(\tilde{u}_n)f(q\tilde{u}_n)} =\Big(\frac{qD_qg(\beta)}{g(\beta)} + c_2\Big)e_n^2 + \left\{\frac{-2c_2D_qg(\beta)}{g(\beta)} - \frac{qD_q^2g(\beta)}{g(\beta)^2} + \frac{qD_q^2g(\beta)}{g(\beta)} - \frac{5qc_2D_qg(\beta)}{g(\beta)} + \frac{2qc_2D_qg(\beta)}{(q+1)g(\beta)} + \frac{2c_2D_qg(\beta)}{(q+1)g(\beta)} - 4c_2^2 - \frac{2q^2D_qg(\beta)^2}{g(\beta)^2} + 2c_3\right\}e_n^3 + O(e_n^4). \tag{3.16}$$

Using (3.14), (3.16) into (2.32), we obtain the error term for the Algorithm B

$$\tilde{u}_{n+1} = \beta + \left\{\frac{3qc_2D_qg(\beta)}{g(\beta)} + \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 2c_2^2\right\}e_n^3 + O(e_n^4),$$

$$e_{n+1} = \left\{\frac{3qc_2D_qg(\beta)}{g(\beta)} + \frac{q^2D_qg(\beta))^2)}{g(\beta)^2} + 2c_2^2\right\}e_n^3 + O(e_n^4). \tag{3.17}$$

By expanding $\tilde{w}_n$, $f(\tilde{w}_n)$ in terms of the $q-$Taylor's series about β

$$\tilde{w}_n = \left\{\beta + \frac{3qc_2D_qg(\beta)}{g(\beta)} + \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 2c_2^2\right\}e_n^3 + O(e_n^4), \tag{3.18}$$

$$f(\tilde{w}_n) = D_qf(\beta)\left\{\Big(\frac{3qc_2D_qg(\beta)}{g(\beta)} + \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 2c_2^2\Big)e_n^3 + O(e_n^4)\right\}. \tag{3.19}$$

From (3.4), (3.9), (3.8) and (3.19) we have

$$\frac{f(\tilde{w}_n)g(\tilde{u}_n)}{D_qf(\tilde{u}_n)g(\tilde{u}_n) + D_qg(\tilde{u}_n)f(q\tilde{u}_n)} = \left\{\frac{3qc_2D_qg(\beta)}{g(\beta)} + \frac{q^2D_qg(\beta)^2}{g(\beta)^2} + 2c_2^2\right\} e_n^3 + O(e_n^4). \tag{3.20}$$

Using (3.18) and (3.20) into (3.12), we obtain the error equation for the *Algorithm C:*

$$\tilde{u}_{n+1} =\tilde{w}_n - \frac{f(\tilde{w}_n)g(\tilde{u}_n)}{D_qf(\tilde{u}_n)g(\tilde{u}_n) + D_qg(\tilde{u}_n)f(q\tilde{u}_n)}, \tag{3.21}$$

$$=\beta + \left(\frac{5q^2c_2D_qg(\beta)^2}{g(\beta)^2} + \frac{8qc_2^2D_qg(\beta)}{g(\beta)} + \frac{q^3(D_qg(\beta))^3}{g(\beta)^3} + 4c_2^3\right) e_n^4 + O(e_n^5), \tag{3.22}$$

$$e_{n+1} = \left(\frac{5q^2 c_2 D_q g(\beta)^2}{g(\beta)^2} + \frac{8qc_2^2 D_q g(\beta)}{g(\beta)} + \frac{q^3 (D_q g(\beta))^3}{g(\beta)^3} + 4c_2^3 \right) e_n^4 + O(e_n^5). \tag{3.23}$$

Equation (3.20) shows the error equation for the Algorithm C and has at least fourth-order convergence. It is noted that Algorithm C is the main iterative scheme and all other schemes investigated from this scheme are at least fourth-order convergent.

Remark 3.1. *Based on the study of convergence analysis of proposed iterative methods, it can be easily observed that various order iterative methods can be developed by choosing appropriately multiple choices of the auxiliary function in Algorithm A, Algorithm B and, Algorithm C respectively.*

$$\text{If} \quad \frac{g(\tilde{u}_n)}{D_q g(\tilde{u}_n)} = -\frac{D_q^2 f(\tilde{u}_n)}{2D_q f(q\tilde{u}_n)}, \tag{3.24}$$

then Algorithm A generates the following iterative method with the initial guess $\tilde{u}_0$.

Algorithm G: For a given initial guess $\tilde{u}_o$, approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$\tilde{u}_{n+1} = \tilde{u}_n - \frac{2f(\tilde{u}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \dots . \tag{3.25}$$

This is $q-$analogue of well known Halley method [32] which has cubic convergence i.e. [3, q], where q represents the $q-$calculus. Now, again using the above stated specified value of an auxiliary function then Algorithm B and Algorithm C reduces to the following iterative procedures.

Algorithm H: For a given initial guess $\tilde{u}_o$, approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$\tilde{v}_n = \tilde{u}_n - \frac{2f(\tilde{u}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \tag{3.26}$$

$$\tilde{u}_{n+1} = \tilde{v}_n - \frac{2f(\tilde{v}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \dots . \tag{3.27}$$

This method is fourth-order convergent for solving nonlinear equations and appears to be a novel one.

Algorithm I: For a given initial guess $\tilde{u}_o$, approximate solution $\tilde{u}_{n+1}$ is computed by the following iterative scheme:

$$\tilde{v}_n = \tilde{u}_n - \frac{2f(\tilde{u}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \tag{3.28}$$

$$\tilde{w}_n = \tilde{v}_n - \frac{2f(\tilde{v}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \tag{3.29}$$

$$\tilde{u}_{n+1} = \tilde{u}_n - \frac{2f(\tilde{w}_n) D_q f(\tilde{u}_n)}{2(D_q f(\tilde{u}_n))^2 - D_q^2 f(\tilde{u}_n) f(q\tilde{u}_n)}, \qquad n = 0, 1, 2, \dots . \tag{3.30}$$

This method emerges as a new method that has fifth-order of convergence.

This completes the proof. □

4. Numerical examples and applications

This section discusses some nonlinear equations. With the support of these examples, we elaborate on the efficacy and performance of newly established methods initiated in this paper. The general algorithm for finding the estimated solution of the given nonlinear function is given as: in Algorithm A, Algorithm B, Algorithm C, we consider $\varepsilon = 10^{-100}$ as tolerance. We obtain an approximate solution relatively than the exact lean on the computational accuracy ε. We adopt the following stopping criterium for computational performance:

$$|\tilde{u}_{n+1} - \tilde{u}_n| < \varepsilon \quad and \quad |f(\tilde{u}_{n+1})| < \varepsilon. \tag{4.1}$$

For convergence criteria, it is prerequisite that the space of two successive estimations for the zero must be less than 10^{-100}. We make use of abbreviations QG & CG for the $q-$iterative methods and traditional iterative methods respectively. We symbolize Algorithm D, Algorithm E and Algorithm F by QG1, QG2 and QG3 respectively and phrase *div* served as the divergence of methods. We develop a comparative analysis between the standard Newton's method (NM) [35], Chun method (CM) [4], Noor method (NR) [8], CG1, CG2 and CG3 [9] and our newly proposed $q-$iterative methods QG1, QG2 and QG3. The computational results of comparative analysis are presented in Tables (4, 8, 12, 14). We exhibit the number of iterations, the final estimated solution and the corresponding functional value by the symbols IT, $\tilde{u}_n$ and $f(\tilde{u}_n)$, whereas, the distance in the middle of two successive estimates is shown by Δ. It is necessary to mention that for the best implementation of results, we choose the value of $q = 0.9999$. We use Maple software to perform all the numerical computations.

Algorithms D, E, F : General roots' finding Algorithm

Input: $f \in$ R–nonlinear function, l–maximal number of iterations, I- recursive method, ε accuracy
Output: Approximate root of given nonlinear function
for $\tilde{u}_0 \in$ A do
$j = 0$
while $i \leq l$ **do**
$\tilde{u}_{n+1} = I(\tilde{u}_n)$
if $|\, \tilde{u}_{n+1} - \tilde{u}_n \,| \leq \varepsilon$ **then**
break
$j = j + 1$
$\tilde{u}_{n+1}$ is the required solution

Now, we recollect the classical *Algorithm 2.1* (CG1) in [9], elucidated as:

$$\tilde{u}_{n+1} = \tilde{v}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g(\tilde{u}_n)}, \quad n = 0, 1, 2, \dots ,$$

and the classical *Algorithm 2.2* (CG2) in [9], described as:

$$\begin{aligned}
\tilde{v}_n &= \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g'(\tilde{u}_n)}, \\
\tilde{u}_{n+1} &= \tilde{v}_n - \frac{f(\tilde{v}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g'(\tilde{u}_n)}, \quad n = 0, 1, 2, \dots ,
\end{aligned}$$

and the classical *Algorithm 2.3* (CG3) in [9], described as:

$$\tilde{v}_n = \tilde{u}_n - \frac{f(\tilde{u}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g'(\tilde{u}_n)},$$
$$\tilde{w}_n = \tilde{v}_n - \frac{f(\tilde{v}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g'(\tilde{u}_n)},$$
$$\tilde{u}_{n+1} = \tilde{w}_n - \frac{f(\tilde{w}_n)g(\tilde{u}_n)}{f'(\tilde{u}_n)g(\tilde{u}_n) + f(\tilde{u}_n)g'(\tilde{u}_n)}, \quad n = 0, 1, 2, \dots .$$

We present some examples of nonlinear equations (4.1–4.4) to illustrate the efficiency of the newly developed one-step, two-step and three-step iterative methods in this article. Firstly, for the sake of simplicity, we investigate the efficacy and credibility of the $q-$recursive schemes for multiple values of q up to three iterations that can be extended to any number of iterations until we achieve the desired accuracy. The results in the Tables (1–3, 5–7, 9–11) demonstrate the calculations of $\tilde{u}_i$ and $f(\tilde{u}_i)$, $i = 1, 2, 3$ by employing QG1, QG2, QG3 for multiple values of q and $\beta = 0.5$. We choose $\beta = 0.5$ for both q and ordinary iterative methods.

Example 4.1 (see [4])**.** *We assume the folllowing nonlinear equation*:

$$f_1(\tilde{u}) = \tilde{u}e^{\tilde{u}^2} - \sin^2 \tilde{u} + 3\cos \tilde{u} + 5. \tag{4.2}$$

We take $\tilde{u}_0 = -2$ as an initial guess for computational evaluations. The quantifiable outcomes for the equation (4.2) are calculated in Tables (1–3) by using QG1, QG2, QG3 for multiple values of q and $\beta = 0.5$. Following the steps of the Tables (1–3), we get the required solution of equation (4.2) i.e. $\tilde{u} = 1.2076478271$.

The results from the Table 1, elaborate that precise values of $\tilde{u}_i$s are achieved subject to the constraint q approaces to one and the parallel functional values $f(\tilde{u}_i)$ tend to zero, where i=1, 2, 3. It is also noted that the values of $f(\tilde{u}_1) = 4.365516e + 01, f(\tilde{u}_2) = 1.736630e + 01, f(\tilde{u}_3) = 6.213934e + 00$ computed by QG1 at $q = 0.9999$ are nearer to zero in comparison with the values $f(\tilde{u}_1) = 4.367558e+01, f(\tilde{u}_2) = 1.738244e + 01, f(\tilde{u}_3) = 6.223602e + 00$ calculated by CG1. And by choosing $q = 0.9999$ and β=0.5 the equation (4.2) converges to the root $\tilde{u}_8 = 1.2076478271$ and corresponding functional value is attained as $f(\tilde{u}_8) = 1.268198e - 10$.

Table 1. The Computational results of Example 4.1 by adopting QG1.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	1.8254771289	4.781399e+01	1.6446487615	2.080730e+01	1.4682546352	8.360155e+00
1.01	1.8156706474	4.573200e+01	1.6261034309	1.904589e+01	1.4443944595	7.240303e+00
0.9999	1.8054350705	4.365516e+01	1.6069907813	1.736630e+01	1.4205163424	6.213934e+00
0.99	1.7950820546	4.164963e+01	1.5879197232	1.581610e+01	1.3974602186	5.303635e+00
0.98	1.7843060898	3.965867e+01	1.5683557800	1.434521e+01	1.3746624563	4.474026e+00
0.97	1.7732141094	3.770686e+01	1.5485309680	1.296746e+01	1.3525035647	3.728577e+00
0.96	1.7618106322	3.579794e+01	1.5284887126	1.167998e+01	1.3311324428	3.061376e+00
0.95	1.7501006871	3.393522e+01	1.5082747516	1.047945e+01	1.3106997348	2.466872e+00
0.9	1.6871602987	2.540046e+01	1.4063812618	5.646958e+00	1.2277744500	4.214155e-01
0.8	1.5421515802	1.254661e+01	1.2218238894	2.941270e-01	1.2001853197	1.498622e-01
0.7	1.3778702120	4.586773e+00	1.1228243978	1.525686e+00	1.3085797228	2.407471e+00
0.6	1.2006929046	1.397745e-01	1.2169781309	1.921632e-01	1.1956606924	2.391136e-01
0.5	1.0148075743	3.019833e+00	1.7800282582	3.889448e+01	0.9416817610	3.825826e+00
0.4	0.8224202727	4.886877e+00	4.4574589961	1.896962e+09	1.7829836172	3.942087e+01

Columns in Table 2 display the more precise values of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ with the constraint that q approaches one and for which $f(\tilde{u}_1), f(\tilde{u}_2), f(\tilde{u}_3)$ tend to zero. Furthermore, it is also noted that the values of $f(\tilde{u}_1) = 3.032134e + 01, f(\tilde{u}_2) = 7.638732e + 00, f(\tilde{u}_3) = 1.084323e + 00$ at $q = 0.9999$ calculated by QG2 exist closely in the neighbourhood of zero in contrast to the values of $f(\tilde{u}_1) = 3.033637e + 01, f(\tilde{u}_2) = 7.646987e+00, f(\tilde{u}_3) = 1.087173e+00$ calculated by CG2. Following the steps of Table 2 and by taking values $q = 0.9999, \beta = 0.5$, the equation (4.2) converges to the root $\tilde{u}_6 = 1.2076478271$ and corresponding function value is attained as $f(\tilde{u}_6) = 1.389449e - 15$.

Table 2. The Computational results of Example 4.1 by adopting QG2.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	1.7469553241	3.345101e+01	1.4893603323	9.437637e+00	1.2844368858	1.759383e+00
1.01	1.7363479211	3.186639e+01	1.4712261908	8.506726e+00	1.2700675438	1.397061e+00
0.9999	1.7255099828	3.032134e+01	1.4531277997	7.638732e+00	1.2571115985	1.084323e+00
0.99	1.7147712370	2.886049e+01	1.4356424131	6.853672e+00	1.2459611644	8.252116e-01
0.98	1.7038129312	2.743800e+01	1.4182786930	6.122244e+00	1.2362606318	6.070097e-01
0.97	1.6927468246	2.606789e+01	1.4012566043	5.448403e+00	1.2281101471	4.286657e-01
0.96	1.6815757292	2.474896e+01	1.3846176710	4.828114e+00	1.2214623281	2.864671e-01
0.95	1.6703014629	2.347984e+01	1.3684031128	4.257621e+00	1.2162424779	1.768145e-01
0.9	1.6123870154	1.782725e+01	1.2951373881	2.040329e+00	1.2071607634	9.883772e-03
0.8	1.4880971668	9.370703e+00	1.2044936727	6.375050e-02	1.2066023590	2.119746e-02
0.7	1.3510185483	3.680628e+00	1.2153042458	1.572886e-01	1.2134104384	1.180445e-01
0.6	1.2017441996	1.188325e-01	1.1966948659	2.188195e-01	1.1861260408	4.233182e-01
0.5	1.0428029928	2.671572e+00	3.4921751247	6.909446e+05	1.7457947641	4.029002e+01
0.4	0.8765710506	4.438584e+00	1.2449990477	*div*	*div*	*div*

Table 3 illustrates the accuracy and precision of results for $\tilde{u}_i$, $i = 1, 2, 3$ whenever $q \to$ one. One can also figure out that for $q = 0.9999$ computed values of $f(\tilde{u}_1) = 2.344843e{+}01$, $f(\tilde{u}_2) = 3.928572e{+}00$, $f(\tilde{u}_3) = 1.242253e-01$ by QG3 give results nearer to zero in comparison to $f(\tilde{u}_1) = 2.345843e+01, f(\tilde{u}_2) = 3.933608e+00, f(\tilde{u}_3) = 1.248415e-01$ calculated by CG3. Following the steps of Table 3 the equation (4.2) converges to the root $\tilde{u}_5 = 1.2076478271$ and $f(\tilde{u}_5) = 3.785784e-18$, for $q = 0.9999$ & β=0.5.

Table 3. The Computational results of Example 4.1 by adopting QG3.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	1.6920209048	2.598026e+01	1.3904466751	5.041269e+00	1.2220019413	2.979023e-01
1.01	1.6810752657	2.469133e+01	1.3743118157	4.461778e+00	1.2172669454	1.981985e-01
0.9999	1.6699968307	2.344638e+01	1.3586170942	3.928572e+00	1.2137094253	1.242253e-01
0.99	1.6591210592	2.227959e+01	1.3438575823	3.452797e+00	1.2112346261	7.323293e-02
0.98	1.6481234444	2.115270e+01	1.3296114931	3.015709e+00	1.2095622688	3.898944e-02
0.97	1.6371173589	2.007559e+01	1.3160631511	2.619025e+00	1.2085336032	1.801178e-02
0.96	1.6261052018	1.904605e+01	1.3032414482	2.259727e+00	1.2079714068	6.574262e-03
0.95	1.6150882454	1.806184e+01	1.2911720932	1.935078e+00	1.2077179285	1.423727e-03
0.9	1.5826834407	1.541101e+01	1.2612508796	1.182847e+00	1.2076472774	1.116271e-05
0.8	1.4477254852	7.390755e+00	1.2057826949	3.777015e-02	1.2079930144	7.013497e-03
0.7	1.3294715959	3.011520e+00	1.2030867033	9.199385e-02	1.2113502472	7.560679e-02
0.6	1.2026379824	1.009764e-01	1.2197934625	2.512202e-01	1.1838911341	4.657518e-01
0.4	*div*	*div*	*div*	*div*	*div*	*div*

In Table 4, we compare our new $q-$iterative methods (QG1, QG2, QG3) with some other methods to examine the reliability and effectiveness of the methods. The second column (IT) in Table 4 exhibits the comparison of different iterative methods with newly established methods concerning to the number of iterations. It is clear from the computational results that proposed methods require less number of iterations compared to other methods to meet the stopping criteria (4.1) or number of iterations are the same in some cases when comparing with (CG1, CG3).

Table 4. Numerical comparison between different algorithms for Example 4.1.

Methods	IT	$\tilde{u}_n$	$f(\tilde{u}_n)$	$\Delta = \|\tilde{u}_n - \tilde{u}_{n-1}\|$
NM	10	1.2076478271309189	3.809499e-81	1.117513e-41
CM	8	1.2076478271309189	4.147459e-71	8.245079e-37
NR	7	1.2076478271309189	1.200000e-126	1.022315e-58
CG1	9	1.2076478271309189	4.046035e-27	9.975651e-15
QG1	9	1.2076478271309189	6.544364e-18	1.333068e-15
CG2	7	1.2076478271309189	2.982998e-73	1.279292e-25
QG2	6	1.2076478271309189	2.272051e-15	1.913599e-09
CG3	5	1.2076478271309189	5.783684e-28	3.280697e-08
QG3	5	1.2076478271309189	8.839817e-18	3.082412e-08

Example 4.2 (Van der Waal's Equation see [37])**.**

We consider the Van der Waal's equation representing the real and ideal behaviour of gass is prescribed as:

$$\Big(\mathcal{P}+\frac{a_1m^2}{V^2}\Big)(\mathcal{V}-mb_1)=m\mathcal{R}\mathcal{T}. \tag{4.3}$$

Eq (4.3) can be transformed to following nonlinear form:

$$\mathcal{P}\mathcal{V}^3-(mb_1\mathcal{P}+m\mathcal{R}\mathcal{T})\mathcal{V}^2+a_1m^2\mathcal{V}-a_1m^3b_1. \tag{4.4}$$

After appropriately choosing the needed parameters and unknown constants we can find the following nonlinear function:

$$0.986\tilde{u}^3-5.181\tilde{u}^2+9.067\tilde{u}-5.289=0, \tag{4.5}$$

where the variable $\tilde{u}$ shows the volume of the gas. We take $\tilde{u}_0 = 3.10$ as an initial guess for computational evaluations. The mathematical computations for the equation (4.5) are calculated in Tables 5–7, for multiple values of q and β=0.5. Following the steps of the Tables 5–7, we get the required solution for equation (4.5) i.e. $\tilde{u} = 1.9298462428$.

Continuing step by step and evaluating values, Table 5 illustrates the accuracy and precision of results for $\tilde{u}_i$, $i = 1, 2, 3$ whenever $q \to$ one. One can also figure out that for $q = 0.9999$ computed values of $f(\tilde{u}_1)$= 4.436290e-01, $f(\tilde{u}_2) = 1.030037e-01$, $f(\tilde{u}_3) = 2.540698e-02$ by adopting QG1 give results more near to zero in comparison to $f(\tilde{u}_1) = 4.437270e-01$, $f(\tilde{u}_2) = 1.030704e-01$, $f(\tilde{u}_3) = 2.544189e-02$ calculated by CG1. And for $q = 0.9999$ & β=0.5, equation (4.5) converges to the root $\tilde{u}_9 = 1.929846242847$ and $f(\tilde{u}_9) = 9.996931e-14$.

Table 5. The Computational results of Example 4.2 by adopting QG1.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	2.5340549915	4.622803e-01	2.2528091052	1.161487e-01	2.0933289950	3.253619e-02
1.01	2.5290275614	4.532706e-01	2.2438490358	1.096752e-01	2.0824083686	2.896476e-02
0.9999	2.5235741872	4.436290e-01	2.2342561950	1.030037e-01	2.0707413296	2.540698e-02
0.99	2.5178563392	4.336656e-01	2.2243016758	9.635846e-02	2.0586445350	2.198946e-02
0.98	2.5116994726	4.231029e-01	2.2136678014	8.956571e-02	2.0457159746	1.863062e-02
0.97	2.5051516759	4.120566e-01	2.2024251150	8.272013e-02	2.0320216583	1.538982e-02
0.95	2.4908481079	3.885886e-01	2.1779892318	6.899282e-02	2.0020810008	9.367367e-03
0.9	2.4475472929	3.229283e-01	2.1036636192	3.613794e-02	1.9082525748	1.630351e-03
0.8	2.3226246157	1.750738e-01	1.8703556778	3.482644e-03	1.9400668627	9.395422e-04
0.7	2.1317302609	4.706104e-02	1.7178745235	4.011128e-03	1.7281465614	4.064557e-03
0.6	1.8631678821	3.710240e-03	1.8713066952	3.449857e-03	1.8790717075	3.157722e-03
0.5	1.5622231117	9.473970e-03	1.5681956056	8.906252e-03	1.5738756949	8.402240e-03
0.4	1.4186354467	3.806147e-02	1.4327916708	3.372384e-02	1.4455832111	3.012982e-02

Table 6 investigate the precision and accuracy of the values of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ when the parameter $q \to$ one parallel functional values $f(\tilde{u}_1), f(\tilde{u}_2), f(\tilde{u}_3)$ tend to zero. Moreover, one can also figure out that for $q = 0.9999$ enumerated values of $f(\tilde{u}_1) = 2.814977e-01, f(\tilde{u}_2) = 4.108624e-02, f(\tilde{u}_3) = 5.485842e-03$ at $q = 0.9999$ by adopting QG2, are more near to zero in comparison with $f(\tilde{u}_1) = 2.815532e-01, f(\tilde{u}_2) = 4.111438e-02, f(\tilde{u}_3) = 5.496367e-03$ computed by CG2. Also, following the steps of Table 6 and for $q = 0.9999$ the equation (4.5) converges towards the root $\tilde{u}_7 = 1.929846242847$ and $f(\tilde{u}_7) = 3.089402e - 19$.

Table 6. The Computational results of Example 4.2 by adopting QG2.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	2.4251903536	2.920793e-01	2.1308490583	4.669183e-02	1.9922592160	7.688946e-03
1.01	2.4213364441	2.869646e-01	2.1240891505	4.391721e-02	1.9852268979	6.571938e-03
0.9999	2.4171671144	2.814977e-01	2.1169241975	4.108624e-02	1.9779347493	5.485842e-03
0.99	2.4128072440	2.758540e-01	2.1095606065	3.829240e-02	1.9706241156	4.468488e-03
0.98	2.4081250695	2.698760e-01	2.1017669619	3.546045e-02	1.9631069315	3.494608e-03
0.97	2.4031586932	2.636282e-01	2.0935999939	3.262785e-02	1.9555001549	2.581114e-03
0.96	2.3979027738	2.571197e-01	2.0850408423	2.980403e-02	1.9478660869	1.734541e-03
0.95	2.3923511342	2.503596e-01	2.0760670068	2.699855e-02	1.9402941069	9.617326e-04
0.9	2.3598750013	2.131229e-01	2.0238013481	1.359491e-02	1.9112433539	1.431686e-03
0.8	2.2659929856	1.261078e-01	1.8466091935	4.108333e-03	1.9343586970	4.008515e-04
0.7	2.1127690921	3.949548e-02	1.7868081653	4.456338e-03	1.8181365605	4.447281e-03
0.6	1.8650773832	3.653196e-03	1.8806096583	3.094552e-03	1.8942169581	2.454490e-03
0.5	1.5682853436	8.898020e-03	1.5793160860	7.951302e-03	1.5894190131	7.193169e-03
0.4	1.4452644311	3.021574e-02	1.4674207779	2.467175e-02	1.4861946088	2.062246e-02

We calculate the values $\tilde{u}_i, f(\tilde{u}_i)$, where i=1,2,3, in the Table 7 that illustrates the accuracy of values $\tilde{u}_i$ subject to constraint q approaches one. Moreover, it is apparently observed that the values of $f(\tilde{u}_1) = 2.021924e-01, f(\tilde{u}_2) = 2.062325e-02, f(\tilde{u}_3) = 1.336893e-03$ at $q = 0.9999$ computed by QG3, are near to zero in comparison with $f(\tilde{u}_1) = 2.022301e-01, f(\tilde{u}_2) = 2.063865e-02, f(\tilde{u}_3) = 1.340965e-03$ determined by CG3. Also, the equation (4.5) converges to root $\tilde{u}_6 = 1.929846242847$ for $q = 0.9999$ and corresponding functional value is attained as $f(\tilde{u}_6) = 2.935505e - 21$.

Table 7. The Computational results of Example 4.2 by adopting QG3.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	2.3564071823	2.093746e-01	2.0648720320	2.371495e-02	1.9524146721	2.230633e-03
1.01	2.3531574276	2.059011e-01	2.0593394441	2.217848e-02	1.9481788722	1.767879e-03
0.9999	2.3496482072	2.021924e-01	2.0535229234	2.062325e-02	1.9440499137	1.336893e-03
0.99	2.3459856668	1.983682e-01	2.0475940715	1.910018e-02	1.9402242968	9.549095e-04
0.98	2.3420601770	1.943218e-01	2.0413708941	1.756771e-02	1.9366793162	6.155390e-04
0.97	2.3379049697	1.900973e-01	2.0349047430	1.604574e-02	1.9335763902	3.297944e-04
0.96	2.3335167607	1.857008e-01	2.0281861877	1.453859e-02	1.9310644099	1.060731e-04
0.95	2.3288915656	1.811387e-01	2.0212030968	1.305038e-02	1.9293165421	4.563520e-05
0.9	2.3020139461	1.560553e-01	1.9815642250	6.017371e-03	1.9270846955	2.346839e-04
0.8	2.2252005155	9.694703e-02	1.8423934407	4.183986e-03	1.9281640453	1.439076e-04
0.7	2.0968561321	3.374084e-02	1.8598344345	3.804075e-03	1.9067527568	1.726845e-03
0.6	1.8669575262	3.594636e-03	1.8890164661	2.716754e-03	1.9060889327	1.768893e-03
0.5	1.5739790349	8.393388e-03	1.5893859137	7.195490e-03	1.6031115465	6.320746e-03
0.4	1.4664042988	2.490755e-02	1.4935938393	1.918070e-02	1.5156493267	1.536937e-02

Table 8 presents the comparative analysis of our new $q-$iterative methods (QG1, QG2, QG3) with some other classical methods. In Table 8 the efficiency of methods is displayed with respect to number of iterations using the stopping criterium (4.1). It is observed that new methods are comparable with other methods and need less number of iterations required by the other methods of the same order.

Table 8. Numerical comparison between different algorithms for Example 4.2.

Methods	IT	$\tilde{u}_n$	$f(\tilde{u}_n)$	$\Delta = \lvert\tilde{u}_n - \tilde{u}_{n-1}\rvert$
NM	12	1.9298462428478622	3.825655e-94	2.693071e-47
CM	10	1.9298462428478621	2.211000e-123	4.575337e-62
NR	8	1.9298462428478621	0.000000e+00	3.271773e-50
CG1	10	1.9298462428478622	4.822492e-43	9.979122e-22
QG1	9	1.9298462428478622	9.050718e-14	9.687156e-10
CG2	8	1.9298462428478622	0.000000e+00	1.931199e-44
QG2	6	1.9298462428500578	1.897669e-13	1.934517e-06
CG3	7	1.9298462428478622	0.000000e+00	4.553643e-61
QG3	6	1.9298462428478622	1.538805e-24	1.404977e-14

Example 4.3 (Motion of particle on an inclined plane see [35])**.** *We consider the nonlinear model created due to the motion of a particle on an inclined plane whose inclination angle θ remodeled at a steady rate $\frac{d(\theta)}{dt} = \gamma < 0$.*

$$\tilde{u}(t) = -\frac{h}{2\gamma^2}\Big(\frac{e^{\gamma t} - e^{-\gamma t}}{2} - sin\gamma t\Big). \tag{4.6}$$

We take $\tilde{u}_o = -1.2$ as an initial guess for computational evaluation. The numerical findings for equation (4.6) are calculated in the Tables (9–11) by using QG1, QG2, QG3 for multiple values of q and $\beta = 0.5$.

The numeric values in Table 9, illustrate that one can obtain more precise values of $\tilde{u}_i$ with the constraint $q \to$ one and $f(\tilde{u}_i)$ attain zero value, where $1 \leq i \leq 3$. It is also noted that the values of $f(\tilde{u}_1) = 1.562110e-01, f(\tilde{u}_2) = 5.384093e-02, f(\tilde{u}_3) = 1.734706e-02$ computed by QG1 at $q = 0.9999$ exist in the neighbourhood of zero more nearly in comparison to the values $f(\tilde{u}_1) = 1.562383e-01, f(\tilde{u}_2) = 5.386087e-02, f(\tilde{u}_3) = 1.735750e-02$ calculated by CG1. Following the steps of the Table 9, the equation (4.6) converges towards the root $\tilde{u}_9 = 0.3170617745$ and $f(\tilde{u}_9) = 8.400168e-13$, for $q = 0.9999$.

Table 9. The Computational results of Example 4.3 by adopting QG1.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	0.9065128305	1.616621e-01	0.6859962729	5.790018e-02	0.5274350537	1.951191e-02
1.01	0.9022072708	1.589572e-01	0.6796900041	5.586816e-02	0.5207645246	1.841821e-02
0.9999	0.8977867194	1.562110e-01	0.6732521632	5.384093e-02	0.5140013161	1.734706e-02
0.99	0.8933824398	1.535059e-01	0.6668750915	5.187932e-02	0.5073486578	1.632989e-02
0.98	0.8888609318	1.507608e-01	0.6603667093	4.992446e-02	0.5006073854	1.533546e-02
0.97	0.8842651916	1.480036e-01	0.6537912994	4.799729e-02	0.4938467857	1.437425e-02
0.96	0.8795940823	1.452351e-01	0.6471491186	4.609874e-02	0.4870691826	1.344626e-02
0.95	0.8748464608	1.424560e-01	0.6404404782	4.422978e-02	0.4802770278	1.255145e-02
0.9	0.8499203175	1.284323e-01	0.6059153687	3.536070e-02	0.4461937277	8.570721e-03
0.8	0.7936638954	1.001489e-01	0.5322413798	2.032320e-02	0.3788240897	2.954718e-03
0.7	0.7279286364	7.261236e-02	0.4534401666	9.348131e-03	0.3172501829	6.321333e-06
0.6	0.6517336303	4.740399e-02	0.3716641373	2.514365e-03	0.2772158130	1.020750e-03
0.5	0.5644679828	2.628337e-02	0.2905129641	7.469249e-04	0.5808625814	2.967734e-02
0.4	0.4661770270	1.080358e-02	0.2165534440	1.571177e-03	0.0187870402	3.509276e-05

The results in Table 10, elaborate that one can obtain more precise values of $\tilde{u}_i$ with the constraint $q \to$ one and $f(\tilde{u}_i)$ attain zero value, where $1 \leq i \leq 3$. By choosing q=0.9999, it is also noted that the values of $f(\tilde{u}_1) = 9.698942e-02, f(\tilde{u}_2) = 1.973729e-02, f(\tilde{u}_3) = 3.342707e-03$ computed by QG2 are nearer to zero in comparison to the values $f(\tilde{u}_1) = 9.700799e-02, f(\tilde{u}_2) = 1.974547e-02, f(\tilde{u}_3) = 3.345242e-03$ calculated by CG2. Continuing the iterative procedure as presented in Table 10, the equation (4.6) converges to the root $\tilde{u}_6 = 0.3170617746$ for $q = 0.9999$ and $f(\tilde{u}_6) = 4.953192e-14$.

Table 10. The Computational results of Example 4.3 by adopting QG2.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	0.7949363458	1.007389e-01	0.5385637930	2.142047e-02	0.3925179431	3.875559e-03
1.01	0.7908881643	9.886946e-02	0.5336993039	2.057318e-02	0.3886467585	3.604555e-03
0.9999	0.7867668819	9.698942e-02	0.5287819917	1.973729e-02	0.3847856611	3.342707e-03
0.99	0.7826952165	9.515481e-02	0.5239585461	1.893725e-02	0.3810508358	3.097343e-03
0.98	0.7785500695	9.331024e-02	0.5190834160	1.814844e-02	0.3773306922	2.860583e-03
0.97	0.7743721845	9.147458e-02	0.5142058577	1.737890e-02	0.3736656864	2.634686e-03
0.96	0.7701613148	8.964812e-02	0.5093264091	1.662854e-02	0.3700584295	2.419381e-03
0.95	0.6987569142	6.215397e-02	0.4352652464	1.589720e-02	0.3665116849	2.214397e-03
0.9	0.7441859116	7.889838e-02	0.4800380977	1.252061e-02	0.3497954433	1.335162e-03
0.8	0.6979650499	6.188441e-02	0.4313667873	7.090723e-03	0.3232085729	2.140527e-04
0.7	0.6473177836	4.614636e-02	0.3831116469	3.231774e-03	0.3127824804	1.395601e-04
0.6	0.5906138511	3.181675e-02	0.3355451322	6.937306e-04	0.3362806048	7.245007e-04
0.5	0.5251859689	1.913898e-02	0.2900837107	7.567601e-04	0.3462805049	1.202954e+05
0.4	0.4475332296	8.711707e-03	0.2556391450	1.338072e-03	0.1755724416	5.061970e-03

Columns in the Table 11 demonstrate that one can obtain more precise values of $\tilde{u}_i$ with the constraint that $q \to 1$ and $f(\tilde{u}_i)$ attain value zero, where $1 \leq i \leq 3$. It is also noted that the values of $f(\tilde{u}_1) = 6.887648e - 02, f(\tilde{u}_2) = 9.422356e - 03, f(\tilde{u}_3) = 8.107219e - 04$ computed by QG3 at $q = 0.9999$ are more adjacent to zero in comparison to the values $f(\tilde{u}_1) = 6.889025e - 02, f(\tilde{u}_2) = 9.426571e - 03, f(\tilde{u}_3) = 8.115710e - 04$ calculated by CG3. Following the steps of Table 11 and for $q = 0.9999, \beta$=0.5, the equation (4.6) converges to the root $\tilde{u}_5 = 0.3170617746$ and parallel functional values are obtained as$f(\tilde{u}_5) = 4.850310e - 15$.

Table 11. The Computational results of Example 4.3 by adopting QG3.

q	$\tilde{u}_1$	$f(\tilde{u}_1)$	$\tilde{u}_2$	$f(\tilde{u}_2)$	$\tilde{u}_3$	$f(\tilde{u}_3)$
1.02	0.7254080389	7.166750e-02	0.4618314256	1.029406e-02	0.3424316921	9.916910e-04
1.01	0.7216490394	7.027308e-02	0.4579839931	9.854191e-03	0.3403446327	8.990485e-04
0.9999	0.7178361000	6.887648e-02	0.4541148102	9.422356e-03	0.3383104763	8.107219e-04
0.99	0.7140827415	6.751913e-02	0.4503391251	9.011018e-03	0.3363907757	7.291314e-04
0.98	0.7102754718	6.615984e-02	0.4465428329	8.607347e-03	0.3345285298	6.516081e-04
0.97	0.7064521071	6.481249e-02	0.4427647200	8.215368e-03	0.3327458730	5.788853e-04
0.96	0.7026126091	6.347717e-02	0.4390053417	7.834888e-03	0.3310452218	5.108533e-04
0.95	0.6987569142	6.215397e-02	0.4352652464	7.465713e-03	0.3294290354	4.474085e-04
0.9	0.6792312557	1.541101e+01	1.2612508796	1.182847e+00	1.2076472774	1.116271e-05
0.8	0.6388304107	4.378843e-02	0.3818321602	3.148034e-03	0.3167479623	1.049548e-05
0.7	0.5960882517	3.305817e-02	0.3495601149	1.323780e-03	0.3161012452	3.199394e-05
0.6	0.5495912988	2.341752e-02	0.3209432007	1.332764e-04	0.2952352918	6.341935e-04
0.4	0.4324993915	7.198641e-03	0.2993416515	5.293015e-04	*div*	*div*

The second column (IT) in Table 12 illustrates the comparsion of different iterative methods with proposed methods in terms of number of iterations. It is clear from the computational results that new methods need less number of iterations as compared to other methods (NM, CM, NR, CG1) to meet the stopping criteria (4.1) or same in some cases when comparing with (CG2, CG3).

Table 12. Numerical comparison between different algorithms for Example 4.3.

Methods	IT	$\tilde{u}_n$	$f(\tilde{u}_n)$	$\Delta = \lvert\tilde{u}_n - \tilde{u}_{n-1}\rvert$
NM	11	0.3170617745729571	4.956350e-75	1.531188e-37
CM	9	0.3170617745729571	1.243498e-77	5.423193e-39
NR	8	0.3170617745729571	1.000000e-128	3.237049e-78
CG1	10	0.3170617745729571	5.694906e-27	1.579893e-13
QG1	9	0.3170617745478905	8.400168e-13	1.165570e-07
CG2	7	0.3170617745729571	1.689005e-33	8.264048e-12
QG2	7	0.3170617745729571	2.308791e-21	1.478062e-12
CG3	6	0.3170617745729571	1.996559e-49	2.670559e-13
QG3	6	0.3170617745729571	4.881111e-26	1.447361e-13

Remark 4.1. *It is worthy to mention that when we evaluate the errors for the $q-$iterative schemes then it oscillate for various values of q. The error reduces when q tends to the highest values uniting* 0 *and* 1. *Therefore, in Table 13 error for equations* [(4.2), (4.5), (4.6)] *are computed by using q=0.9999 and β=0.5 which will estimate the classical methods.*

Table 13. Computational error of multistep $q-$iterative methods for q=0.9999, β=0.5

Algorithm D (QG1)			
Equation	Exact Solution	Estimated solution	Error
(4.2)	1.20764782713091892701	1.20764782713091892669	2.415296e+00
(4.5)	1.92984624284786221849	1.92984624284786221971	1.223749e-18
(4.6)	0.31706177457295709503	0.31706177457295709503	1.524710e+00
Algorithm E (QG2)			
Equation	Exact Solution	Estimated solution	Error
(4.2)	1.20764782713091892701	1.20764782713091903889	2.415296e+00
(4.5)	1.92984624284786221849	1.92984624284786221849	2.469561e-21
(4.6)	0.31706177457295709503	0.31706177457295709503	1.524710e+00
Algorithm F (QG3)			
Equation	Exact Solution	Estimated solution	Error
(4.2)	1.20764782713091892701	1.20764782713091892701	2.415296e+00
(4.5)	1.92984624284786221849	1.92984624284784816872	7.221984e-01
(4.6)	0.31706177457295709503	0.31706177457295709503	1.524710e+00

Example 4.4 (Algebraic and Transedental equations)**.** *This example comprises of a few nonlinear equations which help us to examine the reliability and effectiveness of our new q−iterative methods.*

$$
\begin{aligned}
f_2(\tilde{u}) &= \tilde{u}^3 + 4\tilde{u}^2 - 10,\\
f_3(\tilde{u}) &= \tilde{u}^2 - e^{\tilde{u}} - 3\tilde{u} + 2,\\
f_4\tilde{u}) &= \tilde{u}^2 - (1-\tilde{u})^5,\\
f_5(\tilde{u}) &= \left(\tilde{u}^3 - 9\tilde{u}^2 + 24\tilde{u} - 20\right)^{1/3} + e^{\tilde{u}/2},\\
f_6(\tilde{u}) &= \sqrt[3]{\tilde{u}^3 - 3\tilde{u}^2} + \log(\tilde{u}+1),\\
f_7(\tilde{u}) &= 1000000e^{\tilde{u}} + 435000\left(\frac{e^{\tilde{u}}-1}{\tilde{u}}\right) - 1564000,\\
f_8(\tilde{u}) &= \frac{\tilde{u}}{\tilde{u}-1} - ln\Big[\frac{0.4(1-\tilde{u})}{0.4-0.5\tilde{u}}\Big]^5 + 4.45977,\\
f_9(\tilde{u}) &= \tilde{u}^4 + 11.50\tilde{u}^3 + 47.49\tilde{u}^2 + 83.06325\tilde{u} + 51.23266875.
\end{aligned}
$$

Some of these nonlinear equations are used by Chun [4] & Singh et al. [32] to validate the theoretical results. The last three numerical equations namely; $f_7(\tilde{u})$, $f_8(\tilde{u})$, $f_9(\tilde{u})$ represent some real-world applications of nonlinear equations. These nonlinear equations are the transformations of some mathematical models that appeared in science and engineering. The first one nonlinear equation f_7 is generated as a solution of mathematical modeling of the growth of population over short periods of time that can be written as in the form of differential equation:

$$\frac{d}{dt}(M(\tilde{t})) = \lambda M(\tilde{t}) + \nu, \tag{4.7}$$

where λ denotes the constant birth rate of population and $M(\tilde{t})$ denotes the number in the population at time $\tilde{t}$, for details (see [35]). The second nonlinear equation f_8 represents the physical constraint problem of fractional modification in a chemical reactor. The variable $\tilde{u}$ illustrates a fractional conversion of certain kind in a chemical reactor problem (see [39]). The value of $\tilde{u}$ is chosen between [0,1] because for negative values of $\tilde{u}$ the equation f_8 has no physical meaning. Therefore, within the limited reigion we have to select the initial guess carefully to find the real root of f_8. The third nonlinear equation f_9 is originated from the problem of the fraction conversion of nitrogen-hydrogen to ammonia which was investigated by [38]. The problem has the following form:

$$f(\tilde{u}) = -0.186 - \frac{8\tilde{u}^2(\tilde{u}-4)^2}{(\tilde{u}-2)^3}. \tag{4.8}$$

The values of temperature and pressure have been considered as 500°C and 250 atm respectively which can be easily reduced to the equation f_9.

The Table 14 reflects the comparable outcomes of classical and $q-$iterative methods by implementing the stoping criterium (4.1). From Table 14, it is also noted that we obtain identical results for both $q-$analogue of iterative methods (QG1, QG2 and QG3) and conventional classical iterative methods (CG1, CG2 and CG3). We consider two functions that are non-differentiable at points $\tilde{u}$=2, 3. When, we choose $\tilde{u} = 2, 3$ as an initial guesses for $f_7(\tilde{u})$ & $f_8(\tilde{u})$ consectively then novel schemes QG1, QG2 and QG3 are appropriately implemented and offer quick concurrent solutions whereas, the classical methods crash for these functions which is a major advantage of using $q-$iterative methods over the classical methods.

Table 14. Numerical comparison between different .
algorithms for test problems $f_2 - f_9$.

Methods	IT	$\tilde{u}_n$	$f(\tilde{u}_n)$	$\Delta = \lvert\tilde{u}_n - \tilde{u}_{n-1}\rvert$
		$f_2(\tilde{u}) = \tilde{u}^3 + 4\tilde{u}^2 - 10,\ \tilde{u}_0 = 1$		
NM	7	1.3652300134140968	4.708251e-87	2.411587e-44
CM	41	1.3652300134140968	2.134949e-22	3.631214e-12
NR	4	1.3652300134140968	9.105291e-55	6.121730e-19
CG1	5	1.3652300134140968	4.372042e-66	5.210944e-33
QG1	4	1.3652300134140968	7.767335e-18	3.542734e-13
CG2	3	1.3652300134140968	2.978405e-35	7.274667e-12
QG2	3	1.3652300134140968	2.360238e-22	8.108213e-12
CG3	3	1.3652300134140968	3.921312e-70	1.383883e-10
QG3	3	1.3652300134140968	1.894670e-34	4.902389e-18
		$f_3(\tilde{u}) = \tilde{u}^2 - e^{\tilde{u}} - 3\tilde{u} + 2,\ \tilde{u}_0 = 2$		
NM	7	0.2575302854398608	2.117415e-111	7.743422e-56
CM	8	0.2575302854398608	0.000000e+00	2.446258e-81
NR	5	0.2575302854398608	6.000000e-127	8.934699e-54
CG1	8	0.2575302854398608	8.465472e-43	6.144157e-22
QG1	8	0.2575302854398608	1.286482e-19	2.227650e-15
CG2	7	0.2575302854398608	0.000000e+00	3.075530e-85
QG2	5	0.2575302854398608	5.065368e-19	5.739453e-10
CG3	5	0.2575302854398608	0.000000e+00	3.707751e-57
QG3	4	0.2575302854398608	1.679693e-28	1.245252e-14

$f_4(\tilde{u}) = \tilde{u}^2 - (1-\tilde{u})^5$, $\tilde{u}_0 = 0.9$				
NM	8	0.3459548158482420	0.000000e+00	4.544036e-66
CM	8	0.3459548158482420	1.480844e-85	2.029382e-43
CG1	8	0.3459548158482420	0.000000e+00	2.610406e-88
QG1	6	0.3459548158482420	6.630895e-19	7.367776e-15
NR	5	0.3459548158482420	9.285837e-110	3.587286e-37
CG2	5	0.3459548158482420	0.000000e+00	1.403062e-54
QG2	4	0.3459548158482420	8.037173e-24	1.594632e-15
CG3	4	0.3459548158482420	0.000000e+00	1.331388e-45
QG3	3	0.3459548158482420	1.013570e-24	3.590701e-12
$f_5(\tilde{u}) = \left(\tilde{u}^3 - 9\tilde{u}^2 + 24u - 20\right)^{1/3} + e^{\tilde{u}/2}$, $\tilde{u}_0 = 2$				
NM		Fail		
CM		Fail		
CG1		Fail		
QG1	7	0.9694264485832314	1.547311e-19	1.547311e-19
CG2		Fail		
QG2	4	0.9694264485832314	2.510288e-18	1.084761e-09
CG3		Fail		
QG3	4	0.9694264485832314	1.619641e-29	2.055797e-16
$f_6(\tilde{u}) = \sqrt[3]{\tilde{u}^3 - 3\tilde{u}^2} + \log(\tilde{u}+1)$, $\tilde{u}_0 = 3$				
NM		Fail		
CM		Fail		
CG1		Fail		
QG1	9	2.6925176762621717	8.421577e-18	2.635399e-14
CG2		Fail		
QG2	6	2.6925176762621717	9.040507e-20	1.206573e-12
CG3		Fail		
QG3	5	2.6925176762621717	1.206573e-12	5.006635e-08
$f_7(\tilde{u}) = 1000000e^{\tilde{u}} + 435000\left(\frac{e^{\tilde{u}}-1}{\tilde{u}}\right) - 1564000$, $\tilde{u}_0 = 1.5$				
NM	7	0.1009979296857498	7.697779e-31	1.104193e-18
NR	5	0.1009979296857498	1.334042e-63	1.648664e-23
CM	6	0.1009979296857498	2.569791e-33	4.511244e-20
CG1	6	0.1009979296857498	8.119920e-73	4.614246e-39
QG1	5	0.1009979296857498	4.279781e-16	1.111148e-15
CG2	4	0.1009979296857498	1.016248e-49	1.818599e-18
QG2	3	0.1009979296857498	6.240159e-12	5.241915e-06
CG3	4	0.1009979296857498	0.000000e+00	1.034516e-33
QG3	3	0.1009979296857498	6.206248e-22	2.073657e-08
$f_8(\tilde{u}) = \frac{\tilde{u}}{\tilde{u}-1} - ln\left(\frac{0.4(1-\tilde{u})}{0.4-0.5\tilde{u}}\right)^5 + 4.45977$, $\tilde{u}_0 = 0.78$				
NM	10	0.7293818090058683	3.001866e-53	1.135988e-28
CG1	10	0.1009979296857498	1.331673e-54	2.417731e-29

QG1	10	0.7293818090058684	9.516168e-15	5.735822e-14
NR	7	0.7293818090058683	4.165525e-113	9.043351e-40
CM	8	0.7293818090058683	6.074152e-37	1.142630e-20
CG2	7	0.7293818090058683	8.629663e-63	4.291582e-23
QG2	7	0.7293818090058683	7.870433e-19	2.737598e-15
CG3	6	0.7293818090058683	1.251705e-88	2.211567e-24
QG3	5	0.7293818090058680	3.053887e-14	6.166940e-08
$f_9(\tilde{u}) = \tilde{u}^4 - 7.79075\tilde{u}^3 + 14.7445\tilde{u}^2 + 2.511\tilde{u} - 1.674$, $\tilde{u}_0 = 0.1$				
NM	7	0.2777595428417207	1.147712e-78	3.628854e-4
CM	6	div	div	div
NR	5	0.2777595428417207	1.000000e-127	1.985139e-50
CG1	6	0.2777595428417207	4.565893e-56	1.039764e-28
QG1	6	0.2777595428417207	0.000000e+00	8.647520e-46
CG2	5	0.2777595428417207	0.000000e+00	8.647520e-46
QG2	4	0.2777595428417207	9.989903e-24	6.521164e-15
CG3	3	0.2777595428417207	6.614501e-31	1.657701e-08
QG3	3	0.2777595428417207	3.291692e-22	1.652542e-08

Remark 4.2 (see [16])**.** *The efficiency index is calculated as* $\rho^{\frac{1}{k}}$ *where k determines the required number of estimations per iteration imperatively applied to a step of a recursive method and* ρ *symbolizes the convergence order of the method.*

- *Efficiency index of* CG1 *is* $2^{\frac{1}{4}} = 1.89207$.
- *Efficiency index of* CG2 *is* $3^{\frac{1}{5}} = 1.245731$.
- *Efficiency index of* CG3 *is* $4^{\frac{1}{6}} = 1.259921$.
- *Efficiency index of* QG1 *is* $2^{\frac{1}{4}} = 1.89207$.
- *Efficiency index of* QG2 *is* $3^{\frac{1}{5}} = 1.245731$.
- *Efficiency index of* QG3 *is* $4^{\frac{1}{6}} = 1.259921$.

Finally, we come to an end that the efficiency indexes evaluated by QG1*,* QG2*,* QG3 *and* CG1*,* CG2*,* CG3 *give identical outcomes.*

General formulation of the q-iterative schemes

This section consists of some previous results which are used for the derivation of the generality of the $q-$iterative methods. Now, combining the entries $u_k s'$ ($\forall\ k = 1, 2, 3, \ldots, n$) in (2.13), we acquire

$$\tilde{u}_1 + \tilde{u}_2 + \cdots + \tilde{u}_n = \tilde{u}_0 + N_q\big(\tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1}\big).$$

From (2.16), we have

$$\tilde{u}_0 = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa) f(q\kappa)},$$

and

$$\tilde{u}_0 + \tilde{u}_1 = \tilde{u}_0 - \frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa) f(q\kappa)} = \tilde{u}_0 + N_q(\tilde{u}_0),$$

$$\tilde{u} = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 = \tilde{u}_0 + \tilde{u}_1 - \frac{f(\tilde{u}_0 + \tilde{u}_1)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} = \tilde{u}_0 + N_q(\tilde{u}_0 + \tilde{u}_1).$$

Now, if $\tilde{u}$ is approximated by

$$\begin{aligned}
\tilde{u} &= \tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_n = \tilde{u}_0 + N_q(\tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1}),\\
\tilde{u} &= \tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1} - \frac{f(\tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1})g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},\\
\tilde{u} &= \tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_n = \tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1} - \frac{f(\tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1})g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},
\end{aligned} \tag{4.9}$$

$$\tilde{u}_n = -\frac{f(\tilde{u}_0 + \tilde{u}_1 + \cdots + \tilde{u}_{n-1})g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)} \qquad (\forall\ n = 1, 2, 3, \ldots). \tag{4.10}$$

Therefore, (4.9) gives the following iterative scheme

$$\begin{cases}
\tilde{u}_0 = \kappa - \frac{f(\kappa)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},\\
\tilde{u}_0 + \tilde{u}_1 = \tilde{u}_0 - \frac{f(\tilde{u}_0)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},\\
\tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 = \tilde{u}_0 + \tilde{u}_1 - \frac{f(\tilde{u}_0 + \tilde{u}_1)g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)},\\
\vdots\\
\tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + ... + \tilde{u}_{n-1} + \tilde{u}_n = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + ... + \tilde{u}_{n-1} - \frac{f(\tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + ... + \tilde{u}_{n-1})g(\kappa)}{D_q f(\kappa)g(\kappa) + D_q g(\kappa)f(q\kappa)}.
\end{cases}$$

This relation enables and allows us to propose the subsequent iterative method.

$$\begin{cases}
\tilde{v}_0 = \tilde{u}_m,\\
\tilde{v}_1 = y_0 - \frac{f(\tilde{v}_0)g(\tilde{u}_n)}{D_q f(\tilde{v}_o)g(\tilde{u}_n) + f(q\tilde{v}_o)D_q g(\tilde{u}_n)},\\
\tilde{v}_2 = y_1 - \frac{f(\tilde{v}_1)g(\tilde{u}_n)}{D_q f(\tilde{v}_o)g(\tilde{u}_n) + f(q\tilde{v}_o)D_q g(\tilde{u}_n)},\\
\tilde{v}_3 = y_2 - \frac{f(\tilde{v}_2)g(\tilde{u}_n)}{D_q f(\tilde{v}_o)g(\tilde{u}_n) + f(q\tilde{v}_o)D_q g(\tilde{u}_n)},\\
\vdots\\
\tilde{v}_{n+1} = \tilde{v}_n - \frac{f(\tilde{v}_n)g(\tilde{u}_n)}{D_q f(\tilde{v}_o)g(\tilde{u}_n) + f(q\tilde{v}_o)D_q g(\tilde{u}_n)},\\
\tilde{u}_{m+1} = \tilde{v}_{n+1}.
\end{cases} \tag{4.11}$$

This is convergent generalized $q-$iterative scheme which has order of convergence $n + 2$ for $n = 0, 1, 2, \ldots$ proof is simple and starightforward.

Remark 4.3. *System of nonlinear equations emerges when several scientific and technological challenges are involved. Many integral equations, boundary value problems, minimization problems, and variational problems may also be reduced to the system of nonlinear equations (see [40, 41, 44]). We consider the system of nonlinear equations of the form:*

$$f_1(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) = 0,$$
$$f_2(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) = 0,$$

$$f_3(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) = 0,$$
$$\vdots$$
$$f_n(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) = 0,$$

where each function f_j, $j = 1, 2, \ldots, n$ maps a vector $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n)$ of the n-dimensional space R^n to the real line R. The above system of n nonlinear equations in n-unknowns can also be represented as:

$$F(\tilde{U}) = (f_1(\tilde{U}), f_2(\tilde{U}), \ldots, f_n(\tilde{U}))^t = 0. \tag{4.12}$$

Where $F : R^n \to R^n$ be nonlinear mapping from n-dimensional real linear space R^n into itself. The components f_j, $j = 1, 2, ..., n$, are the coordinate functions of F. The solution of the nonlinear system of equations in (4.12) *may be defined as the process of finding a vector $U^* = (u_1^*, u_2^* \ldots, u_n^*)^t$ such that $F(U^*) = 0$. We feel it worthwhile to mention that considering the methodology and idea of this article, one can present and analyze higher-order multistep iterative methods for solving a system of nonlinear equations* (4.12). *It is an open problem to broaden the concept and ideas of this study for solving the boundary value problems and associated issues. This is another recommendation for prospective research.*

5. Conclusions and observations

Study and formulation of numerical results in quantum calculus induce interest due to the high demand in mathematics and easy implementation. This manuscript introduces some novel iterative schemes to find the estimated solution of nonlinear equations with success in quantum calculus. The key motivation of proposing $q-$iterative schemes is to overcome differentiability and convergence issues while getting solutions of algebraic equations. These new iterative schemes are applicable for different choices of an auxiliary function and derived by considering the valuable Daftardar-Jafari decomposition technique. We develop the comparative analysis of newly proposed methods with the traditional iterative methods to demonstrate the performance and efficiency of $q-$iterative schemes. Moreover, it is shown that the numerical results obtained for both conventional and $q-$iterative methods remain identical. Also, the errors connected with the suggested schemes are relatively marginal by selecting the value of q approaches to one. Hence, it is evident that the transformation of iterative methods in the $q-$calculus framework which we referred to as $q-$analogue of iterative schemes, is better than classical methods, and in limited cases when the parameter $q \to 1$, these methods reduces to the classical iterative methods. The significant challenge of dealing with these schemes which necessitate more exploration is that to get results with high accuracy, we must estimate the value of q in $(0, 1)$.

Our utilization here of the $q-$calculus in the development of the iterative methods are supposed to promote and motivate major future breakthroughs in Mathematical analysis. It is noticed that in (p, q) analysis the extra parameter p is clearly redundant, Srivastava (see [42, p. 340] and [43, pp. 1511-1512]; see also the related recent works [45, 46]) revealed that the so-called (p, q) variations of the suggested $q-$results which are obtained by inconsequentially and trivially adding a redundant parameter p as quite simple and insignificant modification of the standard $q-$calculus. Along these lines, while we reinforce and revitalize the $q-$results introduced in this paper, together with potential $q-$extensions of other similar developments in physical and engineering sciences, we do not encourage and support the so-called (p, q)-variations of the suggested $q-$results which are obtained by inconsequentially and trivially adding a redundant or superfluous parameter p.

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Conflict of interest

The authors agree with the contents of the manuscript and there is no conflict of interest among the authors.

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