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Research article

Dirichlet characters of the rational polynomials

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Abstract: Denote by χ a Dirichlet character modulo $q \geq 3$, and $\overline{a}$ means $a \cdot \overline{a} \equiv 1 \bmod q$. In this paper, we study Dirichlet characters of the rational polynomials in the form

$$\sideset{}{'}\sum_{a=1}^{q} \chi(ma + \overline{a}),$$

where $\sideset{}{'}\sum_{a=1}^{q}$ denotes the summation over all $1 \leq a \leq q$ with $(a, q) = 1$. Relying on the properties of character sums and Gauss sums, we obtain W. P. Zhang and T. T. Wang's identity [6] under a more relaxed situation. We also derive some new identities for the fourth power mean of it by adding some new ingredients.

Keywords: Dirichlet characters; Gauss sums; rational polynomials; fourth power mean; identity
Mathematics Subject Classification: 11L05, 11L10

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1. Introduction and main results

Let $q \geq 3$ be an integer, and χ be a Dirichlet character modulo q. The characters of the rational polynomial are defined as follows:

$$\sum_{x=N+1}^{N+M} \chi\left(f(x)\right),$$

where M and N are any given positive integers, and $f(x)$ is a rational polynomial. For example, when $f(x) = x$, for any non-principal Dirichlet character χ mod q, Pólya [1] and Vinogradov [2] independently proved that

$$\left|\sum_{x=N+1}^{N+M}\chi(x)\right| < \sqrt{q}\ln q,$$

and we call it Pólya-Vinogradov inequality.

When $q = p$ is an odd prime, χ is a p-th order character modulo p, Weil [3] proved

$$\sum_{x=N+1}^{N+M}\chi\left(f(x)\right) \ll p^{\frac{1}{2}}\ln p,$$

where $f(x)$ is not a perfect p-th power modulo p, $A \ll B$ denotes $|A| < kB$ for some constant k, which in this case depends on the degree of f.

Many authors have obtained numerous results for various forms of $f(x)$. For example, W. P. Zhang and Y. Yi [4] constructed a special polynomial as $f(x) = (x-r)^m(x-s)^n$ and deduced

$$\left|\sum_{a=1}^{q}\chi((a-r)^m(a-s)^n)\right| = \sqrt{q},$$

where $(r-s, q) = 1$, and χ is a primitive character modulo q. This shows the power of q in Weil's result is the best possible!

Also, when χ is a primitive character mod q, W. P. Zhang and W. L. Yao [5] obtained

$$\sum_{a=1}^{q}\chi(a^m(1-a)^m) = \sqrt{q}\overline{\chi}(4^m),$$

where q is an odd perfect square and m is any positive integer with $(m, q) = 1$.

When $q = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ is a square full number with $p_i \equiv 3 \bmod 4$, $\chi = \chi_1\chi_2\ldots\chi_s$ with χ_i being any primitive even character mod $p_i^{\alpha_i}(i = 1, 2, \ldots, s)$, W. P. Zhang and T. T. Wang [6] obtained the identity

$$\left|{\sum_{a=1}^{q}}'\chi(ma^{2^k-1}+n\overline{a})\right| = \sqrt{q}\prod_{p|q}\left(1+\left(\frac{mn(2^k-1)}{p}\right)\right), \tag{1.1}$$

where $a\cdot\overline{a} \equiv 1 \bmod q$, and $\left(\frac{*}{p}\right)$ denotes the Legendre symbol. Besides, k, m and n also satisfying some special conditions. Other related work about Dirichlet characters of the rational polynomials can be found in references [7–14]. Inspired by these, we will study the sum

$${\sum_{a=1}^{q}}'\chi(ma+\overline{a}).$$

Following the way in [6], we obtain W. P. Zhang and T. T. Wang's identity (1.1) under a more relaxed situation. Then by adding some new ingredients, we derive some new identities for the fourth power mean of it.

Noting that if χ is an odd character modulo q, m is a positive integer with $(m,q)=1$, we can get

$$\sum_{a=1}^{q}{}' \chi(ma+\overline{a}) = \sum_{a=1}^{q}{}' \chi(-ma+\overline{(-a)}) = -\sum_{a=1}^{q}{}' \chi(ma+\overline{a}).$$

That is to say, under this condition,

$$\sum_{a=1}^{q}{}' \chi(ma+\overline{a}) = 0.$$

So, we will only discuss the case of χ an even character. To the best of our knowledge, the following identities dealing with arbitrary odd square-full number cases are new and have not appeared before.

Theorem 1.1. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd square-full number, χ_i be any primitive even character mod $p_i^{\alpha_i}$ $(i = 1, 2, \ldots, s)$ and $\chi = \chi_1\chi_2\cdots\chi_s$. Then for any integer m with $(m,q)=1$, we have the identity

$$\left|\sum_{a=1}^{q}{}' \chi(ma+\overline{a})\right| = \sqrt{q}\prod_{p\mid q}\left(1+\left(\frac{m}{p}\right)\right),$$

where $\prod_{p\mid q}$ denotes the product over all distinct prime divisors p of q.

Remark 1.1. It is obvious that Theorem 1.1 is W. P. Zhang and T. T. Wang's identity (1.1) with $k=n=1$ by removing the condition $p_i \equiv 3 \bmod 4$ $(i = 1, 2, \ldots, s)$. Besides, using our results, we can directly obtain the absolute values of the sums of Dirichlet characters satisfying some conditions, which avoids complex calculations. What's more, the result of Theorem 1.1 also shows that the order of q in Weil's result can not be improved.

To understand the result better, we give the following examples:

Example 1.1. Let $q = 3^2$, χ be a Dirichlet character modulo 9 defined as follows:

$$\chi(n) = \begin{cases} e^{\frac{2\pi i\cdot \mathrm{ind}_2 n}{3}}, & \text{if } (n,9)=1; \\ 0, & \text{if } (n,9)>1. \end{cases}$$

Obviously, χ is a primitive even character modulo 9. Taking $m = 1, 2$, then we have

$$\left|\sum_{a=1}^{9}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{9}{}' \chi(a+\overline{a})\right| = |3\chi(2)+3\chi(7)| = |3e^{\frac{2\pi i}{3}} + 3e^{\frac{2\pi i\cdot 4}{3}}| = 6,$$

$$\left|\sum_{a=1}^{9}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{9}{}' \chi(2a+\overline{a})\right| = |2\chi(3)+2\chi(6)+2\chi(9)| = 0.$$

Example 1.2. Let $q = 5^2$, χ be a primitive even character modulo 25 defined as follows:

$$\chi(n) = \begin{cases} e^{\frac{2\pi i\cdot \mathrm{ind}_2 n}{5}}, & \text{if } (n,25)=1; \\ 0, & \text{if } (n,25)>1. \end{cases}$$

Taking $m = 1, 2$, then we have

$$\left|\sum_{a=1}^{25}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{25}{}' \chi(a+\overline{a})\right| = |5\chi(2) + 5\chi(23)| = |5e^{\frac{2\pi i}{5}} + 5e^{\frac{2\pi i\cdot 11}{5}}| = 10,$$

$$\left|\sum_{a=1}^{25}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{25}{}' \chi(2a+\overline{a})\right| = |4\chi(2) + 4\chi(3) + 4\chi(7) + 4\chi(8) + 4\chi(12)|$$

$$= \left|4e^{\frac{2\pi i}{5}} + 4e^{\frac{2\pi i\cdot 7}{5}} + 4e^{\frac{2\pi i\cdot 5}{5}} + 4e^{\frac{2\pi i\cdot 3}{5}} + 4e^{\frac{2\pi i\cdot 9}{5}}\right| = 0.$$

Example 1.3. Let $q = 13^2$, χ be a primitive even character modulo 169 defined as follows:

$$\chi(n) = \begin{cases} e^{\frac{2\pi i\cdot \mathrm{ind}_2 n}{13}}, & \text{if } (n, 169) = 1; \\ 0, & \text{if } (n, 169) > 1. \end{cases}$$

Taking $m = 1, 2$, then we have

$$\left|\sum_{a=1}^{169}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{169}{}' \chi(a+\overline{a})\right| = |4\chi(1) + 26\chi(2) + 4\chi(4) + 4\chi(9) + 4\chi(12) + 4\chi(14) + 4\chi(17)$$
$$+ 4\chi(22) + 4\chi(25) + 4\chi(27) + 4\chi(30) + 4\chi(35) + 4\chi(38) + 4\chi(40) + 4\chi(43) + 4\chi(48) + 4\chi(51)$$
$$+ 4\chi(53) + 4\chi(56) + 4\chi(61) + 4\chi(64) + 4\chi(66) + 4\chi(69) + 4\chi(74) + 4\chi(77) + 4\chi(79) + 4\chi(82)|$$
$$= \left|8 + 8e^{\frac{\pi i}{13}} + 34e^{\frac{2\pi i}{13}} + 8e^{\frac{3\pi i}{13}} + 8e^{\frac{4\pi i}{13}} + 8e^{\frac{5\pi i}{13}} + 8e^{\frac{6\pi i}{13}} + 8e^{\frac{7\pi i}{13}} + 8e^{\frac{8\pi i}{13}} + 8e^{\frac{9\pi i}{13}} + 8e^{\frac{10\pi i}{13}} + 8e^{\frac{11\pi i}{13}} + 8e^{\frac{12\pi i}{13}}\right|$$
$$= 26,$$

$$\left|\sum_{a=1}^{169}{}' \chi(ma+\overline{a})\right| = \left|\sum_{a=1}^{169}{}' \chi(2a+\overline{a})\right| = |4\chi(2) + 4\chi(3) + 4\chi(5) + 4\chi(8) + 4\chi(10) + 4\chi(11) + 4\chi(15)$$
$$+ 4\chi(16) + 4\chi(18) + 4\chi(21) + 4\chi(23) + 4\chi(24) + 4\chi(28) + 4\chi(29) + 4\chi(31) + 4\chi(34) + 4\chi(36)$$
$$+ 4\chi(37) + 4\chi(41) + 4\chi(42) + 4\chi(44) + 4\chi(47) + 4\chi(49) + 4\chi(50) + 4\chi(54) + 4\chi(55) + 4\chi(57)$$
$$+ 4\chi(60) + 4\chi(62) + 4\chi(63) + 4\chi(67) + 4\chi(68) + 4\chi(70) + 4\chi(73) + 4\chi(75) + 4\chi(76) + 4\chi(80)$$
$$+ 4\chi(81) + 4\chi(83)|$$
$$= |12 + 12e^{\frac{\pi i}{13}} + 12e^{\frac{2\pi i}{13}} + 12e^{\frac{3\pi i}{13}} + 12e^{\frac{4\pi i}{13}} + 12e^{\frac{5\pi i}{13}} + 12e^{\frac{6\pi i}{13}} + 12e^{\frac{7\pi i}{13}} + 12e^{\frac{8\pi i}{13}} + 12e^{\frac{9\pi i}{13}} + 12e^{\frac{10\pi i}{13}}$$
$$+ 12e^{\frac{11\pi i}{13}} + 12e^{\frac{12\pi i}{13}}| = 0.$$

The above examples can be easily achieved by our Theorem 1.1. From Theorem 1.1, we may immediately obtain the following two corollaries:

Corollary 1.1. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd square-full number, χ_i be any primitive even character mod p_i^{α} $(i = 1, 2, \ldots, s)$ and $\chi = \chi_1\chi_2\cdots\chi_s$. Then for any integer m with $(m, q) = 1$, we have the identity

$$\left|\sum_{a=1}^{q}{}' \chi(ma+\overline{a})\right| = \begin{cases} 2^{\omega(q)}\sqrt{q}, & \text{if } m \text{ is a quadratic residue modulo } q; \\ 0, & \text{otherwise,} \end{cases}$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

Corollary 1.2. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd number with $\alpha_i \geq 1$ ($i = 1, 2, \ldots, s$), χ_i be any primitive even character mod $p_i^{\alpha_i}$ and $\chi = \chi_1 \chi_2 \cdots \chi_s$. Then for any integer m with $(m, q) = 1$, we have the inequality

$$\left| \sum_{a=1}^{q}{}' \chi(ma + \overline{a}) \right| \leq 2^{\omega(q)} \sqrt{q}.$$

Theorem 1.2. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd square-full number, χ_i be any primitive even character mod $p_i^{\alpha_i}$ ($i = 1, 2, \ldots, s$) and $\chi = \chi_1 \chi_2 \cdots \chi_s$. Then for any integers k and m with $k \geq 1$ and $(m, q) = 1$, we have the identity

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{q}{}' \chi(ma + \overline{a}) \right|^{2k} = \frac{q^k}{2^{\omega(q)}} J(q) \prod_{p|q} \left(1 + \left(\frac{m}{p} \right) \right)^{2k},$$

where $J(q)$ denotes the number of primitive characters modulo q, and $\sum\limits_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q.

Example 1.4. Taking $q = 5^2$, $m = 1, 2$, then we have

$$\sum_{\substack{\chi \bmod 25 \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{25}{}' \chi(ma + \overline{a}) \right|^{2k} = \sum_{\substack{\chi \bmod 25 \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{25}{}' \chi(a + \overline{a}) \right|^{2k} = 8 \cdot 10^{2k},$$

$$\sum_{\substack{\chi \bmod 25 \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{25}{}' \chi(ma + \overline{a}) \right|^{2k} = \sum_{\substack{\chi \bmod 25 \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{25}{}' \chi(2a + \overline{a}) \right|^{2k} = 0,$$

which can be easily achieved by our Theorem 1.2.

Taking $k = 2$ in Theorem 1.2, we may immediately obtain the followings:

Corollary 1.3. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd square-full number, χ_i be any primitive even character mod $p_i^{\alpha_i}$ ($i = 1, 2, \ldots, s$) and $\chi = \chi_1 \chi_2 \cdots \chi_s$. Then for any integer m with $(m, q) = 1$, we have the identity

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{q}{}' \chi(ma + \overline{a}) \right|^{4} = \frac{q^2}{2^{\omega(q)}} J(q) \prod_{p|q} \left(1 + \left(\frac{m}{p} \right) \right)^{4}.$$

Corollary 1.4. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be an odd square-full number, χ_i be any primitive even character mod $p_i^{\alpha_i}$ ($i = 1, 2, \ldots, s$) and $\chi = \chi_1 \chi_2 \cdots \chi_s$. Then we have the identity

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \left| \sum_{a=1}^{q}{}' \chi(ma + \overline{a}) \right|^{4} = \begin{cases} 8^{\omega(q)} q^2 J(q), & \text{if } m \text{ is a quadratic residue modulo } q; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.3. Let p be an odd prime, χ be any non-principal character mod p. Then for any integer m with $(m,p)=1$, we have the identity

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|\sum_{a=1}^{p-1}\chi(ma+\overline{a})\right|^4=\begin{cases}2p^3-6p^2+4-4(p^2-3p+2)\left(\dfrac{m}{p}\right)+(p-1)E, & \text{if } p\equiv 3 \bmod 4;\\ 2p^3-6p^2+4-4(p^2+p-2)\left(\dfrac{m}{p}\right)+(p-1)E, & \text{if } p\equiv 1 \bmod 4,\end{cases}$$

where

$$E=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)b}{p}\right)\sum_{d=1}^{p-1}\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right).$$

Remark 1.2. From [8], we know that when $f(x)$ is a polynomial of odd degree $n\geq 3$, Weil's estimate ([15,16])

$$\left|\sum_{x=0}^{p-1}\left(\frac{f(x)}{p}\right)\right|\leq (n-1)\sqrt{p},$$

implies that $E<4p^2-8p$. Noting that $\sum_{a=1}^{q}{}'\chi(ma+\overline{a})$ can be regarded as a dual form of Kloosterman sums, which defined as $\sum_{a=1}^{q}{}' e^{2\pi i\frac{ma+\bar{a}}{q}}$, we can obtain some distributive properties of $\sum_{a=1}^{q}{}'\chi(ma+\overline{a})$ from Theorem 1.2 and 1.3.

From Theorem 1.3, we also have the following corollaries:

Corollary 1.5. Let p be an odd prime, χ be any non-principal character mod p. Then for any quadratic residue m mod p, we have the identity

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|\sum_{a=1}^{p-1}\chi(ma+\overline{a})\right|^4=\begin{cases}2p^3-10p^2+12p-4+(p-1)E, & \text{if } p\equiv 3 \bmod 4;\\ 2p^3-10p^2-4p+12+(p-1)E, & \text{if } p\equiv 1 \bmod 4.\end{cases}$$

Corollary 1.6. Let p be an odd prime, χ be any non-principal character mod p. Then for any quadratic non-residue m mod p, we have the identity

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|\sum_{a=1}^{p-1}\chi(ma+\overline{a})\right|^4=\begin{cases}2p^3-2p^2-12p+4+(p-1)E, & \text{if } p\equiv 3 \bmod 4;\\ 2p^3-2p^2+4p-4+(p-1)E, & \text{if } p\equiv 1 \bmod 4.\end{cases}$$

2. Some Lemmas

To prove our Theorems, we need some Lemmas as the following:

Lemma 2.1. Let q, q_1, q_2 be integers with $q=q_1q_2$ and $(q_1,q_2)=1$, χ_i be any non-principal character mod q_i $(i=1,2)$. Then for any integer m with $(m,q)=1$ and $\chi=\chi_1\chi_2$, we have the identity

$$\sum_{a=1}^{q}{}'\chi(ma+\overline{a})=\sum_{b=1}^{q_1}{}'\chi_1\left(mb+\overline{b}\right)\sum_{c=1}^{q_2}{}'\chi_2\left(mc+\overline{c}\right).$$

Proof. From the properties of Dirichlet characters, we have

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \chi(ma+\overline{a})\\
=&\sum_{a=1}^{q_1q_2}{}' \chi_1\chi_2(ma+\overline{a})\\
=&\sum_{b=1}^{q_1}{}' \sum_{c=1}^{q_2}{}' \chi_1\chi_2\left(m(bq_2+cq_1)+\overline{bq_2+cq_1}\right)\\
=&\sum_{b=1}^{q_1}{}' \sum_{c=1}^{q_2}{}' \chi_1\left(m(bq_2+cq_1)+\overline{bq_2+cq_1}\right)\chi_2\left(m(bq_2+cq_1)+\overline{bq_2+cq_1}\right)\\
=&\sum_{b=1}^{q_1}{}' \chi_1\left(mbq_2+\overline{bq_2}\right)\sum_{c=1}^{q_2}{}' \chi_2\left(mcq_1+\overline{cq_1}\right)\\
=&\sum_{b=1}^{q_1}{}' \chi_1\left(mb+\overline{b}\right)\sum_{c=1}^{q_2}{}' \chi_2\left(mc+\overline{c}\right).
\end{aligned}$$

This completes the proof of Lemma 2.1.

Lemma 2.2. Let p be an odd prime, α and m be integers with $\alpha \geq 1$ and $(m,p)=1$. Then for any primitive even character χ mod p^α, we have the identity

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma+\overline{a}) = \frac{\chi_1(m)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\left(1+\chi_2^0(m)\frac{\tau^2(\chi_2^0\overline{\chi}_1)}{\tau^2(\overline{\chi}_1)}\right),$$

where $\chi_2^0=\left(\frac{*}{p}\right)$, $\tau(\chi)=\sum_{a=1}^{p^\alpha}\chi(a)e\left(\frac{a}{p^\alpha}\right)$, χ_1 is a primitive character mod p^α and $\chi=\chi_1^2$.

Proof. For any primitive even character χ mod p^α, there exists one primitive character χ_1 mod p^α such that $\chi=\chi_1^2$. From the properties of Gauss sum, we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p^\alpha}{}' \chi(ma+\overline{a})\\
=&\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b(ma+\overline{a})}{p^\alpha}\right)\\
=&\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p^\alpha}\overline{\chi}(a)\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b(ma^2+1)}{p^\alpha}\right)\\
=&\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\overline{\chi}(a)e\left(\frac{bma^2}{p^\alpha}\right)\\
=&\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\overline{\chi}_1(a^2)e\left(\frac{bma^2}{p^\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\left(1+\chi_2^0(a)\right)\overline{\chi}_1(a)e\left(\frac{bma}{p^\alpha}\right)\\
&=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\overline{\chi}_1(a)e\left(\frac{bma}{p^\alpha}\right)\\
&\quad+\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\chi_2^0(a)\overline{\chi}_1(a)e\left(\frac{bma}{p^\alpha}\right)\\
&:=B_1+B_2.
\end{aligned}
$$

Now we compute B_1 and B_2 respectively.

$$
\begin{aligned}
B_1 &=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\overline{\chi}_1(a)e\left(\frac{bma}{p^\alpha}\right)\\
&=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)\chi_1(bm)e\left(\frac{b}{p^\alpha}\right)\sum_{a=1}^{p^\alpha}\overline{\chi}_1(bma)e\left(\frac{bma}{p^\alpha}\right)\\
&=\frac{\chi_1(m)\tau(\overline{\chi}_1)}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}(b)\chi_1(b)e\left(\frac{b}{p^\alpha}\right)\\
&=\frac{\chi_1(m)\tau(\overline{\chi}_1)}{\tau(\overline{\chi})}\sum_{b=1}^{p^\alpha}\overline{\chi}_1(b)e\left(\frac{b}{p^\alpha}\right)\\
&=\frac{\chi_1(m)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}.
\end{aligned}
$$

Similarly, we have

$$
B_2=\frac{\chi_1(m)\chi_2^0(m)\tau^2(\chi_2^0\overline{\chi}_1)}{\tau(\overline{\chi})}.
$$

Therefore, we can obtain

$$
\sum_{a=1}^{p^\alpha}{}' \chi(ma+\overline{a})=\frac{\chi_1(m)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\left(1+\chi_2^0(m)\frac{\tau^2(\chi_2^0\overline{\chi}_1)}{\tau^2(\overline{\chi}_1)}\right).
$$

Lemma 2.3. Let p be an odd prime. Then for any integer n, we have the identity

$$
\sum_{a=1}^{p}\left(\frac{a^2+n}{p}\right)=\begin{cases}-1, & \text{if } (n,p)=1;\\ p-1, & \text{if } (n,p)=p.\end{cases}
$$

Proof. See Theorem 8.2 of [17].

Lemma 2.4. Let p be an odd prime. Then we have the identity

$$
\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)b}{p}\right)=2\times(-1)^{\frac{p-1}{2}}+2.
$$

Proof. From the properties of character sum, we have

$$
\begin{aligned}
&\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)b}{p}\right)\\
=&\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\sum_{a=2}^{p-2}\left(\frac{(a^2b-1)b}{p}\right)\\
=&\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\sum_{a=2}^{p-2}\left(\frac{b^2(a^2-\overline{b})}{p}\right)\\
=&\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\sum_{a=2}^{p-2}\left(\frac{a^2-\overline{b}}{p}\right)\\
=&\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\left(\sum_{a=1}^{p}\left(\frac{a^2-\overline{b}}{p}\right)-\left(\frac{1-\overline{b}}{p}\right)-\left(\frac{(p-1)^2-\overline{b}}{p}\right)-\left(\frac{p^2-\overline{b}}{p}\right)\right)\\
=&\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\left(-1-2\left(\frac{1-\overline{b}}{p}\right)-\left(\frac{-\overline{b}}{p}\right)\right)\\
=&-\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)-2\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\left(\frac{1-\overline{b}}{p}\right)-\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\left(\frac{-\overline{b}}{p}\right)\\
=&-\sum_{b=0}^{p-2}\left(\frac{b}{p}\right)-2\sum_{b=1}^{p-1}\left(\frac{b-1}{p}\right)\left(\frac{(1-\overline{b})b^2}{p}\right)-\sum_{b=1}^{p-1}\left(\frac{\overline{b}-1}{p}\right)\\
=&-2\sum_{b=0}^{p-2}\left(\frac{b}{p}\right)-2\sum_{b=1}^{p-1}\left(\frac{(b-1)^2b}{p}\right)\\
=&-2\left(\sum_{b=0}^{p-1}\left(\frac{b}{p}\right)-\left(\frac{p-1}{p}\right)\right)-2\times(-1)\\
=&2\times(-1)^{\frac{p-1}{2}}+2.
\end{aligned}
$$

This completes the proof of Lemma 2.4.

3. Proof of Theorems

Now we come to prove our Theorems.

Firstly, we prove Theorem 1.1. With the help of Lemma 2 in [6], when $\alpha \geq 2$, we have

$$\frac{\tau^2(\chi_2^0\overline{\chi}_1)}{\tau^2(\overline{\chi}_1)} = \left(\frac{1}{p}\right)^2 = 1,$$

which implies from Lemma 2.2, we can obtain

$$\left|{\sum_{a=1}^{p^\alpha}}'\chi(ma+\overline{a})\right| = \left|\frac{\chi_1(m)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\left(1+\left(\frac{m}{p}\right)\right)\right| = \sqrt{p^\alpha}\left(1+\left(\frac{m}{p}\right)\right).$$

Then, applying Lemma 2.1, we can obtain

$$\begin{aligned}
&\left|\sum_{a=1}^{q}{}' \chi(ma+\overline{a})\right| \\
=&\left|\sum_{a_1=1}^{p_1^{\alpha_1}}{}' \chi_1(ma_1+\overline{a_1})\right|\cdots\left|\sum_{a_s=1}^{p_s^{\alpha_s}}{}' \chi_s(ma_s+\overline{a_s})\right| \\
=&\sqrt{q}\prod_{p|q}\left(1+\left(\frac{m}{p}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 1.1.

Then, from Lemma 2.1 and Lemma 2.2, we can prove Theorem 1.2 as following:

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}^{*}\left|\sum_{a=1}^{q}{}' \chi(ma+\overline{a})\right|^{2k} \\
=&\sum_{\substack{\chi_1 \bmod p_1^{\alpha_1}\\ \chi_1(-1)=1}}^{*}\left|\sum_{a_1=1}^{p_1^{\alpha_1}}{}' \chi_1(ma_1+\overline{a_1})\right|^{2k}\cdots\sum_{\substack{\chi_s \bmod p_s^{\alpha_s}\\ \chi_s(-1)=1}}^{*}\left|\sum_{a_s=1}^{p_s^{\alpha_s}}{}' \chi_s(ma_s+\overline{a_s})\right|^{2k} \\
=&\prod_{i=1}^{s}\left[\frac{1}{2}J(p_i^{\alpha_i})p_i^{k\alpha_i}\left|1+\left(\frac{m}{p_i}\right)\right|^{2k}\right] \\
=&\frac{q^k}{2^{\omega(q)}}J(q)\prod_{p|q}\left(1+\left(\frac{m}{p}\right)\right)^{2k}.
\end{aligned}$$

Finally, we prove Theorem 1.3. For any integer m with $(m,p)=1$, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\chi(ma+\overline{a}) \\
=&\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ am+\overline{a}\equiv u \bmod p}}^{p-1}1 \\
=&\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ a^2m^2-amu+m\equiv 0 \bmod p}}^{p-1}1 \\
=&\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ (2am-u)^2\equiv u^2-4m \bmod p}}^{p-1}1 \\
=&\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ a^2\equiv u^2-4m \bmod p}}^{p-1}1
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{u=1}^{p-1}\chi(u)\left(1+\left(\frac{u^2-4m}{p}\right)\right)\\
&=\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-4m}{p}\right)\\
&=\chi(2)\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-m}{p}\right).
\end{aligned}
$$

So from the orthogonality of Dirichlet characters and the properties of reduced residue system modulo p, we have

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|\sum_{a=1}^{p-1}\chi(ma+\overline{a})\right|^4\\
=&\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|\chi(2)\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-m}{p}\right)\right|^2\left|\chi(2)\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-m}{p}\right)\right|^2\\
=&\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ac\overline{bd})\left(\frac{a^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{c^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\\
=&\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ac)\left(\frac{a^2b^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{c^2d^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^2b^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{c^2d^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\chi(ac)\\
=&\frac{p-1}{2}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a\equiv\overline{c} \bmod p}}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^2b^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{c^2d^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\\
&+\frac{p-1}{2}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a\equiv-\overline{c} \bmod p}}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^2b^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{c^2d^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\\
=&(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^2b^2-m}{p}\right)\left(\frac{b^2-m}{p}\right)\left(\frac{\overline{a}^2d^2-m}{p}\right)\left(\frac{d^2-m}{p}\right)\\
=&(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(1+\left(\frac{b}{p}\right)\right)\left(\frac{a^2b-m}{p}\right)\left(\frac{b-m}{p}\right)\sum_{d=1}^{p-1}\left(1+\left(\frac{d}{p}\right)\right)\left(\frac{\overline{a}^2d-m}{p}\right)\left(\frac{d-m}{p}\right)\\
=&(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\sum_{d=1}^{p-1}\left(\frac{\overline{a}^2d-1}{p}\right)\left(\frac{d-1}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)\\
&+(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(a^2b-1)(b-1)b}{p}\right)\sum_{d=1}^{p-1}\left(\frac{\overline{a}^2d-1}{p}\right)\left(\frac{d-1}{p}\right)\\
&+(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(a^2b-1)(b-1)b}{p}\right)\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)\\
:=&A_1+A_2+A_3+A_4.
\end{aligned}
$$

Now we compute A_1, A_2, A_3, A_4 respectively. Noticing that $\chi(-1)=1$, from the properties of the complete residue system modulo p, we have

$$
\begin{aligned}
&\sum_{b=1}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\\
=&\sum_{b=0}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)-1\\
=&\sum_{b=0}^{p-1}\left(\frac{4a^2}{p}\right)\left(\frac{(a^2b-1)(b-1)}{p}\right)-1\\
=&\sum_{b=0}^{p-1}\left(\frac{(2a^2b-a^2-1)^2-(a^2-1)^2}{p}\right)-1\\
=&\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)-1.
\end{aligned}
$$

Applying Lemma 2.3, we can get

$$
\begin{aligned}
A_1=&(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\sum_{d=1}^{p-1}\left(\frac{\overline{a}^2d-1}{p}\right)\left(\frac{d-1}{p}\right)\\
=&(p-1)\sum_{a=1}^{p-1}\left(\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)-1\right)\left(\sum_{d=0}^{p-1}\left(\frac{d^2-(\overline{a}^2-1)^2}{p}\right)-1\right)\\
=&(p-1)\left[2\sum_{b=0}^{p-1}\left(\frac{b^2}{p}\right)\sum_{d=0}^{p-1}\left(\frac{d^2}{p}\right)+\sum_{a=2}^{p-2}\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)\sum_{d=0}^{p-1}\left(\frac{d^2-(\overline{a}^2-1)^2}{p}\right)\right]\\
&-2(p-1)\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)+(p-1)^2\\
=&2p^3-6p^2+4.
\end{aligned}
$$

Then, we compute A_2. With the aid of Lemma 2.4, we have

$$
A_2=(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)
$$

$$
\begin{aligned}
=&(p-1)\sum_{a=1}^{p-1}\left[\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)-1\right]\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)\\
=&(p-1)^2\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(d-1)^2d}{p}\right)-(p-1)\sum_{a=2}^{p-2}\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)\\
&+(p-1)^2\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{\left((p-1)^2d-1\right)(d-1)d}{p}\right)-(p-1)\sum_{a=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right)\\
=&(p^2-3p+2)\left[\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(d-1)^2d}{p}\right)+\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{\left((p-1)^2d-1\right)(d-1)d}{p}\right)\right]\\
&-2(p-1)\sum_{a=2}^{p-2}\sum_{d=1}^{p-1}\left(\frac{m}{p}\right)\left(\frac{(a^2d-1)(d-1)d}{p}\right)\\
=&2(p^2-3p+2)\left(\frac{m}{p}\right)\sum_{d=1}^{p-1}\left(\frac{(d-1)^2d}{p}\right)-4(p-1)\left[(-1)^{\frac{p-1}{2}}+1\right]\left(\frac{m}{p}\right)\\
=&2(p^2-3p+2)\left(\frac{m}{p}\right)\sum_{b=2}^{p-1}\left(\frac{b}{p}\right)-4(p-1)\left[(-1)^{\frac{p-1}{2}}+1\right]\left(\frac{m}{p}\right)\\
=&-2(p^2-3p+2)\left(\frac{m}{p}\right)-4(p-1)\left[(-1)^{\frac{p-1}{2}}+1\right]\left(\frac{m}{p}\right).
\end{aligned}
$$

Similarly, we have

$$
A_3=-2(p^2-3p+2)\left(\frac{m}{p}\right)-4(p-1)\left[(-1)^{\frac{p-1}{2}}+1\right]\left(\frac{m}{p}\right).
$$

Note that

$$
A_4=(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)b}{p}\right)\sum_{d=1}^{p-1}\left(\frac{(\overline{a}^2d-1)(d-1)d}{p}\right),
$$

which completes the proof of Theorem 1.3.

4. Conclusions

Three Theorems are stated in the main results. The Theorem 1.1 obtains an exact computational formula for $\sum_{a=1}^{q}{}' \chi(ma+\overline{a})$, which broadens the scope of q by removing the condition $p\equiv 3 \bmod 4$ in the previous article, where p is the prime divisor of q. The Theorem 1.2 derives a new identity for the mean value of it by adding some different ingredients. What's more, the Theorem 1.3 bridges the fourth power of Dirichlet characters with Legendre symbols of certain polynomials, which may be useful in the related future research. However, due to some technical reasons, we can only deal with the odd square-full number q case.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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