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Research article

On a class of differential inclusions in the frame of generalized Hilfer fractional derivative

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Abstract: In the present paper, we extend and develop a qualitative analysis for a class of nonlinear fractional inclusion problems subjected to nonlocal integral boundary conditions (nonlocal IBC) under the φ-Hilfer operator. Both claims of convex valued and nonconvex valued right-hand sides are investigated. The obtained existence results of the proposed problem are new in the frame of a φ-Hilfer fractional derivative with nonlocal IBC, which are derived via the fixed point theorems (FPT's) for set-valued analysis. Eventually, we give some illustrative examples for the acquired results.

Keywords: fractional differential inclusions; φ-Hilfer fractional derivative; existence; fixed point theorem
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1. Introduction

The theory of fractional differential equations (FDEs) and fractional differential inclusions (FDIs) have recently received significant attention in various fields of engineering and science, see [1–4], with many applications to name a few [5–14]. Recently, many diverse definitions of fractional derivatives (or fractional integrals) (FDs or FIs), the most common of which are Riemann-Liouville [2], Caputo [15] and Hilfer [10], have emerged. This is followed by numerous

generalized fractional operators [16–21]. Moreover, new fabulous generalizations have emerged that combine a broad classes of the aforementioned fractional operators such as φ-Caputo [22], and φ-Hilfer [23].

Over the years, many researchers are interested in debating the qualitative analysis of FDEs and FDIs like existence, uniqueness, controllability, stability, and optimizations, etc, see [24–33]. Some authors have consecrated their efforts to debate more qualitative analysis of this kinds of equations and inclusions, while others focused on applications and numerical solutions. A lot of related articles about the existence, and uniqueness of FDEs (FDIs) under the different types of FDs, can be found at [34–42]. For the recent development of fractional calculus theory and the importance of application of Hilfer FD, see [43–45].

The authors in [46] have started the investigation of the following Hilfer-type FDEs

$$\begin{cases} {}^{H}\mathfrak{D}^{\varrho_1,\varrho_2}\phi(\tau) = f(\tau,\phi(\tau)),\ \tau\in[\mathbf{a},b],\ 1<\varrho_1<2,\ 0\le\varrho_2\le 1,\\ \phi(\mathbf{a}) = 0,\ \phi(b) = \sum\limits_{i=1}^{m}\delta_i\mathfrak{I}^{\lambda_i}_{\mathbf{a}+}\phi(\theta_i),\ \theta_i\in[\mathbf{a},b], \end{cases} \tag{1.1}$$

where $\lambda_i > 0$, $\delta_i \in \mathbb{R}$, ${}^{H}\mathfrak{D}^{\varrho_1,\varrho_2}$ and $\mathfrak{I}^{\lambda_i}_{\mathbf{a}+}$ are the Hilfer FD of order (ϱ_1,ϱ_2) and the Riemann-Liouville FI of order λ_i,respectively. The existence and stability of solutions for implicit-type FDEs (1.1) in the ψ-Hilfer FD sense have been investigated by [47]. In this regard, Wongcharoen et al., in [48] studied the problem (1.1) with set-valued case, that is

$$\begin{cases} {}^{H}\mathfrak{D}^{\varrho_1,\varrho_2}\phi(\tau)\in\mathbb{F}(\tau,\phi(\tau)),\ \tau\in[\mathbf{a},b],\\ \phi(\mathbf{a}) = 0,\ \phi(b) = \sum\limits_{i=1}^{m}\delta_i\mathfrak{I}^{\lambda_i}_{\mathbf{a}+}\phi(\theta_i),\ \theta_i\in[\mathbf{a},b], \end{cases} \tag{1.2}$$

where $\mathbb{F}:[\mathbf{a},b]\times\mathbb{R}\to\mathbb{P}(\mathbb{R})$ is a set-valued map. Motivated by aforesaid works, we prove the existence of solutions for the following nonlinear FDI in the frame of φ-Hilfer FD with nonlocal IBCs

$$\begin{cases} {}^{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varphi}_{\mathbf{a}+}\phi(\tau)\in\mathbb{F}(\tau,\phi(\tau)),\ \tau\in(\mathbf{a},b),\ \mathbf{a}>0,\\ \phi(\mathbf{a}) = 0,\ \phi(b) = \sum\limits_{i=1}^{m}\delta_i\mathfrak{I}^{\lambda_i;\varphi}_{\mathbf{a}+}\phi(\theta_i), \end{cases} \tag{1.3}$$

where ${}^{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varphi}_{\mathbf{a}+}$ is the φ-Hilfer FD of order $\varrho_1\in(1,2)$ and type $\varrho_2\in[0,1]$, $\mathfrak{I}^{\lambda_i;\varphi}$ is the φ-Riemann-Liouville FI of order $\lambda_i>0$, $\mathbb{F}$ is a set-valued map from $[\mathbf{a},b]\times\mathbb{R}$ to the collection of $\mathbb{P}(\mathbb{R})\subset\mathbb{R}$, $-\infty<\mathbf{a}<b<\infty$, $\delta_i\in\mathbb{R}$, $i=1,2,...,m$, $0\le\mathbf{a}\le\theta_1<\theta_2<\theta_3<...<\theta_m\le b$.

Remark 1.1. **i)** The FDI (1.3) involving φ-Hilfer FD is the more wide category of BVPs that combines the FDI involving φ-Riemann–Liouville FD (for $\varrho_2=0$, $\varphi(\tau)=\tau$) and the FDI involving φ -Caputo FD (for $\varrho_2=1$, $\varphi(\tau)=\tau$).

ii) For various values of ϱ_2 and φ, our problem (1.3) reduces to FDIs involving the FDs like Hilfer, Katugampola, Erdélyi-Kober, Hadamard, and many other FDs.

iii) The acquired results in the current article include the results of Asawasamrit, et al. [46] (when $\varphi(\tau)=\tau$ and $\mathbb{F}(\tau,\phi(\tau))=\{f(\tau,\phi(\tau))\}$) and Wongcharoen et al. [48] (when $\varphi(\tau)=\tau$).

The novelty of this work lies in that the obtained results in this work unify most of the preceding results concerning FDIs.

This article is framed as follows. In Section 2, we provide some essentials concepts of advanced fractional calculus, set-valued analysis, and FP methods. The existence results for a φ-Hilfer type inclusion problem (1.3) are obtained in Section 3. The results obtained will be illustrated by examples in the Section 4.

2. Preliminary notions

2.1. Fractional calculus (FC)

In this portion, we introduce some notations and definitions of FC. Let $\mho = [\mathbf{a}, b]$, $\varrho_1 \in (1,2)$, $\varrho_2 \in [0,1]$ where $\mathfrak{p} = \varrho_1 + \varrho_2 (2 - \varrho_1) \in (1,2]$. Set

$$\mathbb{C} := C(\mho, \mathbb{R}) = \{g : \mho \to \mathbb{R};\ g \text{ is continuous}\}.$$

Clearly, $\mathbb{C}$ is a Banach space with norm

$$\|g\| = \sup\{|g(\tau)| : \tau \in \mho\}.$$

Denote $L^1(\mho, \mathbb{R})$ be the Banach space of Lebesgue-integrable functions $g : \mho \to \mathbb{R}$ with the norm

$$\|g\|_{L^1} = \int_{\mho} |g(\tau)|\, d\tau.$$

Let $g \in L^1(\mho, \mathbb{R})$ and $\varphi \in C^n(\mho, \mathbb{R})$ be increasing such that $\varphi'(\tau) \neq 0$ for each $\tau \in \mho$.

Definition 2.1 ([2]). The ϱ_1^{th}-φ-Riemann-Liouville FI of g is given by

$$\mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} g(\tau) = \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(\tau) - \varphi(\zeta))^{\varrho_1 - 1} g(\zeta)\, d\zeta.$$

Definition 2.2 ([2]). The ϱ_1^{th}-φ-Riemann-Liouville FD of g is given by

$$\mathfrak{D}_{\mathbf{a}+}^{\varrho_1;\varphi} g(\tau) = \left(\frac{1}{\varphi'(\tau)} \frac{d}{d\tau}\right)^n \mathfrak{I}_{\mathbf{a}+}^{(n-\varrho_1);\varphi} g(\tau),\ n = [\varrho_1] + 1, n \in \mathbb{N}.$$

Definition 2.3 ([23]). *T*he φ-Hilfer FD of g of order ϱ_1 and type ϱ_2 is given by

$${}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi} g(\tau) = \mathfrak{I}_{\mathbf{a}+}^{\varrho_2(n-\varrho_1);\varphi}\ \mathfrak{D}_{\varphi}^{[n]}\ \mathfrak{I}_{\mathbf{a}+}^{(1-\varrho_2)(n-\varrho_1);\varphi} g(\tau),$$

where $\mathfrak{D}_{\varphi}^{[n]} = \left(\frac{1}{\varphi'(\tau)} \frac{d}{d\tau}\right)^n$.

Lemma 2.4 ([2, 23]). *Let $\varrho_1, \varrho_2, \kappa > 0$. Then*

1) $\mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} \mathfrak{I}_{\mathbf{a}+}^{\varrho_2;\varphi} g(\tau) = \mathfrak{I}_{\mathbf{a}+}^{\varrho_1+\varrho_2;\varphi} g(\tau)$.

2) $\mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} (\varphi(\tau) - \varphi(\mathbf{a}))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\varrho_1+\kappa)} (\varphi(\tau) - \varphi(\mathbf{a}))^{\varrho_1+\kappa-1}$.

Lemma 2.5 ([23]). *For $\kappa > 0$, $\varrho_1 \in (n-1, n)$ and $\varrho_2 \in [0,1]$,*

$${}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi} (\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{z}-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa - \varrho_1)} (\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{z}-\varrho_1-1},\ \kappa > n.$$

In case, if $\varrho_1 \in (1,2)$ and $\kappa \in (1,2]$, then

$${}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi} (\varphi(\tau) - \varphi(\mathbf{a}))^{\kappa-1} = 0.$$

Lemma 2.6 ([23]). *If $g \in C^n(\mho, \mathbb{R})$, $n-1 < \varrho_1 < n$ and $\varrho_2 \in (0,1)$, we have*

1) $\mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi}\ {}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi}\, g(\tau) = g(\tau) - \sum_{k=1}^{n} \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-k}}{\Gamma(\mathfrak{p}-k+1)} \left(\frac{1}{\varphi'(\tau)}\frac{d}{d\tau}\right)^{n-k} \mathfrak{I}_{\mathbf{a}+}^{(1-\varrho_2)(n-\varrho_1);\varphi} g(\mathbf{a})$.

2) ${}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi}\ \mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\,\varphi} g(\tau) = g(\tau)$.

In regard to the problem (1.3), the next lemma is needed which was demonstrated in [47].

Lemma 2.7 ([47]). *Let $\mathcal{F} \in \mathbb{C}$ and*

$$\Omega = \frac{(\varphi(b)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Gamma(\mathfrak{p})} - \sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\mathfrak{p}+\lambda_i)} (\varphi(\theta_i)-\varphi(\mathbf{a}))^{\mathfrak{p}+\lambda_i-1} \neq 0, \tag{2.1}$$

then, the solution of nonlocal BVP

$$\begin{cases} {}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi}\, \phi(\tau) = \mathcal{F}(\tau),\ \tau \in (\mathbf{a}, b), \\ \phi(\mathbf{a}) = 0,\ \phi(b) = \sum_{i=1}^{m} \delta_i \mathfrak{I}_{\mathbf{a}+}^{\lambda_i;\varphi} \phi(\theta_i), \end{cases} \tag{2.2}$$

is obtained as

$$\phi(\tau) = \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \delta_i \mathfrak{I}_{\mathbf{a}+}^{\varrho_1+\lambda_i;\varphi} \mathcal{F}(\theta_i) - \mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} \mathcal{F}(b) \right) + \mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} \mathcal{F}(\tau). \tag{2.3}$$

2.2. Set-valued analysis

We requisition some basics related to the theory of set-valued maps. To this purpose, consider the Banach space $(\mathbb{E}, \|.\|)$ and the multi-valued map $\mathfrak{M} : \mathbb{E} \to \mathbb{P}(\mathbb{E})$, (i) is closed (convex) valued if $\mathfrak{M}(\phi)$ is closed (convex) $\forall \phi \in \mathbb{E}$; (ii) is bounded if $\mathfrak{M}(\mathcal{D}) = \cup_{\phi\in\mathcal{D}} \mathfrak{M}(\phi)$ is bounded in $\mathbb{E}$ for all bounded set $\mathcal{D}$ of $\mathbb{E}$, i.e.,$\sup_{\phi\in\mathcal{D}} \{\sup\{|\alpha| : \alpha \in \mathfrak{M}(\phi)\}\} < \infty$; (iii) is measurable if $\forall\ \alpha \in \mathbb{R}$, the function $\tau \to d(\alpha, \mathfrak{M}(\tau)) = \inf\{|\alpha - \lambda| : \lambda \in \mathfrak{M}(\tau)\}$ is measurable.

For other definitions such as completely continuous, upper semi-continuity (u.s.c.), we indicate to [49]. Further, the set of selections of $\mathbb{F}$ is given by

$$\mathfrak{R}_{\mathbb{F},\alpha} = \left\{ \varkappa \in L^1(\mho, \mathbb{R}) : \varkappa(\tau) \in \mathbb{F}(\tau, \alpha) \text{ for a.e. } \tau \in \mho \right\}.$$

Consider

$$\mathbb{P}_\sigma(\mathbb{E}) = \{\mathcal{M} \in \mathbb{P}(\mathbb{E}) : \mathcal{M} \neq \emptyset \text{ and has property } \sigma\},$$

where $\mathbb{P}_b$, $\mathbb{P}_{cl}$, $\mathbb{P}_{cp}$, and $\mathbb{P}_c$ are the categories of all closed, bounded, compact and convex subsets of $\mathbb{E}$, respectively.

Definition 2.8. Set-valued map $\mathbb{F} : \mho \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ is a Carathéodory if $\tau \to \mathbb{F}(\tau, \phi)$ is measurable for any $\phi \in \mathbb{R}$, and $\phi \to \mathbb{F}(\tau, \phi)$ is u.s.c., for (a.e.) all $\tau \in \mho$.

Besides, a set-valued map $\mathbb{F}$ is called L^1-Carathéodory if $\forall\ w > 0$, there exists $\Phi \in L^1(\mho, \mathbb{R}^+)$ such that

$$\|\mathbb{F}(\tau, \phi)\| = \sup\{|\varkappa| : \varkappa \in \mathbb{F}(\tau, \phi)\} \leq \Phi(\tau),$$

for a.e. $\tau \in \mho$, and for all $\|\phi\| \leq w$.

Now, we offer the next essential lemmas:

Lemma 2.9 ([7]). *Let* $\mathrm{Gr}(\mathfrak{M}) = \{(\phi, \alpha) \in \mathbb{E} \times Z, \alpha \in \mathfrak{M}(\phi)\}$ *be a graph of* $\mathfrak{M}$. *If* $\mathfrak{M} : \mathbb{E} \to \mathbb{P}_{cl}(Z)$ *is u.s.c., then* $\mathrm{Gr}(\mathfrak{M})$ *is a closed subset of* $\mathbb{E} \times Z$. *Conversely, if* $\mathfrak{M}$ *is completely continuous and has a closed graph, then it is u.s.c.*

Lemma 2.10 ([50]). *Let* $\mathbb{E}$ *be a separable Banach space.* $\mathbb{F} : \mathfrak{U} \times \mathbb{R} \to \mathbb{P}_{cp,c}(\mathbb{E})$ *be an* L^1*-Carathéodory set-valued map, and* $\mathcal{T} : L^1(\mathfrak{U}, \mathbb{E}) \to C(\mathfrak{U}, \mathbb{E})$ *be a linear continuous mapping. Then the operator*

$$\mathcal{T} \circ \Re_{\mathbb{F}} : C(\mathfrak{U}, \mathbb{E}) \to \mathbb{P}_{cp,c}(C(\mathfrak{U}, \mathbb{E})), \ \phi \to (\mathcal{T} \circ \Re_{\mathbb{F}})(\phi) = \mathcal{T}\left(\Re_{\mathbb{F},\phi}\right),$$

is a closed graph operator in $C(\mathfrak{U}, \mathbb{E}) \times C(\mathfrak{U}, \mathbb{E})$.

3. Existence results for set-valued problem

Definition 3.1. A function $\phi \in \mathbb{C}$ is a solution of (1.3), if there is $\varkappa \in L^1(\mathfrak{U}, \mathbb{R})$ with $\varkappa(\tau) \in \mathbb{F}(\tau, \phi)$ $\forall \tau \in \mathfrak{U}$ fulfilling the nonlocal IBC

$$\phi(\mathbf{a}) = 0, \ \phi(b) = \sum_{i=1}^{m} \delta_i \mathfrak{I}_{\mathbf{a}+}^{\lambda_i;\Upsilon} \phi(\theta_i),$$

and

$$\begin{aligned}
\phi(\tau) &= \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \delta_i \mathfrak{I}_{\mathbf{a}+}^{\varrho_1+\lambda_i;\varphi} \varkappa(\theta_i) - \mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} \varkappa(b) \right) + \mathfrak{I}_{\mathbf{a}+}^{\varrho_1;\varphi} \varkappa(\tau) \\
&= \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1 + \lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa(\zeta) \, d\zeta \right. \\
&\left. - \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta) \, d\zeta \right) + \frac{1}{\Gamma(\alpha)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(\tau) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta) \, d\zeta.
\end{aligned}$$

3.1. The U.S.C. case

The first consequence transacts with the convex valued $\mathbb{F}$ depending on Leray-Schauder-type for set-valued maps [51].

Theorem 3.2. *Let*

$$\eta = \sum_{i=1}^{m} |\delta_i| \frac{(\varphi(b) - \varphi(\mathbf{a}))^{\varrho_1+\lambda_i+\mathfrak{p}-1}}{|\Omega| \Gamma(\mathfrak{p}) \Gamma(\varrho_1 + \lambda_i + 1)} + \frac{(\varphi(b) - \varphi(\mathbf{a}))^{\mathfrak{p}+\varrho_1-1}}{|\Omega| \Gamma(\mathfrak{p}) \Gamma(\varrho_1 + 1)} + \frac{(\varphi(b) - \varphi(\mathbf{a}))^{\varrho_1}}{\Gamma(\varrho_1 + 1)}, \tag{3.1}$$

and suppose that

(As1) $\mathbb{F} : \mathfrak{U} \times \mathbb{R} \to \mathbb{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set-valued map.*

(As2) $\exists \ \widetilde{\mathfrak{Z}}_1 \in C(\mathfrak{U}, [0, \infty))$ *and a nondecreasing* $\widetilde{\mathfrak{Z}}_2 \in C([0, \infty), [0, \infty))$ *with*

$$\|\mathbb{F}(\tau, \phi)\|_{\mathbb{P}} = \sup\{|\alpha| : \alpha \in \mathbb{F}(\tau, \phi)\} \leq \widetilde{\mathfrak{Z}}_1(\tau) \widetilde{\mathfrak{Z}}_2(\|\phi\|), \ \forall (\tau, \phi) \in \mathfrak{U} \times \mathbb{R}.$$

(As3) There is a constant $\mathcal{K} > 0$ *such that*

$$\frac{\mathcal{K}}{\eta \left\| \widetilde{\mathfrak{Z}}_1 \right\| \widetilde{\mathfrak{Z}}_2(\mathcal{K})} > 1. \tag{3.2}$$

Then the problem (1.3) has at least one solution on $\mathfrak{U}$.

Proof. At first, to convert (1.3) into a FP problem, we define the operator $\tilde{\mathfrak{B}} : \mathbb{C} \to \mathbb{P}(\mathbb{C})$ by

$$\tilde{\mathfrak{B}}(\phi) = \left\{ \widetilde{\mathfrak{p}} \in \mathbb{C} : \widetilde{\mathfrak{p}}(\tau) = \left\{ \begin{array}{l} \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum\limits_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa(\zeta)\, d\zeta \right. \\ \left. -\frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta)\, d\zeta \right) \\ +\frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(\tau) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta)\, d\zeta \end{array} \right\}, \quad (3.3)$$

for $\varkappa \in \mathfrak{R}_{\mathbb{F},\phi}$. Clearly, the solution of (1.3) is the FP of the operator $\tilde{\mathfrak{B}}$. Proof cases will be given in a number of steps as:

Case 1. $\tilde{\mathfrak{B}}(\phi)$ is convex for any $\phi \in \mathbb{C}$.

Let $\widetilde{\mathfrak{p}}_1, \widetilde{\mathfrak{p}}_2 \in \tilde{\mathfrak{B}}(\phi)$. Then there exist $\varkappa_1, \varkappa_2 \in \mathfrak{R}_{\mathbb{F},\phi}$ such that for each $\tau \in \mathfrak{U}$

$$\begin{aligned} \widetilde{\mathfrak{p}}_j(\tau) = \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} & \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa_j(\zeta)\, d\zeta \right. \\ & \left. -\frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa_j(\zeta)\, d\zeta \right) \\ & + \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(\tau) - \varphi(\zeta))^{\varrho_1-1} \varkappa_j(\zeta)\, d\zeta, \ j = 1, 2. \end{aligned}$$

Let $\eta \in [0, 1]$. Then for each $\tau \in \mathfrak{U}$

$$\begin{aligned} & [\eta\widetilde{\mathfrak{p}}_1 + (1-\eta)\widetilde{\mathfrak{p}}_2](\tau) \\ & = \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} [\eta\varkappa_1(\zeta) + (1-\eta)\varkappa_2(\zeta)]\, d\zeta \right. \\ & \left. -\frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} [\eta\varkappa_1(\zeta) + (1-\eta)\varkappa_2(\zeta)]\, d\zeta \right) \\ & + \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(\tau) - \varphi(\zeta))^{\varrho_1-1} [\eta\varkappa_1(\zeta) + (1-\eta)\varkappa_2(\zeta)]\, d\zeta. \end{aligned}$$

As $\mathbb{F}$ possesses convex values, $\mathfrak{R}_{\mathbb{F},\phi}$ is convex and $[\eta\varkappa_1(\zeta) + (1-\eta)\varkappa_2(\zeta)] \in \mathfrak{R}_{\mathbb{F},\phi}$. Thus, $\eta\widetilde{\mathfrak{p}}_1 + (1-\eta)\widetilde{\mathfrak{p}}_2 \in \tilde{\mathfrak{B}}(\phi)$.

Case 2. The image of a bounded set under $\tilde{\mathfrak{B}}$ is bounded in $\mathbb{C}$.

For $r \in \mathbb{R}^+$, let $\mathfrak{D}_r = \{\phi \in \mathbb{C} : \|\phi\| \leq r\}$ be a bounded set in $\mathbb{C}$. Then for each $\widetilde{\mathfrak{p}} \in \tilde{\mathfrak{B}}(\phi)$ and $\phi \in \mathfrak{D}_r$, there exists $\varkappa \in \mathfrak{R}_{\mathbb{F},\phi}$ such that

$$\begin{aligned} \widetilde{\mathfrak{p}}(\tau) = \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} & \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa(\zeta)\, d\zeta \right. \\ & \left. -\frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta)\, d\zeta \right) \\ & + \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta)\, d\zeta. \end{aligned}$$

From the hypothesis (As2) and $\forall \tau \in \mho$, we get

$$\begin{aligned}
\left|\widetilde{\mathfrak{p}}(\tau)\right| &\leq \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}|\varkappa(\zeta)|\,d\zeta\right.\\
&\left.+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}|\varkappa(\zeta)|\,d\zeta\right)\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}|\varkappa(\zeta)|\,d\zeta\\
&\leq \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r)}{|\Omega|\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{|\delta_i|}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\,d\zeta\right.\\
&\left.+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\,d\zeta\right)+\frac{\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r)}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\,d\zeta\\
&\leq \left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r)\left(\sum_{i=1}^{m}|\delta_i|\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\varrho_1+\lambda_i+\mathfrak{p}-1}}{|\Omega|\Gamma(\mathfrak{p})\Gamma(\varrho_1+\lambda_i+1)}+\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\mathfrak{p}+\varrho_1-1}}{|\Omega|\Gamma(\mathfrak{p})\Gamma(\varrho_1+1)}+\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right).
\end{aligned}$$

Thus

$$\left\|\widetilde{\mathfrak{p}}\right\| \leq \eta\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r).$$

Case 3. We prove that $\tilde{\mathfrak{B}}(\mathfrak{D}_r)$ is equicontinuous.

Let $\phi \in \mathfrak{D}_r$ and $\widetilde{\mathfrak{p}} \in \tilde{\mathfrak{B}}(\phi)$. Then there is a function $\varkappa \in \Re_{\mathbb{F},\phi}$ such that

$$\begin{aligned}
\widetilde{\mathfrak{p}}(\tau) &= \frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa(\zeta)\,d\zeta\right.\\
&\left.-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta\right)\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta,\ \tau\in\mho.
\end{aligned}$$

Let $\tau_1, \tau_2 \in \mho$, $\tau_1 < \tau_2$. Then

$$\begin{aligned}
&\left|\widetilde{\mathfrak{p}}(\tau_2)-\widetilde{\mathfrak{p}}(\tau_1)\right|\\
&\leq \frac{(\varphi(\tau_2)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}-(\varphi(\tau_1)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{|\Omega|\Gamma(\mathfrak{p})}\\
&\left(\sum_{i=1}^{m}\frac{|\delta_i|}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}|\varkappa(\zeta)|\,d\zeta\right.\\
&\left.+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}|\varkappa(\zeta)|\,d\zeta\right)\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau_1}\varphi'(\zeta)\left((\varphi(\tau_2)-\varphi(\zeta))^{\varrho_1-1}-(\varphi(\tau_1)-\varphi(\zeta))^{\varrho_1-1}\right)|\varkappa(\zeta)|\,d\zeta\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\tau_1}^{\tau_2}\varphi'(s)(\varphi(\tau_2)-\varphi(\zeta))^{\varrho_1-1}|\varkappa(\zeta)|\,d\zeta
\end{aligned}$$

$$\leq \frac{\left((\varphi(\tau_2)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}-(\varphi(\tau_1)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}\right)\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r)}{|\Omega|\,\Gamma(\mathfrak{p})}$$
$$\left(\sum_{i=1}^{m}\frac{|\delta_i|}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\,d\zeta\right.$$
$$\left.+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\,d\zeta\right)$$
$$+\frac{\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(r)}{\Gamma(\varrho_1+1)}\left(((\varphi(\tau_2)-\varphi(\mathbf{a}))^{\varrho_1}-(\varphi(\tau_1)-\varphi(\mathbf{a}))^{\varrho_1})\right).$$

As $\tau_1 \to \tau_2$, we obtain

$$\left|\widetilde{\mathfrak{p}}(\tau_2)-\widetilde{\mathfrak{p}}(\tau_1)\right| \to 0.$$

So, $\tilde{\mathfrak{B}}(\mathfrak{D}_r)$ is equicontinuous. Based on Arzela-Ascoli theorem and above cases $(2-3)$, we conclude that $\tilde{\mathfrak{B}}$ is completely continuous.

Case 4. The graph of $\tilde{\mathfrak{B}}$ is closed.

Let $\phi_n \to \phi_*$, $\widetilde{\mathfrak{p}}_n \in \tilde{\mathfrak{B}}(\phi_n)$ and $\widetilde{\mathfrak{p}}_n$ converges to $\widetilde{\mathfrak{p}}_*$. We prove that $\widetilde{\mathfrak{p}}_* \in \tilde{\mathfrak{B}}(\phi_*)$. Since $\widetilde{\mathfrak{p}}_n \in \tilde{\mathfrak{B}}(\phi_n)$, there exists $\varkappa_n \in \Re_{\mathbb{F},\phi_n}$ such that

$$\widetilde{\mathfrak{p}}_n(\tau)=\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa_n(\zeta)\,d\zeta\right.$$
$$\left.-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_n(\zeta)\,d\zeta\right)$$
$$+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_n(\zeta)\,d\zeta,\ \tau\in\mathfrak{V}.$$

Thus, we need to show that there exists $\varkappa_* \in \Re_{\mathbb{F},\phi_*}$ such that, for each $\tau \in \mathfrak{V}$,

$$\widetilde{\mathfrak{p}}_*(\tau)=\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa_*(\zeta)\,d\zeta\right.$$
$$\left.-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_*(\zeta)\,d\zeta\right)$$
$$+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_*(\zeta)\,d\zeta.$$

Define $\mathcal{T}: L^1(\mathfrak{V},\mathbb{R}) \to C(\mathfrak{V},\mathbb{R})$ such that be continuous linear operator by

$$\varkappa \to \mathcal{T}(\varkappa)(\tau)=\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa(\zeta)\,d\zeta\right.$$
$$\left.-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta\right)$$
$$+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta,\ \tau\in\mathfrak{V}.$$

Observe that

$$\begin{aligned}
&\left\|\widetilde{\mathfrak{p}}_n-\widetilde{\mathfrak{p}}_*\right\|\\
&=\left\|\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}(\varkappa_n(\zeta)-\varkappa_*(\zeta))\,d\zeta\right.\right.\\
&-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}(\varkappa_n(\zeta)-\varkappa_*(\zeta))\,d\zeta\Bigg)\\
&\left.+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}(\varkappa_n(\zeta)-\varkappa_*(\zeta))\,d\zeta\to 0\right\|,
\end{aligned}$$

when $n\to\infty$. So in light of Lemma (2.10) that $\mathcal{T}\circ\Re_{\mathbb{F},\phi}$ is a closed graph operator. Besides, we have

$$\widetilde{\mathfrak{p}}_n\in\mathcal{T}\left(\Re_{\mathbb{F},\phi_n}\right).$$

Since $\phi_n\to\phi_*$, Lemma (2.10) gives

$$\begin{aligned}
\widetilde{\mathfrak{p}}_*(\tau)=\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}&\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa_*(\zeta)\,d\zeta\right.\\
&-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_*(\zeta)\,d\zeta\Bigg)\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_*(\zeta)\,d\zeta.
\end{aligned}$$

for some $\varkappa_*\in\Re_{\mathbb{F},\phi_*}$.

Case 5. There exists an open set $\mathcal{N}\subseteq\mathbb{C}$ with $\phi\notin\delta\tilde{\mathfrak{B}}(\phi)$ for every $\delta\in(0,1)$ and $\forall\phi\in\partial\mathcal{N}$.

Let $\delta\in(0,1)$ and $\phi\in\delta\tilde{\mathfrak{B}}(\phi)$. Then there exists $\varkappa\in\Re_{\mathbb{F},\phi}$ such that

$$\begin{aligned}
|\phi(\tau)|=\Bigg|&\frac{\delta(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}\varkappa(\zeta)\,d\zeta\right.\\
&-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta\Bigg)\\
&\left.+\frac{\delta}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa(\zeta)\,d\zeta\right|\\
&\leq\eta\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(\|\phi\|).
\end{aligned}$$

Thus, we have

$$|\phi(\tau)|\leq\eta\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(\|\phi\|),\ \forall\tau\in\mho.$$

Hence, we obtain

$$\frac{\|\phi\|}{\eta\left\|\widetilde{\beth}_1\right\|\widetilde{\beth}_2(\|\phi\|)}\leq 1.$$

From (As3), there is a positive constant $\mathcal{K}$ such that $\|\phi\|\neq\mathcal{K}$. We define the set $\mathcal{N}$ by

$$\mathcal{N}=\{\phi\in\mathbb{C}:\|\phi\|<\mathcal{K}\}.$$

From previous cases, $\tilde{\mathfrak{B}} : \overline{\mathcal{N}} \to \mathbb{P}(\mathbb{C})$ is completely continuous and u.s.c. Depending on the choice of $\mathcal{N}$, there is no $\phi \in \partial\mathcal{N}$ such that $\phi \in \delta\tilde{\mathfrak{B}}(\phi)$ for some $\delta \in (0,1)$. Therefore, We can infer that problem (1.3) possesses at least one solution $\phi \in \overline{\mathcal{N}}$ according to Leray-Schauder theorem for multi-valued maps. □

3.2. The Lipschitz case

In this part, we give another existence criterion for φ-Hilfer FDI (1.3) according to new assumptions. In what follows, we prove the existence result when $\mathbb{F}$ has a non convex-valued using Covitz and Nadler theorem [52].

Let $(\mathbb{E}, d)$ be a metric space. Consider $\mathbb{H}_d : \mathbb{P}(\mathbb{E}) \times \mathbb{P}(\mathbb{E}) \to \mathbb{R}^+ \cup \{\infty\}$ defined by

$$\mathbb{H}_d\left(\tilde{M}, \tilde{N}\right) = \max\left\{\sup_{\tilde{m}\in\tilde{M}} d\left(\tilde{m}, \tilde{N}\right), \sup_{\tilde{n}\in\tilde{N}} d\left(\tilde{M}, \tilde{n}\right)\right\},$$

where $d\left(\tilde{M}, \tilde{n}\right) = \inf_{\tilde{m}\in\tilde{M}} d(\tilde{m}, \tilde{n})$ and $d\left(\tilde{m}, \tilde{N}\right) = \inf_{\tilde{n}\in\tilde{N}} d(\tilde{m}, \tilde{n})$. Then $\left(\mathbb{P}_{b,cl}(\mathbb{E}), \mathbb{H}_d\right)$ is a metric space (see [53]).

Definition 3.3. A set-valued operator $\tilde{\mathfrak{B}} : \mathbb{E} \to \mathbb{P}_{cl}(\mathbb{E})$ is κ-Lipschitz iff $\exists\, \kappa > 0$ such that

$$\mathbb{H}_d\left(\tilde{\mathfrak{B}}(\phi), \tilde{\mathfrak{B}}(\alpha)\right) \leq \kappa d(\phi, \alpha) \text{ for any } \phi,\ \alpha \in \mathbb{E}.$$

Particularly, if $\kappa < 1$, then $\tilde{\mathfrak{B}}$ is a contraction.

Theorem 3.4. *Suppose that*

(As4) $\mathbb{F} : \mho \times \mathbb{R} \to \mathbb{P}_{cp}(\mathbb{R})$ *is such that* $\mathbb{F}(., \phi) : \mho \to \mathbb{P}_{cp}(\mathbb{R})$ *is measurable for each* $\phi \in \mathbb{R}$.

(As5) $\mathbb{H}_d\left(\mathbb{F}(\tau, \phi), \mathbb{F}\left(\tau, \overline{\phi}\right)\right) \leq \tilde{r}(\tau)\left|\phi - \overline{\phi}\right|$ *for (a.e.) all* $\tau \in \mho$ *and* $\phi, \overline{\phi} \in \mathbb{R}$ *with* $\tilde{r} \in C(\mho, \mathbb{R}^+)$ *and* $d(0, \mathbb{F}(\tau, 0)) \leq \tilde{r}(\tau)$ *for (a.e.) all* $\tau \in \mho$.

Then the problem (1.3) has at least one solution on $\mho$ *if*

$$\eta \|\tilde{r}\| < 1,$$

where η *is defined in (3.1).*

Proof. In view of Theorem III.6 in [8] and the assumption (As4), $\mathbb{F}$ has a measurable selection $\varkappa : \mho \to \mathbb{R}$, $\varkappa \in L^1(\mho, \mathbb{R})$, as well as $\mathbb{F}$ is integrably bounded. Thus, $\mathfrak{R}_{\mathbb{F},\phi} \neq \varnothing$. Now, we prove that $\tilde{\mathfrak{B}} : \mathbb{C} \to \mathbb{P}(\mathbb{C})$ defined in (3.3) satisfies the assumptions of FPT of Nadler and Covitz. To show that $\tilde{\mathfrak{B}}(\phi)$ is closed for any $\phi \in \mathbb{C}$. Let $\{u_n\}_{n\geq 0} \in \tilde{\mathfrak{B}}(\phi)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathbb{C}$. Then $u \in \mathbb{C}$ and there is $\varkappa_n \in \mathfrak{R}_{\mathbb{F},\phi_n}$ such that

$$\begin{aligned} u_n(\tau) = {} & \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1 + \lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta)(\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa_n(\zeta)\, d\zeta \right. \\ & \left. - \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta)(\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa_n(\zeta)\, d\zeta\right) \\ & + \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta)(\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa_n(\zeta)\, d\zeta,\ \forall \tau \in \mho. \end{aligned}$$

As $\mathbb{F}$ possesses compact values, so there is a subsequence $\varkappa_n \to \varkappa$ in $L^1(\mathfrak{U}, \mathbb{R})$. Consequently. $\varkappa \in \mathfrak{R}_{\mathbb{F},\phi}$ and we get

$$u_n(\tau) \to u(\tau) = \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1 + \lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa(\zeta) \, d\zeta \right.$$
$$\left. - \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta) \, d\zeta \right)$$
$$+ \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa(\zeta) \, d\zeta, \ \forall \tau \in \mathfrak{U}.$$

Hence $u \in \tilde{\mathfrak{B}}(\phi)$.

Next, we show that there is a $\vartheta \in (0,1)$, $(\vartheta = \eta \|\tilde{r}\|)$ such that

$$\mathbb{H}_d\left(\tilde{\mathfrak{B}}(\phi), \tilde{\mathfrak{B}}\left(\overline{\phi}\right)\right) \leq \vartheta \left\|\phi - \overline{\phi}\right\| \text{ for each } \phi, \ \overline{\phi} \in \mathbb{C}.$$

Let $\phi, \overline{\phi} \in \mathbb{C}$ and $\widetilde{\mathfrak{p}}_1 \in \tilde{\mathfrak{B}}(\phi)$. Then there exists $\varkappa_1(\tau) \in \mathbb{F}(\tau, \phi(\tau))$ such that, for each $\tau \in \mathfrak{U}$

$$\widetilde{\mathfrak{p}}_1(\tau) = \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1 + \lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa_1(\zeta) \, d\zeta \right.$$
$$\left. - \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{b} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa_1(\zeta) \, d\zeta \right)$$
$$+ \frac{1}{\Gamma(\varrho_1)} \int_{\mathbf{a}}^{\tau} \varphi'(\zeta) (\varphi(b) - \varphi(\zeta))^{\varrho_1-1} \varkappa_1(\zeta) \, d\zeta.$$

By (As5), we have

$$\mathbb{H}_d\left(\mathbb{F}(\tau, \phi), \mathbb{F}\left(\tau, \overline{\phi}\right)\right) \leq \tilde{r}(\tau) \left|\phi(\tau) - \overline{\phi}(\tau)\right|.$$

Thus, there exists $\tilde{w}(\tau) \in \mathbb{F}\left(\tau, \overline{\phi}\right)$ such that

$$|\varkappa_1(\tau) - \tilde{w}| \leq \tilde{r}(\tau) \left|\phi(\tau) - \overline{\phi}(\tau)\right|, \ \tau \in \mathfrak{U}.$$

Constructing a set-valued map $\mathcal{E} : \mathfrak{U} \to \mathbb{P}(\mathbb{R})$ as

$$\mathcal{E}(\tau) = \left\{\tilde{w} \in \mathbb{R} : |\varkappa_1(\tau) - \tilde{w}| \leq \tilde{r}(\tau) \left|\phi(\tau) - \overline{\phi}(\tau)\right|\right\}.$$

We can infer that the set-valued map $\mathcal{E}(\tau) \cap \mathbb{F}\left(\tau, \overline{\phi}\right)$ is measurable, because $\varkappa_1$ and $\Delta = \tilde{r}\left|\phi - \overline{\phi}\right|$ are both measurable. Now, we choose $\varkappa_2(\tau) \in \mathbb{F}\left(\tau, \overline{\phi}\right)$ with

$$|\varkappa_1(\tau) - \varkappa_2(\tau)| \leq \tilde{r}(\tau) \left|\phi(\tau) - \overline{\phi}(\tau)\right|, \ \forall \tau \in \mathfrak{U}.$$

Define

$$\widetilde{\mathfrak{p}}_2(\tau) = \frac{(\varphi(\tau) - \varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})} \left(\sum_{i=1}^{m} \frac{\delta_i}{\Gamma(\varrho_1 + \lambda_i)} \int_{\mathbf{a}}^{\theta_i} \varphi'(\zeta) (\varphi(\theta_i) - \varphi(\zeta))^{\varrho_1+\lambda_i-1} \varkappa_2(\zeta) \, d\zeta \right.$$

$$-\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_2(\zeta)\,d\zeta\Big)$$
$$+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}\varkappa_2(\zeta)\,d\zeta,\ \forall\tau\in\mathfrak{V}.$$

As a result, we get

$$\begin{aligned}
&\left|\widetilde{\mathfrak{p}}_1(\tau)-\widetilde{\mathfrak{p}}_2(\tau)\right|\\
&\leq\frac{(\varphi(\tau)-\varphi(\mathbf{a}))^{\mathfrak{p}-1}}{\Omega\Gamma(\mathfrak{p})}\Bigg(\sum_{i=1}^{m}\frac{\delta_i}{\Gamma(\varrho_1+\lambda_i)}\int_{\mathbf{a}}^{\theta_i}\varphi'(\zeta)(\varphi(\theta_i)-\varphi(\zeta))^{\varrho_1+\lambda_i-1}|\varkappa_1(\zeta)-\varkappa_2(\zeta)|\,d\zeta\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{b}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}|\varkappa_1(\zeta)-\varkappa_2(\zeta)|\,d\zeta\Bigg)\\
&+\frac{1}{\Gamma(\varrho_1)}\int_{\mathbf{a}}^{\tau}\varphi'(\zeta)(\varphi(b)-\varphi(\zeta))^{\varrho_1-1}|\varkappa_1(\zeta)-\varkappa_2(\zeta)|\,d\zeta\\
&\leq\|m\|\left\|\phi-\overline{\phi}\right\|\sum_{i=1}^{m}|\delta_i|\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\varrho_1+\lambda_i+\mathfrak{p}-1}}{|\Omega|\Gamma(\mathfrak{p})\Gamma(\varrho_1+\lambda_i+1)}+\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\mathfrak{p}+\varrho_1-1}}{|\Omega|\Gamma(\mathfrak{p})\Gamma(\varrho_1+1)}+\frac{(\varphi(b)-\varphi(\mathbf{a}))^{\varrho_1}}{\Gamma(\varrho_1+1)}.
\end{aligned}$$

Hence

$$\left\|\widetilde{\mathfrak{p}}_1-\widetilde{\mathfrak{p}}_2\right\|\leq\eta\|\tilde{r}\|\left\|\phi-\overline{\phi}\right\|.$$

Analogously, interchanging the roles of ϕ and $\overline{\phi}$, we get

$$\mathbb{H}_d\left(\tilde{\mathfrak{B}}(\phi),\tilde{\mathfrak{B}}\left(\overline{\phi}\right)\right)\leq\eta\|\tilde{r}\|\left\|\phi-\overline{\phi}\right\|.$$

As $\tilde{\mathfrak{B}}$ is a contraction, we conclude that it has a FP ϕ which is a solution of (1.3) according to the Covitz and Nadler theorem. □

4. Examples

In this section, we give some special cases of FDIs to illustrate the obtained outcomes. Consider the FDIs of the following type

$$\begin{cases}{}^{H}\mathfrak{D}_{\mathbf{a}+}^{\varrho_1,\varrho_2;\varphi}\phi(\tau)\in\mathbb{F}(\tau,\phi),\ \tau\in(\mathbf{a},b),\\ \phi(\mathbf{a})=0,\ \phi(b)=\sum\limits_{i=1}^{m}\delta_i\mathfrak{I}_{\mathbf{a}+}^{\lambda_i,\varphi}\phi(\theta_i).\end{cases}\tag{4.1}$$

The following instances are special cases of FDIs defined by (4.1).

Example 4.1. Using the following data $\varphi(\tau)=\log\tau$, $\varrho_2\to 0$, $\mathbf{a}=1$, $b=e$, $\varrho_1=\frac{3}{2}$, $\delta_1=\frac{1}{2}$, $\delta_2=\frac{1}{10}$, $\lambda_1=\frac{1}{4}$, $\lambda_2=\frac{5}{2}$, $\theta_1=\frac{3}{2}$, $\theta_2=2$ in (4.1). Thus, the problem (4.1) convert to

$$\begin{cases}{}^{H}\mathfrak{D}_{1+}^{\frac{3}{2},0;\log\tau}\phi(\tau)\in\mathbb{F}(\tau,\phi),\ \tau\in(1,e),\\ \phi(1)=0,\ \phi(e)=\frac{1}{2}\mathfrak{I}_{1+}^{\frac{1}{4},\log\tau}\phi\left(\frac{3}{2}\right)+\frac{1}{10}\mathfrak{I}_{1+}^{\frac{5}{2},\log\tau}\phi(2),\end{cases}\tag{4.2}$$

with $\mathfrak{p}=\frac{3}{2}$. Let $\mathbb{F}:[1,e]\times\mathbb{R}\to\mathbb{P}(\mathbb{R})$ defined by

$$\phi \to \mathbb{F}(\tau,\phi) = \left[\frac{1}{(\tau^3+6\exp(\tau^2))}\frac{\phi^2}{3(\phi^2+2)}, \frac{1}{\sqrt{\tau+8}}\frac{|\phi|}{|\phi|+1}\right]. \tag{4.3}$$

From above data we get $\Omega = 0.84640 \neq 0$. Clearly $\mathbb{F}$ fulfills (As1) and

$$\|\mathbb{F}(\tau,\phi)\|_{\mathbb{P}} = \sup\{|\alpha| : \alpha \in \mathbb{F}(\tau,\phi)\} \leq \frac{1}{\sqrt{\tau+8}} = \widetilde{\beth}_1(\tau)\widetilde{\beth}_2(\|\phi\|),$$

which yields $\left\|\widetilde{\beth}_1\right\| = \frac{1}{3}$ and $\widetilde{\beth}_2(\|\phi\|) = 1$. Therefore, the condition (As2) is fulfilled, and by (As3), it found that $\mathcal{K} > 0.72503$.

Hence all suppositions of Theorem 3.2 hold, and so there is at least one solution of the problem (4.2) on $[1, e]$.

Example 4.2. Using the following data $\varphi(\tau) = \tau$, $\varrho_2 \to 0$, $\mathbf{a} = 0$, $b = 1$, $\varrho_1 = \frac{5}{4}$, $\delta_1 = 3$, $\delta_2 = 5$, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{2}$ in (4.1). Thus, the problem (4.1) convert to

$$\begin{cases} {}^{H}\mathfrak{D}_{0+}^{\frac{5}{4},0;\tau}\phi(\tau) \in \mathbb{F}(\tau,\phi), \tau \in (0,1), \\ \phi(0) = 0,\ \phi(1) = 3\mathfrak{I}_{0+}^{\frac{1}{4},\tau}\phi\left(\frac{1}{4}\right) + 5\mathfrak{I}_{0+}^{\frac{1}{2},\tau}\phi\left(\frac{1}{2}\right), \end{cases} \tag{4.4}$$

with $\mathfrak{p} = \frac{5}{4}$. Let $\mathbb{F} : [0,1] \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ defined by

$$\phi \to \mathbb{F}(\tau,\phi) = \left[\exp\left(-\phi^4\right) + \tau + 4, \frac{|\phi|}{|\phi|+1} + \tau + 2\right]. \tag{4.5}$$

From above data we get $\Omega = -3.8241 \neq 0$. Clearly $\mathbb{F}$ fulfills (As1) and

$$\|\mathbb{F}(\tau,\phi)\|_{\mathbb{P}} = \sup\{|\alpha| : \alpha \in \mathbb{F}(\tau,\phi)\} \leq 6 = \widetilde{\beth}_1(\tau)\widetilde{\beth}_2(\|\phi\|),$$

where $\left\|\widetilde{\beth}_1\right\| = 1$ and $\widetilde{\beth}_2(\|\phi\|) = 6$. Therefore, the condition (As2) is valid, and by (As3), it follows that $\mathcal{K} > 16.111$.

Hence all suppositions of Theorem 3.2 hold, and so there is at least one solution of (4.4) on $[0, 1]$.

Example 4.3. Using the following data $\varphi(\tau) = \tau$, $\varrho_2 \to \frac{1}{2}$, $\mathbf{a} = 0$, $b = 1$, $\varrho_1 = \frac{7}{4}$, $\delta_1 = 3$, $\delta_2 = 5$, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{2}$ in (4.1). Thus, the problem (4.1) convert to

$$\begin{cases} {}^{H}\mathfrak{D}_{0+}^{\frac{7}{4},\frac{1}{2};\tau}\phi(\tau) \in \mathbb{F}(\tau,\phi), \tau \in (0,1), \\ \phi(0) = 0,\ \phi(1) = 3\mathfrak{I}_{0+}^{\frac{1}{4},\tau}\phi\left(\frac{1}{4}\right) + 5\mathfrak{I}_{0+}^{\frac{1}{2},\tau}\phi\left(\frac{1}{2}\right), \end{cases} \tag{4.6}$$

with $\mathfrak{p} = \frac{15}{8}$. Let $\mathbb{F} : [0,1] \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ given by

$$\phi \to \mathbb{F}(\tau,\phi) = \left[0, \frac{2\sin(\phi)}{(\tau^2+16)} + \frac{1}{20}\right]. \tag{4.7}$$

From above data we get $\Omega = -1.1237 \neq 0$. Obviously $\mathbb{H}_d\left(\mathbb{F}(\tau,\phi), \mathbb{F}\left(\tau,\overline{\phi}\right)\right) \leq \tilde{r}(\tau)\left|\phi - \overline{\phi}\right|$, where $\tilde{r}(\tau) = \frac{2}{(\tau^2+16)}$ and $d(0, \mathbb{F}(\tau,0)) = \frac{1}{20} \leq \tilde{r}(\tau)$ for (a.e.) all $\tau \in [0,1]$. Additionally, we obtain $\|\tilde{r}\| = \frac{1}{8}$ which leads to $\eta\|\tilde{r}\| \approx 0.55 < 1$. Accordingly, all hypotheses of Theorem (3.4) are satisfied, and so there exists at least one solution of the problem (4.6) on $[0, 1]$.

5. Conclusions

In this article, we have considered a class of BVP's for φ-Hilfer-type FDIs subjected to nonlocal IBC. The existence results have been proved by considering the kinds when the set-valued map has convex or nonconvex values. In the case of a convex set-valued map, we have applied the Leray-Schauder FPT, whereas the Nadler's and Covitz's FPT concern set-valued contractions are used in the case of a nonconvex set-valued map. The obtained outcomes are well explained through many relevant illustrative examples. We have settled that current results are new in the frame of φ-Hilfer FDIs and it covers many findings in the existing literature as a special case as shown in the Remark 1.1.

In future studies. We will try to expand the problem presented in this article to a general structure using the Mittag-Leffler power law [21] and fractal fractional operators [54].

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Conflict of interest

The authors declare that they have no conflict of interest.

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