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Research article

Explicit iteration and unique solution for ϕ-Hilfer type fractional Langevin equations

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Abstract: This paper proves that the monotone iterative method is an effective method to find the approximate solution of fractional nonlinear Langevin equation involving ϕ-Hilfer fractional derivative with multi-point boundary conditions. First, we apply a approach based on the properties of the Mittag-Leffler function to derive the formula of explicit solutions for the proposed problem. Next, by using the fixed point technique and some properties of Mittag-Leffler functions, we establish the sufficient conditions of existence of a unique solution for the considered problem. Moreover, we discuss the lower and upper explicit monotone iterative sequences that converge to the extremal solution by using the monotone iterative method. Finally, we construct a pertinent example that includes some graphics to show the applicability of our results.

Keywords: fractional Langevin equation; existence; ϕ-Hilfer fractional derivative; extremal solutions; nonlocal boundary conditions
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1. Introduction

Fractional differential equations (FDEs) have a profound physical background and rich theoretical connotations and have been particularly eye-catching in recent years. Fractional order differential equations refer to equations that contain fractional derivatives or integrals. Currently, fractional derivatives and integrals have a wide range of applications in many disciplines such as physics, biology, and chemistry, etc. For more information see [1–5].

Langevin equation is an important tool of many areas such as mathematical physics, protein dynamics [6], deuteron-cluster dynamics, and described anomalous diffusion [7]. In 1908, Langevin established first the Langevin equation with a view to describe the advancement of physical phenomena in fluctuating conditions [8]. Some evolution processes are characterized by the fact that they change of state abruptly at certain moments of time. These perturbations are short-term in comparison with the duration of the processes. So, the Langevin equations are a suitable tool to describe such problems. Besides the intensive improvement of fractional derivatives, the Langevin (FDEs) have been presented in 1990 by Mainardi and Pironi [9], which was trailed by numerous works interested in some properties of solutions like existence and uniqueness for Langevin FDEs [10–19]. We also refer here to some recent works that deal with a qualitative analysis of such problems, including the generalized Hilfer operator, see [20–24]. Recent works related to our work were done by [25–30]. The monotone iterative technique is one of the important techniques used to obtain explicit solutions for some differential equations. For more details about the monotone iterative technique, we refer the reader to the classical monographs [31, 32].

Lakshmikantham and Vatsala [25] studied the general existence and uniqueness results for the following FDE

$$\begin{cases} D_{0^+}^{\mu}\left(\upsilon(\varkappa)-\upsilon(0)\right)=f\left(\varkappa,\upsilon(\varkappa)\right), \ \varkappa\in[0,b], \\ \upsilon(0)=\upsilon_0, \end{cases}$$

by the monotone iterative technique and comparison principle. Fazli et al. [26] investigated the existence of extremal solutions of a nonlinear Langevin FDE described as follows

$$\begin{cases} D_{0^+}^{\mu_1}\left(D_{0^+}^{\mu_2}+\lambda\right)\upsilon(\varkappa)=f\left(\varkappa,\upsilon(\varkappa)\right), \ \varkappa\in[0,b], \\ g(\upsilon(0),\upsilon(b))=0, D_{0^+}^{\mu_2}\upsilon(0)=\upsilon_{\mu_2}, \end{cases}$$

via a constructive technique that produces monotone sequences that converge to the extremal solutions. Wang et al. [27], used the monotone iterative method to prove the existence of extremal solutions for the following nonlinear Langevin FDE

$$\begin{cases} {}^{\beta}D_{0^+}^{\mu}\left({}^{\gamma}D_{0^+}^{\mu}+\lambda\right)\upsilon(\varkappa)=f\left(\varkappa,\upsilon(\varkappa),\left({}^{\gamma}D_{0^+}^{\mu}+\lambda\right)\right), \ \varkappa\in(0,b], \\ \varkappa^{\mu(1-\gamma)}\upsilon(0)=\tau_1\int_0^{\eta}\upsilon(s)ds+\sum_{i=1}^{m}\mu_i\upsilon(\sigma_i), \\ \varkappa^{\mu(1-\beta)}\left({}^{\gamma}D_{0^+}^{\mu}+\lambda\right)\upsilon(0)=\tau_2\int_0^{\eta}{}^{\gamma}D_{0^+}^{\mu}\upsilon(s)ds+\sum_{i=1}^{m}\rho_i^{\gamma}D_{0^+}^{\mu}\upsilon(\sigma_i), \end{cases}$$

Motivated by the novel advancements of the Langevin equation and its applications, also by the above argumentations, in this work, we apply the monotone iterative method to investigate the lower and upper explicit monotone iterative sequences that converge to the extremal solution of a fractional Langevin equation (FLE) with multi-point sub-strip boundary conditions described by

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon(\varkappa)=f\left(\varkappa,\upsilon(\varkappa)\right), \ \varkappa\in(0,b], \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa)\Big|_{\varkappa=0}=0, \upsilon(0)=0, \upsilon(b)=\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\upsilon(\zeta_i), \end{cases} \quad (1.1)$$

where ${}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}$ and ${}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}$ are the ϕ-Hilfer fractional derivatives of order $\mu_1\in(0,1]$ and $\mu_2\in(1,2]$ respectively, and type $\beta_1,\beta_2\in[0,1], \sigma_i>0, \lambda_1,\lambda_2\in\mathbb{R}^+, \delta_i>0, m\geq1, 0<\zeta_1<\zeta_2<......<1$, $f:(0,b]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and ϕ is an increasing function, having a continuous derivative ϕ' on $(0,b)$ such that $\phi'(\varkappa)\neq0$, for all $\varkappa\in(0,b]$. Our main contributions to this work are as follows:

• By adopting the same techniques used in [26, 27], we derive the formula of explicit solutions for ϕ-Hilfer-FLEs (1.1) involving two parameters Mittag-Leffler functions.

• We use the monotone iterative method to study the extremal of solutions of ϕ-Hilfer-FLE (1.1).

• We investigate the lower and upper explicit monotone iterative sequences that converge to the extremal solution.

• The proposed problem (1.1) covers some problems involving many classical fractional derivative operators, for different values of function ϕ and parameter $\mu_i, i = 1, 2$. For instance:

• If $\phi(\varkappa) = \varkappa$ and $\mu_i = 1$, then the FLE (1.1) reduces to Caputo-type FLE.

• If $\phi(\varkappa) = \varkappa$ and $\mu_i = 0$, then the FLE (1.1) reduces to Riemann-Liouville-type FLE.

• If $\mu_i = 0$, then the FLE (1.1) reduces to FLE with the ϕ-Riemann-Liouville fractional derivative.

• If $\phi(\varkappa) = \varkappa$, then the FLE (1.1) reduces to classical Hilfer-type FLE.

• If $\phi(\varkappa) = \log \varkappa$, then the FLE (1.1) reduces to Hilfer-Hadamard-type FLE.

• If $\phi(\varkappa) = \varkappa^{\rho}$, then the FLE (1.1) reduces to Katugampola-type FLE.

• The results obtained in this work includes the results of Fazli et al. [26], Wang et al. [27] and cover many problems which do not study yet.

The structure of our paper is as follows: In the second section, we present some notations, auxiliary lemmas and some basic definitions which are used throughout the paper. Moreover, we derive the formula of the explicit solution for FLE (1.1) in the term of Mittag-Leffler with two parameters. In the third section, we discuss the existence of extremal solutions to our FLE (1.1) and prove lower and upper explicit monotone iterative sequences which converge to the extremal solution. In the fourth section, we provide a numerical example to illustrate the validity of our results. The concluding remarks will be given in the last section.

2. Auxiliary notions

To achieve our main purpose, we present here some definitions and basic auxiliary results that are required throughout our paper. Let $\mathcal{J} := [0, b]$, and $C(\mathcal{J})$ be the Banach space of continuous functions $\upsilon : \mathcal{J} \to \mathbb{R}$ equipped with the norm $\|\upsilon\| = \sup\{|\upsilon(\varkappa)| : \varkappa \in \mathcal{J}\}$.

Definition 2.1. *[2] Let f be an integrable function and $\mu > 0$. Also, let ϕ be an increasing and positive monotone function on $(0, b)$, having a continuous derivative ϕ' on $(0, b)$ such that $\phi'(\varkappa) \neq 0$, for all $\varkappa \in \mathcal{J}$. Then the ϕ-Riemann-Liouville fractional integral of f of order μ is defined by*

$$I_{0^+}^{\mu,\phi} f(\varkappa) = \int_0^{\varkappa} \frac{\phi'(s)\,(\phi(\varkappa) - \phi(s))^{\mu-1}}{\Gamma(\mu)} f(s)ds, \ 0 < \varkappa \leq b.$$

Definition 2.2. *[33] Let $n - 1 < \mu < n$, $(n \in \mathbb{N})$, and $f, \phi \in C^n(\mathcal{J})$ such that $\phi'(\varkappa)$ is continuous and satisfying $\phi'(\varkappa) \neq 0$ for all $\varkappa \in \mathcal{J}$. Then the left-sided ϕ-Hilfer fractional derivative of a function f of order μ and type $\beta \in [0, 1]$ is defined by*

$${}^{H}\mathcal{D}_{0^+}^{\mu,\beta,\phi} f(\varkappa) = I_{0^+}^{\beta(n-\mu);\phi} \mathcal{D}_{a^+}^{\gamma;\phi} f(\varkappa), \quad \gamma = \mu + n\beta - \mu\beta,$$

where

$$\mathcal{D}_{0^+}^{\gamma;\phi} f(\varkappa) = f_{\phi}^{[n]} I_{0^+}^{(1-\beta)(n-\mu);\phi} f(\varkappa), \ \ and \ \ f_{\phi}^{[n]} = \left[\frac{1}{\phi'(\varkappa)} \frac{d}{d\varkappa}\right]^n.$$

Lemma 2.3. *[2, 33] Let $n-1<\mu<n$, $0\leq\beta\leq 1$, and $n<\delta\in\mathbb{R}$. For a given function $f:\mathcal{J}\to\mathbb{R}$, we have*

$$I_{0^+}^{\mu,\phi} I_{0^+}^{\beta,\phi} f(\varkappa) = I_{0^+}^{\mu+\beta,\phi} f(\varkappa),$$

$$I_{0^+}^{\mu,\phi} (\phi(\varkappa)-\phi(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mu+\delta)} (\phi(\varkappa)-\phi(0))^{\mu+\delta-1},$$

and

$${}^{H}\mathcal{D}_{0^+}^{\mu,\beta,\phi} (\phi(\varkappa)-\phi(0))^{\delta-1} = 0, \quad \delta<n.$$

Lemma 2.4. *[33] Let $f:\mathcal{J}\to\mathbb{R}$, $n-1<\mu<n$, and $0\leq\beta\leq 1$. Then*

(1) If $f\in C^{n-1}(\mathcal{J})$, then

$$I_{0^+}^{\mu;\phi}\, {}^{H}\mathcal{D}_{0^+}^{\mu,\beta,\phi} f(\varkappa) = f(\varkappa) - \sum_{k=1}^{n-1} \frac{(\phi(\varkappa)-\phi(0))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\phi}^{[n-k]} I_{0^+}^{(1-\beta)(n-\mu);\phi} f(0),$$

(2) If $f\in C(\mathcal{J})$, then

$${}^{H}\mathcal{D}_{0^+}^{\mu,\beta,\phi} I_{0^+}^{\mu;\phi} f(\varkappa) = f(\varkappa).$$

Lemma 2.5. *For $\mu,\beta,\gamma>0$ and $\lambda\in\mathbb{R}$, we have*

$$I_{0^+}^{\mu,\phi} [\phi(\varkappa)-\phi(0)]^{\beta-1} E_{\gamma,\beta}[\lambda(\phi(\varkappa)-\phi(0))^{\gamma}] = [\phi(\varkappa)-\phi(0)]^{\beta+\mu-1} E_{\gamma,\beta+\mu}[\lambda(\phi(\varkappa)-\phi(0))^{\gamma}],$$

where $E_{\gamma,\beta}$ is Mittag-Leffler function with two-parameter defined by

$$E_{\gamma,\beta}(\upsilon) = \sum_{i=1}^{\infty} \frac{\upsilon^i}{\Gamma(\gamma i+\beta)}, \upsilon\in\mathbb{C}.$$

Proof. See [34]. □

Lemma 2.6. *[27] Let $\mu\in(1,2]$ and $\beta>0$ be arbitrary. Then the functions $E_\mu(\cdot)$, $E_{\mu,\mu}(\cdot)$ and $E_{\mu,\beta}(\cdot)$ are nonnegative. Furthermore,*

$$E_\mu(\chi) := E_{\mu,1}(\chi)\leq 1, \quad E_{\mu,\mu}(\chi)\leq\frac{1}{\Gamma(\mu)}, \quad E_{\mu,\beta}(\chi)\leq\frac{1}{\Gamma(\beta)},$$

for $\chi<0$.

Lemma 2.7. *Let $\mu,k,\beta>0$, $\lambda\in\mathbb{R}$ and $f\in C(\mathcal{J})$. Then*

$$I_{0^+}^{k,\phi}\left[I_{0^+}^{\mu,\phi} E_{\mu,\mu}(\lambda(\phi(\varkappa)-\phi(0))^{\mu})\right] = I_{0^+}^{\mu+k,\phi} E_{\mu,\mu+k}(\lambda(\phi(\varkappa)-\phi(0))^{\mu}).$$

Proof. See [34]. □

For some analysis techniques, we will suffice with indication to the classical Banach contraction principle (see [35]).

To transform the ϕ-Hilfer type FLE (1.1) into a fixed point problem, we will present the following Lemma.

Lemma 2.8. *Let $\gamma_j = \mu_j + j\beta_j - \mu_j\beta_j$, $(j = 1, 2)$ such that $\mu_1 \in (0, 1]$, $\mu_2 \in (1, 2]$, $\beta_j \in [0, 1]$, $\lambda_1, \lambda_2 \geq 0$ and $\hbar$ is a function in the space $C(J)$. Then, υ is a solution of the ϕ-Hilfer linear FLE of the form*

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa) = \hbar(\varkappa), \ \varkappa \in (0, b], \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa)\Big|_{\varkappa=0} = 0, \upsilon(0) = 0, \upsilon(b) = \sum_{i=1}^{m} \delta_i I_{0^+}^{\sigma_i,\phi}\upsilon(\zeta_i), \end{cases} \tag{2.1}$$

if and only if υ satisfies the following equation

$$\begin{aligned} \upsilon(\varkappa) = & \frac{[\phi(\varkappa) - \phi(0)]^{\gamma_2-1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right)}{\Theta} \\ & \Big[\Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(b) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(b) - \phi(0)]^{\mu_1}\right)\hbar(b)\right) \\ & - \sum_{i=1}^{m} \delta_i \Gamma(\mu_2) I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2 [\phi(\zeta_i) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\zeta_i) - \phi(0)]^{\mu_1}\right)\hbar(\zeta_i)\right)\Big] \\ & + \Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right) \\ & \Gamma(\mu_1) I_{0^+}^{\mu_1,\phi}\left[E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\varkappa) - \phi(0)]^{\mu_1}\right)\hbar(\varkappa)\right]. \end{aligned} \tag{2.2}$$

where

$$\Theta := \begin{pmatrix} \sum_{i=1}^{m} \delta_i [\phi(\zeta_i) - \phi(0)]^{\gamma_2+\sigma_i-1} E_{\mu_2,\gamma_2+\sigma_i}\left(-\lambda_2 [\phi(\zeta_i) - \phi(0)]^{\mu_2}\right) \\ - [\phi(b) - \phi(0)]^{\gamma_2-1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(b) - \phi(0)]^{\mu_2}\right) \end{pmatrix} \neq 0. \tag{2.3}$$

Proof. Let $\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa) = P(\varkappa)$. Then, the problem (2.1) is equivalent to the following problem

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)P(\varkappa) = \hbar(\varkappa), \ \varkappa \in (0, b], \\ P(0) = 0. \end{cases} \tag{2.4}$$

Applying the operator $I_{0^+}^{\mu_1,\phi}$ to both sides of the first equation of (2.4) and using Lemma 2.4, we obtain

$$P(\varkappa) = \frac{c_0}{\Gamma(\gamma_1)}[\phi(\varkappa) - \phi(0)]^{\gamma_1-1} - \lambda_1 I_{0^+}^{\mu_1,\phi}P(\varkappa) + I_{0^+}^{\mu_1,\phi}\hbar(\varkappa), \tag{2.5}$$

where c_0 is an arbitrary constant. For explicit solutions of Eq (2.4), we use the method of successive approximations, that is

$$P_0(\varkappa) = \frac{c_0}{\Gamma(\gamma_1)}[\phi(\varkappa) - \phi(0)]^{\gamma_1-1}, \tag{2.6}$$

and

$$P_k(\varkappa) = P_0(\varkappa) - \lambda_1 I_{0^+}^{\mu_1,\phi}P_{k-1}(\varkappa) + I_{0^+}^{\mu_1,\phi}\hbar(\varkappa). \tag{2.7}$$

By Definition 2.1 and Lemma 2.3 along with Eq (2.6), we obtain

$$\begin{aligned} P_1(\varkappa) &= P_0(\varkappa) - \lambda_1 I_{0^+}^{\mu_1,\phi}P_0(\varkappa) + I_{0^+}^{\mu_1,\phi}\hbar(\varkappa) \\ &= \frac{c_0}{\Gamma(\gamma_1)}[\phi(\varkappa) - \phi(0)]^{\gamma_1-1} - \lambda_1 I_{0^+}^{\mu_1,\phi}\left(\frac{c_0}{\Gamma(\gamma_1)}[\phi(\varkappa) - \phi(0)]^{\gamma_1-1}\right) + I_{0^+}^{\mu_1,\phi}\hbar(\varkappa) \end{aligned}$$

$$
\begin{aligned}
&= \frac{c_0}{\Gamma(\gamma_1)}\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_1-1}-\lambda_1\frac{c_0}{\Gamma(\gamma_1+\mu_1)}\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_1+\mu_1-1}+I_{0^+}^{\mu_1,\phi}\hbar(\varkappa)\\
&= c_0\sum_{i=1}^{2}\frac{(-\lambda_1)^{i-1}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\beta_1(1-\mu_1)-1}}{\Gamma(i\mu_1+\beta_1(1-\mu_1))}+I_{0^+}^{\mu_1,\phi}\hbar(\varkappa). \qquad (2.8)
\end{aligned}
$$

Similarly, by using Eqs (2.6)–(2.8), we get

$$
\begin{aligned}
P_2(\varkappa) &= P_0(\varkappa)-\lambda_1 I_{0^+}^{\mu_1,\phi}P_1(\varkappa)+I_{0^+}^{\mu_1,\phi}\hbar(\varkappa)\\
&= \frac{c_0}{\Gamma(\gamma_1)}\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_1-1}-\lambda_1 I_{0^+}^{\mu_1,\phi}\\
&\quad \left(c_0\sum_{i=1}^{2}\frac{(-\lambda_1)^{i-1}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\beta_1(1-\mu_1)-1}}{\Gamma(i\mu_1+\beta_1(1-\mu_1))}+I_{0^+}^{\mu_1,\phi}\hbar(\varkappa)\right)+I_{0^+}^{\mu_1,\phi}\hbar(\varkappa)\\
&= c_0\sum_{i=1}^{3}\frac{(-\lambda_1)^{i-1}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\beta_1(1-\mu_1)-1}}{\Gamma(i\mu_1+\beta_1(1-\mu_1))}+\sum_{i=1}^{2}(-\lambda_1)^{i-1}I_{0^+}^{i\mu_1,\phi}\hbar(\varkappa).
\end{aligned}
$$

Repeating this process, we get $P_k(\varkappa)$ as

$$
P_k(\varkappa)=c_0\sum_{i=1}^{k+1}\frac{(-\lambda_1)^{i-1}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\beta_1(1-\mu_1)-1}}{\Gamma(i\mu_1+\beta_1(1-\mu_1))}+\sum_{i=1}^{k}(-\lambda_1)^{i-1}I_{0^+}^{i\mu_1,\phi}\hbar(\varkappa).
$$

Taking the limit $k\to\infty$, we obtain the expression for $P_k(\varkappa)$, that is

$$
P(\varkappa)=c_0\sum_{i=1}^{\infty}\frac{(-\lambda_1)^{i-1}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\beta_1(1-\mu_1)-1}}{\Gamma(i\mu_1+\beta_1(1-\mu_1))}+\sum_{i=1}^{\infty}(-\lambda_1)^{i-1}I_{0^+}^{i\mu_1,\phi}\hbar(\varkappa).
$$

Changing the summation index in the last expression, $i\to i+1$, we have

$$
P(\varkappa)=c_0\sum_{i=0}^{\infty}\frac{(-\lambda_1)^{i}\left[\phi(\varkappa)-\phi(0)\right]^{i\mu_1+\gamma_1-1}}{\Gamma(i\mu_1+\gamma_1)}+\sum_{i=0}^{\infty}(-\lambda_1)^{i}I_{0^+}^{i\mu_1+\mu_1,\phi}\hbar(\varkappa).
$$

From the definition of Mittag-Leffler function, we get

$$
\begin{aligned}
P(\varkappa) &= c_0\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_1-1}E_{\mu_1,\gamma_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right) \qquad (2.9)\\
&\quad +\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)\hbar(\varkappa).
\end{aligned}
$$

By the condition $P(0)=0$, we get $c_0=0$ and hence

Equation (2.9) reduces to

$$
P(\varkappa)=\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)\hbar(\varkappa). \qquad (2.10a)
$$

Similarly, the following equation

$$
\begin{cases}
\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon(\varkappa)=P(\varkappa),\ \varkappa\in(0,b],\\
\upsilon(0)=0,\upsilon(b)=\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\upsilon(\zeta_i)
\end{cases}
$$

is equivalent to

$$\begin{aligned}\upsilon(\varkappa) &= c_1\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)\\ &\quad +c_2\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-2}E_{\mu_2,\gamma_2-1}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)\\ &\quad +\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)P(\varkappa). \end{aligned} \tag{2.11}$$

By the condition $\upsilon(0)=0$, we obtain $c_2=0$ and hence Eq (2.11) reduces to

$$\begin{aligned}\upsilon(\varkappa) &= c_1\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)\\ &\quad +\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)P(\varkappa). \end{aligned} \tag{2.12}$$

By the condition $\upsilon(b)=\sum_{i=1}^{m}\delta_i\, I_{0^+}^{\sigma_i,\phi}\upsilon(\zeta_i)$, we get

$$c_1=\frac{1}{\Theta}\left(\begin{array}{c}\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(b)-\phi(0)\right]^{\mu_2}\right)P(b)\\ -\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\, I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_2}\right)P(\zeta_i)\end{array}\right). \tag{2.13}$$

Put c_0 in Eq (2.12), we obtain

$$\begin{aligned}\upsilon(\varkappa) &= \frac{\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)}{\Theta}\\ &\quad \Big[\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(b)-\phi(0)\right]^{\mu_2}\right)P(b)\\ &\quad -\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\, I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_2}\right)P(\zeta_i)\Big]\\ &\quad +\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}\Big[E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)P(\varkappa)\Big]. \end{aligned} \tag{2.14}$$

Substituting Eq (2.10a) into Eq (2.14), we can get Eq (2.2).

On the other hand, we assume that the solution υ satisfies Eq (2.2). Then, one can get $\upsilon(0)=0$. Applying ${}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}$on both sides of Eq (2.2), we get

$$\begin{aligned}{}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa) &= {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\frac{\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)}{\Theta}\\ &\quad \Big[\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(b)-\phi(0)\right]^{\mu_2}\right)\\ &\quad \Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(b)-\phi(0)\right]^{\mu_1}\right)\hbar(b)\Big)\\ &\quad -\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\, I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_2}\right)\\ &\quad \Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_1}\right)\hbar(\zeta_i)\Big)\Big]\\ &\quad +{}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)\\ &\quad \Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}\Big[E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)\hbar(\varkappa)\Big]. \end{aligned} \tag{2.15}$$

Since $\gamma_2=\mu_2+\beta_2-\mu_2\beta_2$, then, by Lemma 2.3, we have ${}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\left[\phi(\varkappa)-\phi(0)\right]^{\gamma_2-1}=0$ and hence Eq (2.15) reduces to the following equation

$${}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa) = {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)$$

$$\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi}\left[E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)\hbar(\varkappa)\right].$$

By using some properties of Mittag-Leffler function and taking $\varkappa = 0$, we obtain

$$^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(0) = 0.$$

Thus, the derivative condition is satisfied. The proof of Lemma 2.8 is completed. □

Lemma 2.9. *(Comparison Theorem). For $j = 1, 2$, let $\gamma_j = \mu_j + j\beta_j - \mu_j\beta_j$, $\mu_1 \in (0, 1], \mu_2 \in (1, 2]$, $\beta_j \in [0, 1], \lambda_1 \geq 0$ and $\upsilon \in C(\mathcal{J})$ be a continuous function satisfies*

$$\begin{cases} \left(^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\left(^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa) \geq 0, \\ \quad ^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa)\Big|_{\varkappa=0} \geq 0, \upsilon(0) \geq 0, \upsilon(b) \geq 0, \end{cases}$$

then $\upsilon(\varkappa) \geq 0$, $\varkappa \in (0, b]$.

Proof. If $z \geq 0$, then from Lemma 2.6, we have $E_{\mu,\beta}(z) \geq 0$. If $z < 0$, then $E_{\mu,\beta}(z)$ is completely monotonic function [35], that means $E_{\mu,\beta}(z)$ possesses derivatives for all arbitrary integer order and $(-1)^n \frac{d^n}{dz^n}E_{\mu,\beta}(z) \geq 0$. Hence, $E_{\mu,\beta}(z) \geq 0$ for all $z \in \mathbb{R}$. In view of Eq (2.2), Eq (2.9), and from fact that $E_{\mu_1,\gamma_1}(\cdot) \geq 0$ and $E_{\mu,\mu}(\cdot) \geq 0$ with help the definition of ϕ, we obtain $\upsilon(\varkappa) \geq 0$, for $\varkappa \in (0, b]$. (Alternative proof). Let $\left(^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa) = P(\varkappa)$. Then, we have

$$\begin{cases} \left(^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)P(\varkappa) \geq 0, \\ P(0) \geq 0. \end{cases}$$

Assume that $P(\varkappa) \geq 0$ (for all $\varkappa \in (0, b]$) is not true. Then, there exist $\varkappa_1, \varkappa_2$, $(0 < \varkappa_1 < \varkappa_2 \leq b)$ such that $P(\varkappa_2) < 0, P(\varkappa_1) = 0$ and

$$\begin{cases} P(\varkappa) \geq 0, \varkappa \in (0, \varkappa_1), \\ P(\varkappa) < 0, \varkappa \in (\varkappa_1, \varkappa_2). \end{cases}$$

Since $\lambda_1 \geq 0$, we have $\left(^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)P(\varkappa) \geq 0$ for all $\varkappa \in (\varkappa_1, \varkappa_2)$. In view of

$$^{H}\mathcal{D}_{0^+}^{\mu_1,\beta_1,\phi}P(\varkappa) = I_{0^+}^{\beta_1(1-\mu_1);\phi}\left(\frac{1}{\phi'(\varkappa)}\frac{d}{d\varkappa}\right)I_{0^+}^{1-\gamma_1;\phi}P(\varkappa),$$

the operator $I_{0^+}^{1-\gamma_1;\phi}P(\varkappa)$ is nondecreasing on $(\varkappa_1, \varkappa_2)$. Hence

$$I_{0^+}^{1-\gamma_1;\phi}P(\varkappa) - I_{0^+}^{1-\gamma_1;\phi}P(\varkappa_1) \geq 0, \varkappa \in (\varkappa_1, \varkappa_2).$$

On the other hand, for all $\varkappa \in (\varkappa_1, \varkappa_2)$, we have

$$\begin{aligned} I_{0^+}^{1-\gamma_1;\phi}P(\varkappa) - I_{0^+}^{1-\gamma_1;\phi}P(\varkappa_1) &= \frac{1}{\Gamma(1-\gamma_1)}\int_0^{\varkappa}\phi'(s)\left(\phi(\varkappa)-\phi(s)\right)^{1-\gamma_1-1}P(s)ds \\ &\quad -\frac{1}{\Gamma(1-\gamma_1)}\int_0^{\varkappa_1}\phi'(s)\left(\phi(\varkappa_1)-\phi(s)\right)^{1-\gamma_1-1}P(s)ds \\ &= \frac{1}{\Gamma(1-\gamma_1)}\int_0^{\varkappa_1}\phi'(s)\left[\left(\phi(\varkappa)-\phi(s)\right)^{-\gamma_1} - \left(\phi(\varkappa_1)-\phi(s)\right)^{-\gamma_1}\right]P(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(1-\gamma_1)}\int_{\varkappa_1}^{\varkappa}\phi'(s)\left(\phi(\varkappa)-\phi(s)\right)^{-\gamma_1}P(s)ds \\
<\quad & 0,\ \text{for all } \varkappa\in(\varkappa_1,\varkappa_2),
\end{aligned}
$$

which is a contradiction. Therefore, $P(\varkappa)\geq 0$ $(\varkappa\in(0,b])$. By the same technique, one can prove that $\upsilon(\varkappa)\geq 0$, for all $\varkappa\in(0,b]$. □

As a result of Lemma 2.8, we have the following Lemma.

Lemma 2.10. *For $j=1,2$, let $\gamma_j=\mu_j+j\beta_j-\mu_j\beta_j$, $\mu_1\in(0,1]$, $\mu_2\in(1,2]$, $\beta_j\in[0,1]$ and $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ is continuous function. If $\upsilon\in C(\mathcal{J})$ satisfies the problem (1.1), then, υ satisfies the following integral equation*

$$
\begin{aligned}
\upsilon(\varkappa)\quad=\quad & \frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)}{\Theta}\\
&\Big[\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(b)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(b)-\phi(0)]^{\mu_1}\right)f(b,\upsilon(b))\Big)\\
&-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2[\phi(\zeta_i)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\zeta_i)-\phi(0)]^{\mu_1}\right)f(\zeta_i,\upsilon(\zeta_i))\Big)\Big]\\
&+\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\varkappa)-\phi(0)]^{\mu_1}\right)f(\varkappa,\upsilon(\varkappa))\Big).
\end{aligned}
$$

3. Existence of extremal solutions

In this part, we focus on the existence of lower and upper explicit monotone iterative sequences that converge to the extremal solution for the nonlinear ϕ-Hilfer FLE (1.1). The existence of unique solution for the problem (1.1) is based on Banach fixed point theorem. Now, let us give the following definitions:

Definition 3.1. *For $\mathcal{J}=[0,b]\subset\mathbb{R}_+$. Let $\upsilon\in C(\mathcal{J})$. Then, the upper and lower-control functions are defined by*

$$
\overline{f}(\varkappa,\upsilon(\varkappa))=\sup_{0\leq\mathcal{Y}\leq\upsilon}\{f(\varkappa,\mathcal{Y}(\varkappa))\},
$$

and

$$
\underline{f}(\varkappa,\upsilon(\varkappa))=\inf_{\upsilon\leq\mathcal{Y}\leq b}\{f(\varkappa,\mathcal{Y}(\varkappa))\},
$$

respectively. Clearly, $\overline{f}(\varkappa,\upsilon(\varkappa))$ and $\underline{f}(\varkappa,\upsilon(\varkappa))$ are monotonous non-decreasing on $[a,b]$ and

$$
\underline{f}(\varkappa,\upsilon(\varkappa))\leq f(\varkappa,\upsilon(\varkappa))\leq\overline{f}(\varkappa,\upsilon(\varkappa))
$$

Definition 3.2. *Let $\overline{\upsilon}$, $\underline{\upsilon}\in C(\mathcal{J})$ be upper and lower solutions of the problem (1.1) respectively. Then*

$$
\begin{cases}
\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\overline{\upsilon}(\varkappa)\geq\overline{f}(\varkappa,\overline{\upsilon}(\varkappa)),\ \varkappa\in(0,b],\\
{}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\overline{\upsilon}(\varkappa)\big|_{\varkappa=0}\geq 0,\overline{\upsilon}(0)\geq 0,\overline{\upsilon}(b)\geq\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\overline{\upsilon}(\zeta_i),
\end{cases}
$$

and

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}(\varkappa)\le \underline{f}\left(\varkappa,\underline{\upsilon}(\varkappa)\right), \ \varkappa\in(0,b], \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\underline{\upsilon}(\varkappa)\Big|_{\varkappa=0}\le 0, \underline{\upsilon}(0)\le 0, \underline{\upsilon}(b)\le \sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\underline{\upsilon}(\zeta_i). \end{cases}$$

According to Lemma 2.8, we have

$$\begin{aligned} \overline{\upsilon}(\varkappa) \ &\ge \ \frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)}{\Theta} \\ &\Big[\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(b)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(b)-\phi(0)\right]^{\mu_1}\right)f(b,\overline{\upsilon}(b)\right) \\ &-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_1}\right)f(\zeta_i,\overline{\upsilon}(\zeta_i))\right)\Big] \\ &+\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)f(\varkappa,\overline{\upsilon}(\varkappa))\right) \end{aligned}$$

and

$$\begin{aligned} \underline{\upsilon}(\varkappa) \ &\le \ \frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right)}{\Theta} \\ &\Big[\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(b)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(b)-\phi(0)\right]^{\mu_1}\right)f(b,\underline{\upsilon}(b)\right) \\ &-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\zeta_i)-\phi(0)\right]^{\mu_1}\right)f(\zeta_i,\underline{\upsilon}(\zeta_i))\right)\Big] \\ &+\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2\left[\phi(\varkappa)-\phi(0)\right]^{\mu_2}\right) \\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1\left[\phi(\varkappa)-\phi(0)\right]^{\mu_1}\right)f(\varkappa,\underline{\upsilon}(\varkappa))\right). \end{aligned}$$

Theorem 3.3. *Let $\overline{\upsilon}(\varkappa)$ and $\underline{\upsilon}(\varkappa)$ be upper and lower solutions of the problem (1.1), respectively such that $\underline{\upsilon}(\varkappa)\le\overline{\upsilon}(\varkappa)$ on $\mathcal{J}$. Moreover, the function $f(\varkappa,\upsilon)$ is continuous on $\mathcal{J}$ and there exists a constant number $\kappa>0$ such that $|f(\varkappa,\upsilon)-f(\varkappa,v)|\le\kappa|\upsilon-v|$, for $\upsilon,v\in\mathbb{R}^+$, $\varkappa\in\mathcal{J}$. If*

$$\begin{aligned} Q_1 \ &= \ \kappa\frac{[\phi(b)-\phi(0)]^{\gamma_2-1}}{\Gamma(\gamma_2)\Theta}\left[\frac{[\phi(b)-\phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2+1)\Gamma(\mu_1+1)}\right. \\ &\left.+\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\frac{[\phi(\zeta_i)-\phi(0)]^{\mu_2+\mu_1+\sigma_i}}{\Gamma(\mu_2+\sigma_i+1)\Gamma(\mu_2+\sigma_i)\Gamma(\mu_1+1)}\right] \\ &+\kappa\frac{[\phi(b)-\phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2+1)\Gamma(\mu_1+1)}<1, \end{aligned}$$

then the problem (1.1) has a unique solution $\upsilon\in C(\mathcal{J})$.

Proof. Let $\Xi = P - \underline{P}$, where $P(\varkappa) = \left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa)$ and $\underline{P}(\varkappa) = \left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\underline{\upsilon}(\varkappa)$. Then, we get

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\Xi \geq 0, \ \varkappa \in (0,b], \\ \Xi(0) = 0. \end{cases}$$

In view of Lemma 2.9, we have $\Xi(\varkappa) \geq 0$ on $\mathcal{J}$ and hence $\underline{P}(\varkappa) \leq P(\varkappa)$. Since $P(\varkappa) = \left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\upsilon(\varkappa)$ and $\underline{P}(\varkappa) = \left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\underline{\upsilon}(\varkappa)$, by the same technique, we get $\underline{\upsilon}(\varkappa) \leq \upsilon(\varkappa)$. Similarly, we can show that $\upsilon(\varkappa) \leq \overline{\upsilon}(\varkappa)$. Consider the continuous operator $\mathcal{G} : C(\mathcal{J}) \to C(\mathcal{J})$ defined by

$$\begin{aligned} \mathcal{G}\upsilon(\varkappa) = & \frac{[\phi(\varkappa) - \phi(0)]^{\gamma_2 - 1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right)}{\Theta} \\ & \Big[\Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(b) - \phi(0)]^{\mu_2}\right) \\ & \Big(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(b) - \phi(0)]^{\mu_1}\right) f(b, \upsilon(b))\Big) \\ & - \sum_{i=1}^{m} \delta_i \Gamma(\mu_2) I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(-\lambda_2 [\phi(\zeta_i) - \phi(0)]^{\mu_2}\right) \\ & \Big(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\zeta_i) - \phi(0)]^{\mu_1}\right) f(\zeta_i, \upsilon(\zeta_i))\Big)\Big] \\ & + \Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right) \\ & \Big(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\varkappa) - \phi(0)]^{\mu_1}\right) f(\varkappa, \upsilon(\varkappa))\Big). \end{aligned}$$

Clearly, the fixed point of $\mathcal{G}$ is a solution to problem (1.1). Define a closed ball $\mathbb{B}_R$ as

$$\mathbb{B}_R = \left\{\upsilon \in C(\mathcal{J}) : \|\upsilon\|_{C(\mathcal{J})} \leq R, \right\}$$

with

$$R \geq \frac{Q_2}{1 - Q_1},$$

where

$$\begin{aligned} Q_2 = & \mathcal{P}\frac{[\phi(b) - \phi(0)]^{\gamma_2 - 1}}{\Gamma(\gamma_2)\Theta}\Bigg[\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)} \\ & + \sum_{i=1}^{m} \delta_i \Gamma(\mu_2) \frac{[\phi(\zeta_i) - \phi(0)]^{\mu_2+\mu_1+\sigma_i}}{\Gamma(\mu_2 + \sigma_i + 1)\Gamma(\mu_2 + \sigma_i)\Gamma(\mu_1 + 1)}\Bigg] \\ & + \mathcal{P}\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)} \end{aligned}$$

and $\mathcal{P} = \sup_{s\in\mathcal{J}} |f(s, 0)|$. Let $\upsilon \in \mathbb{B}_R$ and $\varkappa \in \mathcal{J}$. Then by Lemma 2.6, we have

$$\begin{aligned} |f(\varkappa, \upsilon(\varkappa))| &= |f(\varkappa, \upsilon(\varkappa)) - f(\varkappa, 0) + f(\varkappa, 0)| \\ &\leq |f(\varkappa, \upsilon(\varkappa)) - f(\varkappa, 0)| + |f(\varkappa, 0)| \\ &\leq \kappa |\upsilon(\varkappa)| + \mathcal{P} \\ &\leq (\kappa \|\upsilon\| + \mathcal{P}). \end{aligned}$$

Now, we will present the proof in two steps:

First step: We will show that $\mathcal{G}(\mathbb{B}_R) \subset \mathbb{B}_R$. First, by Lemma 2.6 and Definition 2.1, we have

$$I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(\lambda_2 \left[\phi(\varkappa) - \phi(0)\right]^{\mu_2}\right) \leq \frac{\left[\phi(\varkappa) - \phi(0)\right]^{\mu_2}}{\Gamma(\mu_2 + 1)\Gamma(\mu_2)}.$$

Next, for $\upsilon \in \mathbb{B}_R$, we obtain

$$\begin{aligned}
&|\mathcal{G}\upsilon(\varkappa)| \\
&\leq \frac{[\phi(b) - \phi(0)]^{\gamma_2 - 1}}{\Gamma(\gamma_2)\Theta}\left[(\kappa\|\upsilon\| + \mathcal{P})\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)}\right. \\
&\left. + \sum_{i=1}^{m} \delta_i \Gamma(\mu_2)\frac{[\phi(\zeta_i) - \phi(0)]^{\mu_2+\mu_1+\sigma_i}}{\Gamma(\mu_2 + \sigma_i + 1)\Gamma(\mu_2 + \sigma_i)\Gamma(\mu_1 + 1)}(\kappa\|\upsilon\| + \mathcal{P})\right] \\
&+ (\kappa\|\upsilon\| + \mathcal{P})\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)} \\
&\leq Q_1 R + Q_2 \\
&\leq R.
\end{aligned}$$

Thus $\mathcal{G}(\mathbb{B}_R) \subset \mathbb{B}_R$.

Second step: We shall prove that $\mathcal{G}$ is contraction. Let $\upsilon, \widehat{\upsilon} \in \mathbb{B}_R$ and $\varkappa \in \mathcal{J}$. Then by Lemma 2.6 and Definition 2.1, we obtain

$$\begin{aligned}
\left\|\mathcal{G}\upsilon - \mathcal{G}\widehat{\upsilon}\right\| &\leq \kappa\left\|\upsilon - \widehat{\upsilon}\right\|\frac{(\phi(b\varkappa) - \phi(0))^{\gamma_2 - 1}}{\Gamma(\gamma_2)\Theta}\left[\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)}\right. \\
&\quad\left. + \sum_{i=1}^{m} \delta_i \Gamma(\mu_2)\frac{[\phi(\zeta_i) - \phi(0)]^{\mu_2+\mu_1+\sigma_i}}{\Gamma(\mu_2 + \sigma_i + 1)\Gamma(\mu_2 + \sigma_i)\Gamma(\mu_1 + 1)}\right] \\
&\quad + \kappa\left\|\upsilon - \widehat{\upsilon}\right\|\frac{[\phi(b) - \phi(0)]^{\mu_2+\mu_1}}{\Gamma(\mu_2 + 1)\Gamma(\mu_1 + 1)} \\
&\leq Q_1\left\|\upsilon - \widehat{\upsilon}\right\|.
\end{aligned}$$

Thus, $\mathcal{G}$ is a contraction. Hence, the Banach contraction principle theorem [35] shows that the problem (1.1) has a unique solution. □

Theorem 3.4. *Assume that $\overline{\upsilon}, \underline{\upsilon} \in C(\mathcal{J})$ be upper and lower solutions of the problem (1.1), respectively, and $\underline{\upsilon}(\varkappa) \leq \overline{\upsilon}(\varkappa)$ on $\mathcal{J}$. In addition, If the continuous function $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ satisfies $f(\varkappa, \upsilon(\varkappa)) \leq f(\varkappa, y(\varkappa))$ for all $\underline{\upsilon}(\varkappa) \leq \upsilon(\varkappa) \leq y(\varkappa) \leq \overline{\upsilon}(\varkappa), \varkappa \in \mathcal{J}$ then there exist monotone iterative sequences $\left\{\underline{\upsilon}_j\right\}_{j=0}^{\infty}$ and $\left\{\overline{\upsilon}_j\right\}_{j=0}^{\infty}$ which uniformly converges on $\mathcal{J}$ to the extremal solutions of problem (1.1) in $\Phi = \left\{\upsilon \in C(\mathcal{J}) : \underline{\upsilon}(\varkappa) \leq \upsilon(\varkappa) \leq \overline{\upsilon}(\varkappa), \varkappa \in \mathcal{J}\right\}$.*

Proof. Step (1): Setting $\underline{\upsilon}_0 = \underline{\upsilon}$ and $\overline{\upsilon}_0 = \overline{\upsilon}$, then given $\left\{\underline{\upsilon}_j\right\}_{j=0}^{\infty}$ and $\left\{\overline{\upsilon}_j\right\}_{j=0}^{\infty}$ inductively define $\underline{\upsilon}_{j+1}$ and $\overline{\upsilon}_{j+1}$ to be the unique solutions of the following problem

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\underline{\upsilon}_{j+1}(\varkappa) = f\left(\varkappa, \underline{\upsilon}_j(\varkappa)\right), \ \varkappa \in \mathcal{J}, \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\underline{\upsilon}_{j+1}(\varkappa)\Big|_{\varkappa=0} = 0, \underline{\upsilon}_{j+1}(0) = 0, \underline{\upsilon}_{j+1}(b) = \sum_{i=1}^{m} \delta_i I_{0^+}^{\sigma_i,\phi}\underline{\upsilon}_{j+1}(\zeta_i). \end{cases} \tag{3.1}$$

and

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\overline{\upsilon}_{j+1}(\varkappa)=f\left(\varkappa,\overline{\upsilon}_j(\varkappa)\right),\ \varkappa\in\mathcal{J}, \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\overline{\upsilon}_{j+1}(\varkappa)\Big|_{\varkappa=0}=0,\overline{\upsilon}_{j+1}(0)=0,\overline{\upsilon}_{j+1}(b)=\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\overline{\upsilon}_{j+1}(\zeta_i). \end{cases} \tag{3.2}$$

By Theorem 3.3, we know that the above problems have a unique solutions in $C(\mathcal{J})$.

Step (2): Now, for $\varkappa\in\mathcal{J}$, we claim that

$$\begin{aligned} \underline{\upsilon}(\varkappa) &= \underline{\upsilon}_0(\varkappa)\le\underline{\upsilon}_1(\varkappa)\le ....\le\underline{\upsilon}_j(\varkappa)\le\underline{\upsilon}_{j+1}(\varkappa) \\ &\le ......\le\overline{\upsilon}_{j+1}(\varkappa)\le\overline{\upsilon}_j(\varkappa)\le ......\le\overline{\upsilon}_1(\varkappa)\le\overline{\upsilon}_0(\varkappa)=\overline{\upsilon}(\varkappa). \end{aligned} \tag{3.3}$$

To confirm this claim, from (3.1) for $j=0$, we have

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}_1(\varkappa)=f\left(\varkappa,\underline{\upsilon}_0(\varkappa)\right),\ j\ge 0, \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\underline{\upsilon}_1(\varkappa)\Big|_{\varkappa=0}=0,\underline{\upsilon}_1(0)=0,\underline{\upsilon}_1(b)=\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\underline{\upsilon}_1(\zeta_i). \end{cases} \tag{3.4}$$

With reference to the definitions of the lower solution $\underline{\upsilon}(\varkappa)=\underline{\upsilon}_0(\varkappa)$ and putting $\Xi(\varkappa)=P_1(\varkappa)-\underline{P}_0(\varkappa)$, where $P_1(\varkappa)=\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon_1(\varkappa)$ and $\underline{P}_0(\varkappa)=\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}_0(\varkappa)$. Then, we get

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\Xi\ge 0,\ \varkappa\in(0,b], \\ \Xi(0)\ge 0. \end{cases}$$

Consequently, Lemma 2.9 implies $\Xi(\varkappa)\ge 0$, that means $\underline{P}_0(\varkappa)\le P_1(\varkappa),\varkappa\in\mathcal{J}$ and by the same technique, where $P(\varkappa)=\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon(\varkappa)$ we get $\upsilon(\varkappa)\ge 0$. Hence, $\underline{\upsilon}_0(\varkappa)\le\underline{\upsilon}_1(\varkappa),\varkappa\in\mathcal{J}$. Now, from Eq (3.4) and our assumptions, we infer that

$$\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}_1(\varkappa)=f\left(\varkappa,\underline{\upsilon}_0(\varkappa)\right)\le f\left(\varkappa,\underline{\upsilon}_1(\varkappa)\right).$$

Therefore, $\underline{\upsilon}_1$ is a lower solution of problem (1.1). In the same way of the above argument, we conclude that $\underline{\upsilon}_1(\varkappa)\le\underline{\upsilon}_2(\varkappa),\varkappa\in\mathcal{J}$. By mathematical induction, we get $\underline{\upsilon}_j(\varkappa)\le\underline{\upsilon}_{j+1}(\varkappa),\varkappa\in\mathcal{J}, j\ge 2$.

Similarly, we put $\Xi(\varkappa)=\overline{P}_1(\varkappa)-\underline{P}_1(\varkappa)$, where $\overline{P}_1(\varkappa)=\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\overline{\upsilon}_1(\varkappa)$ and $\underline{P}_1(\varkappa)=\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}_1(\varkappa)$. Then, we get

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\Xi(\varkappa)\ge 0,\ \varkappa\in(0,b], \\ \Xi(0)\ge 0. \end{cases}$$

Consequently, Lemma 2.9 implies $\Xi(\varkappa)\ge 0$, that means $\overline{P}_1(\varkappa)\le\underline{P}_1(\varkappa),\varkappa\in\mathcal{J}$ and by the same technique, we get $\overline{\upsilon}_1(\varkappa)\ge\underline{\upsilon}_1(\varkappa),\varkappa\in\mathcal{J}$. By mathematical induction, we get $\overline{\upsilon}_j(\varkappa)\ge\underline{\upsilon}_j(\varkappa)$, $\varkappa\in\mathcal{J}$, $j\ge 0$.

Step (3): In view of Eq (3.3), one can show that the sequences $\left\{\underline{\upsilon}_j\right\}_{j=0}^{\infty}$ and $\left\{\overline{\upsilon}_j\right\}_{j=0}^{\infty}$ are equicontinuous and uniformly bounded. In view of Arzela-Ascoli Theorem, we have $\lim_{j\to\infty}\underline{\upsilon}_j=\upsilon_*$ and $\lim_{j\to\infty}\overline{\upsilon}_j=\upsilon^*$ uniformly on J and the limit of the solutions υ_* and υ^* satisfy the problem (1.1). Moreover, υ_*, $\upsilon^*\in\Phi$.

Step (4): We will prove that υ_* and υ^* are the extremal solutions of the problem (1.1) in Φ. For this end, let $\upsilon\in\Phi$ be a solution of the problem (1.1) such that $\overline{\upsilon}_j(\varkappa)\ge\upsilon(\varkappa)\ge\underline{\upsilon}_j(\varkappa),\varkappa\in\mathcal{J}$, for some $j\in\mathbb{N}$. Therefore, by our assumption, we find that

$$f\left(\varkappa,\overline{\upsilon}_j(\varkappa)\right)\ge f(\varkappa,\upsilon(\varkappa))\ge f\left(\varkappa,\underline{\upsilon}_j(\varkappa)\right).$$

Hence

$$\begin{aligned}&\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\overline{\upsilon}_{j+1}(\varkappa)\\ &\geq\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon(\varkappa)\\ &\geq\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\underline{\upsilon}_{j+1}(\varkappa),\end{aligned}$$

and

$${}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\overline{\upsilon}_{j+1}(\varkappa)\Big|_{\varkappa=0}={}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa)\Big|_{\varkappa=0}={}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\underline{\upsilon}_{j+1}(\varkappa)\Big|_{\varkappa=0}=0.$$

Consequently, $\overline{\upsilon}_{j+1}(\varkappa)\geq\upsilon(\varkappa)\geq\underline{\upsilon}_{j+1}(\varkappa),\varkappa\in\mathcal{J}$. It follows that

$$\overline{\upsilon}_{j}(\varkappa)\geq\upsilon(\varkappa)\geq\underline{\upsilon}_{j}(\varkappa),\varkappa\in\mathcal{J},\ j\in\mathbb{N}.\tag{3.5}$$

Taking the limit of Eq (3.5) as $j\to\infty$, we get $\upsilon^*(\varkappa)\geq\upsilon(\varkappa)\geq\upsilon_*(\varkappa)$, $\varkappa\in\mathcal{J}$. That is, υ^* and υ_* are the extremal solutions of the problem (1.1) in Φ. □

Corollary 3.5. *Assume that $f:\mathcal{J}\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous, and there exist $\aleph_1,\aleph_2>0$ such that*

$$\aleph_1\leq f(\varkappa,\upsilon)\leq\aleph_2,\ \forall(\varkappa,\upsilon)\in\mathcal{J}\times\mathbb{R}^+.\tag{3.6}$$

Then the problem (1.1) has at least one solution $\upsilon(\varkappa)\in C(\mathcal{J})$. Moreover

$$\begin{aligned}\upsilon(\varkappa)\ \leq\ &\frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)}{\Theta}\\ &\Big[\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(b)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(b)-\phi(0)]^{\mu_1}\right)\aleph_2\right)\\ &-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2[\phi(\zeta_i)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\zeta_i)-\phi(0)]^{\mu_1}\right)\aleph_2\right)\Big]\\ &+\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\varkappa)-\phi(0)]^{\mu_1}\right)\aleph_2\right)\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\upsilon(\varkappa)\ \geq\ &\frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1}E_{\mu_2,\gamma_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)}{\Theta}\\ &\Big[\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(b)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(b)-\phi(0)]^{\mu_1}\right)\aleph_1\right)\\ &-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)I_{0^+}^{\mu_2+\sigma_i,\phi}E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2[\phi(\zeta_i)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\zeta_i)-\phi(0)]^{\mu_1}\right)\aleph_1\right)\Big]\\ &+\Gamma(\mu_2)I_{0^+}^{\mu_2,\phi}E_{\mu_2,\mu_2}\left(-\lambda_2[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)\\ &\left(\Gamma(\mu_1)I_{0^+}^{\mu_1,\phi}E_{\mu_1,\mu_1}\left(-\lambda_1[\phi(\varkappa)-\phi(0)]^{\mu_1}\right)\aleph_1\right).\end{aligned}\tag{3.8}$$

Proof. From Eq (3.6) and definition of control functions, we get

$$\aleph_1 \leq \underline{f}(\varkappa, \upsilon(\varkappa)) \leq \overline{f}(\varkappa, \upsilon(\varkappa)) \leq \aleph_2, \ \forall (\varkappa, \upsilon) \in \mathcal{J} \times \mathbb{R}^+. \tag{3.9}$$

Now, we consider the following problem

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\overline{\upsilon}(\varkappa) = \aleph_2, \ \varkappa \in (0, b], \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\overline{\upsilon}(\varkappa)\Big|_{\varkappa=0} = 0, \ \overline{\upsilon}(0) = 0, \ \overline{\upsilon}(b) = \sum_{i=1}^{m} \delta_i I_{0^+}^{\sigma_i,\phi}\overline{\upsilon}(\zeta_i). \end{cases} \tag{3.10}$$

In view of Lemma 2.8, the problem (3.10) has a solution

$$\begin{aligned} \overline{\upsilon}(\varkappa) \ = \ & \frac{[\phi(\varkappa) - \phi(0)]^{\gamma_2 - 1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right)}{\Theta} \\ & \Big[\Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(b) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(b) - \phi(0)]^{\mu_1}\right)\aleph_2\right) \\ & - \sum_{i=1}^{m} \delta_i \Gamma(\mu_2) I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2 [\phi(\zeta_i) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\zeta_i) - \phi(0)]^{\mu_1}\right)\aleph_2\right)\Big] \\ & + \Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\varkappa) - \phi(0)]^{\mu_1}\right)\aleph_2\right). \end{aligned}$$

Taking into account Eq (3.9), we obtain

$$\begin{aligned} \overline{\upsilon}(\varkappa) \ \geq \ & \frac{[\phi(\varkappa) - \phi(0)]^{\gamma_2 - 1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right)}{\Theta} \\ & \Big[\Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(b) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(b) - \phi(0)]^{\mu_1}\right)\overline{f}(b, \overline{\upsilon}(b))\right) \\ & - \sum_{i=1}^{m} \delta_i \Gamma(\mu_2) I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2 [\phi(\zeta_i) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\zeta_i) - \phi(0)]^{\mu_1}\right)\overline{f}(\zeta_i, \overline{\upsilon}(\zeta_i))\right)\Big] \\ & + \Gamma(\mu_2) I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right) \\ & \left(\Gamma(\mu_1) I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1 [\phi(\varkappa) - \phi(0)]^{\mu_1}\right)\overline{f}(\varkappa, \overline{\upsilon}(\varkappa))\right). \end{aligned}$$

It is obvious that $\overline{\upsilon}(\varkappa)$ is the upper solution of problem (1.1). Also, we consider the following problem

$$\begin{cases} \left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi} + \lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi} + \lambda_2\right)\underline{\upsilon}(\varkappa) = \aleph_1, \ \varkappa \in (0, b], \\ {}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\underline{\upsilon}(\varkappa)\Big|_{\varkappa=0} = 0, \ \underline{\upsilon}(0) = 0, \ \underline{\upsilon}(b) = \sum_{i=1}^{m} \delta_i I_{0^+}^{\sigma_i,\phi}\underline{\upsilon}(\zeta_i). \end{cases} \tag{3.11}$$

In view of Lemma 2.8, the problem (3.11) has a solution

$$\underline{\upsilon}(\varkappa) \ = \ \frac{[\phi(\varkappa) - \phi(0)]^{\gamma_2 - 1} E_{\mu_2,\gamma_2}\left(-\lambda_2 [\phi(\varkappa) - \phi(0)]^{\mu_2}\right)}{\Theta}$$

$$
\begin{aligned}
&\Big[\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2\,[\phi(b)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(b)-\phi(0)]^{\mu_1}\right)\aleph_1\Big)\\
&-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\, I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2\,[\phi(\zeta_i)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(\zeta_i)-\phi(0)]^{\mu_1}\right)\aleph_1\Big)\Big]\\
&+\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2\,[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(\varkappa)-\phi(0)]^{\mu_1}\right)\aleph_1\Big).
\end{aligned}
$$

Taking into account Eq (3.9), we obtain

$$
\begin{aligned}
\underline{\upsilon}(\varkappa) \;\le\;& \frac{[\phi(\varkappa)-\phi(0)]^{\gamma_2-1} E_{\mu_2,\gamma_2}\left(-\lambda_2\,[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)}{\Theta}\\
&\Big[\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2\,[\phi(b)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(b)-\phi(0)]^{\mu_1}\right)\underline{f}\left(b,\underline{\upsilon}(b)\right)\Big)\\
&-\sum_{i=1}^{m}\delta_i\Gamma(\mu_2)\, I_{0^+}^{\mu_2+\sigma_i,\phi} E_{\mu_2,\mu_2+\sigma_i}\left(\lambda_2\,[\phi(\zeta_i)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(\zeta_i)-\phi(0)]^{\mu_1}\right)\underline{f}\left(\zeta_i,\underline{\upsilon}(\zeta_i)\right)\Big)\Big]\\
&+\Gamma(\mu_2)\, I_{0^+}^{\mu_2,\phi} E_{\mu_2,\mu_2}\left(-\lambda_2\,[\phi(\varkappa)-\phi(0)]^{\mu_2}\right)\\
&\Big(\Gamma(\mu_1)\, I_{0^+}^{\mu_1,\phi} E_{\mu_1,\mu_1}\left(-\lambda_1\,[\phi(\varkappa)-\phi(0)]^{\mu_1}\right)\underline{f}\left(\varkappa,\underline{\upsilon}(\varkappa)\right)\Big).
\end{aligned}
$$

Thus, $\underline{\upsilon}(\varkappa)$ is the lower solution of problem (1.1).

The application of Theorem 3.4 results that problem (1.1) has at least one solution $\upsilon(\varkappa) \in C(\mathcal{J})$ that satisfies the inequalities (3.7) and (3.8). □

4. An example

Example 4.1. *Let us consider the following problem*

$$
\begin{cases}
\left({}^{H}D_{0^+}^{\mu_1,\beta_1;\phi}+\lambda_1\right)\left({}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}+\lambda_2\right)\upsilon(\varkappa)=f(\varkappa,\upsilon(\varkappa)), \;\varkappa\in[0,1],\\
{}^{H}D_{0^+}^{\mu_2,\beta_2;\phi}\upsilon(\varkappa)\Big|_{\varkappa=0}=0,\upsilon(0)=0,\upsilon(b)=\sum_{i=1}^{m}\delta_i I_{0^+}^{\sigma_i,\phi}\upsilon(\zeta_i),
\end{cases}
\tag{4.1}
$$

Here $\mu_1=\frac{1}{2},\mu_2=\frac{3}{2},\beta_1=\beta_2=\frac{1}{3},\gamma_1=\frac{2}{3},\gamma_2=\frac{4}{3},\lambda_1=\lambda_2=10,m=1,\delta_1=\frac{1}{4},\sigma_1=\frac{2}{3},\zeta_1=\frac{3}{4},b=1,$ $\phi=e^{\varkappa},\lambda_1=\lambda_2=10$ *and we set* $f(\varkappa,\upsilon(\varkappa))=2+\varkappa^2+\frac{\varkappa^3}{5(1+\upsilon(\varkappa))}\upsilon(\varkappa)$*. For* $\upsilon,w\in\mathbb{R}^+,\varkappa\in\mathcal{J}$*, we have*

$$
\begin{aligned}
|f(\varkappa,\upsilon)-f(\varkappa,w)| &= \left|\left(2+\varkappa^2+\frac{\varkappa^3}{5(1+\upsilon(\varkappa))}\upsilon(\varkappa)\right)-\left(2+\varkappa^2+\frac{\varkappa^3}{5(1+w(\varkappa))}w(\varkappa)\right)\right|\\
&\le \frac{1}{5}\,|\upsilon(\varkappa)-w(\varkappa)|\,.
\end{aligned}
$$

By the given data, we get $Q_1\approx 0.9<1$ *and hence all conditions in Theorem 3.3 are satisfied with* $\kappa=\frac{1}{5}>0$*. Thus, the problem (4.1) has a unique solution* $\upsilon\in C(\mathcal{J})$*. On the other hand, from*

Theorem 3.4 and Theorem 3.3, the sequences $\left\{\underline{\upsilon}_n\right\}_{n=0}^{\infty}$ *and* $\{\overline{\upsilon}_n\}_{n=0}^{\infty}$ *can be obtained as*

$$\begin{aligned}\overline{\upsilon}_{n+1}(\varkappa) &= \Gamma\left(\frac{3}{2}\right) I_{0^+}^{\frac{3}{2},e^{\varkappa}} E_{\frac{3}{2},\frac{3}{2}}\left(10\left[e^{\varkappa}-1\right]^{\frac{3}{2}}\right) \\ &\quad \left(\Gamma\left(\frac{1}{2}\right) I_{0^+}^{\frac{1}{2},e^{\varkappa}} E_{\frac{1}{2},\frac{1}{2}}\left(10\left[e^{\varkappa}-1\right]^{\frac{1}{2}}\right)\left(2+\varkappa^2+\frac{1}{5\left(1+\overline{\upsilon}_n(\varkappa)\right)}\varkappa^3\overline{\upsilon}_n(\varkappa)\right)\right). \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\underline{\upsilon}_{n+1}(\varkappa) &= \Gamma\left(\frac{3}{2}\right) I_{0^+}^{\frac{3}{2},e^{\varkappa}} E_{\frac{3}{2},\frac{3}{2}}\left(10\left[e^{\varkappa}-1\right]^{\frac{3}{2}}\right) \\ &\quad \left(\Gamma\left(\frac{1}{2}\right) I_{0^+}^{\frac{1}{2},e^{\varkappa}} E_{\frac{1}{2},\frac{1}{2}}\left(10\left[e^{\varkappa}-1\right]^{\frac{1}{2}}\right)\left(2+\varkappa^2+\frac{1}{5\left(1+\underline{\upsilon}_n(\varkappa)\right)}\varkappa^3\underline{\upsilon}_n(\varkappa)\right)\right). \end{aligned} \tag{4.3}$$

Moreover, for any $\upsilon \in \mathbb{R}^+$ *and* $\varkappa \in [0,1]$, *we have*

$$\begin{aligned}\lim_{\upsilon\to+\infty} f\left(\varkappa,\upsilon(\varkappa)\right) &= \lim_{\upsilon\to+\infty}\left(2+\varkappa^2+\frac{\varkappa^3}{5\left(1+\upsilon(\varkappa)\right)}\upsilon(\varkappa)\right) \\ &= 2+\varkappa^2+\frac{\varkappa^3}{5}.\end{aligned}$$

It follows that

$$2 < f\left(\varkappa,\upsilon(\varkappa)\right) < \frac{16}{5}.$$

Thus, by Corollary 3.5, we get $\aleph_1 = 2$ *and* $\aleph_2 = \frac{16}{5}$. *Then by Definitions 3.1 and 3.2, the problem (4.1) has a solution which verifies* $\underline{\upsilon}(\varkappa) \le \upsilon(\varkappa) \le \overline{\upsilon}(\varkappa)$ *where*

$$\begin{aligned}\overline{\upsilon}(\varkappa) &= \frac{\left(e^{\varkappa}-1\right)^{\frac{4}{3}-1} E_{\frac{3}{2},\frac{4}{3}}\left(-10\left(e^{\varkappa}-1\right)^{\frac{3}{2}}\right)}{\Theta} \\ &\quad 2\Bigg[\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)(e-1)^2 E_{\frac{3}{2},3}\left(-10\left(e-1\right)^{\frac{3}{2}}\right) E_{\frac{1}{2},1}\left(-10\left(e-1\right)^{\frac{1}{2}}\right) \\ &\quad -\frac{4}{5}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(e^{\frac{3}{4}}-1\right)^{\frac{7}{3}} E_{\frac{3}{2},\frac{21}{6}}\left(-10\left(e^{\frac{3}{4}}-1\right)^{\frac{3}{2}}\right) E_{\frac{1}{2},1}\left(-10\left(e^{\frac{3}{4}}-1\right)^{\frac{1}{2}}\right)\Bigg] \\ &\quad +\frac{16}{5}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(e^{\varkappa}-1\right)^2 E_{\frac{3}{2},3}\left(-10\left(e-1\right)^{\frac{3}{2}}\right) E_{\frac{1}{2},1}\left(-10\left(e^{\varkappa}-1\right)^{\frac{1}{2}}\right),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\underline{\upsilon}(\varkappa) &= \frac{\left(e^{\varkappa}-1\right)^{\frac{4}{3}-1} E_{\frac{3}{2},\frac{4}{3}}\left(-10\left(e^{\varkappa}-1\right)^{\frac{3}{2}}\right)}{\Theta} \\ &\quad \frac{16}{5}\Bigg[\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)(e-1)^2 E_{\frac{3}{2},3}\left(-10\left(e-1\right)^{\frac{3}{2}}\right) E_{\frac{1}{2},1}\left(-10\left(e-1\right)^{\frac{1}{2}}\right) \\ &\quad -\frac{1}{2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(e^{\frac{3}{4}}-1\right)^{\frac{7}{3}} E_{\frac{3}{2},\frac{21}{6}}\left(-10\left(e^{\frac{3}{4}}-1\right)^{\frac{3}{2}}\right) E_{\frac{1}{2},1}\left(-10\left(e^{\frac{3}{4}}-1\right)^{\frac{1}{2}}\right)\Bigg]\end{aligned}$$

$$+2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)(e^{\varkappa}-1)^{2}E_{\frac{3}{2},3}\left(-10\left(e-1\right)^{\frac{3}{2}}\right)E_{\frac{1}{2},1}\left(-10\left(e^{\varkappa}-1\right)^{\frac{1}{2}}\right), \tag{4.5}$$

are respectively the upper and lower solutions of the problem (4.1) and

$$\Theta := \left(\frac{1}{4}\left[e^{\frac{3}{4}}-1\right]^{1}E_{\frac{3}{2},2}\left(-10\left(e^{\frac{3}{4}}-1\right)^{\frac{3}{2}}\right)-\left[e-1\right]^{\frac{4}{3}-1}E_{\frac{3}{2},\frac{4}{3}}\left(-10\left(e-1\right)^{\frac{3}{2}}\right)\right)\neq 0.$$

Let us see graphically, we plot in Figure 1 the behavior of the upper solution $\overline{\upsilon}$ and lower solution $\underline{\upsilon}$ of the problem (4.1) with given data above.

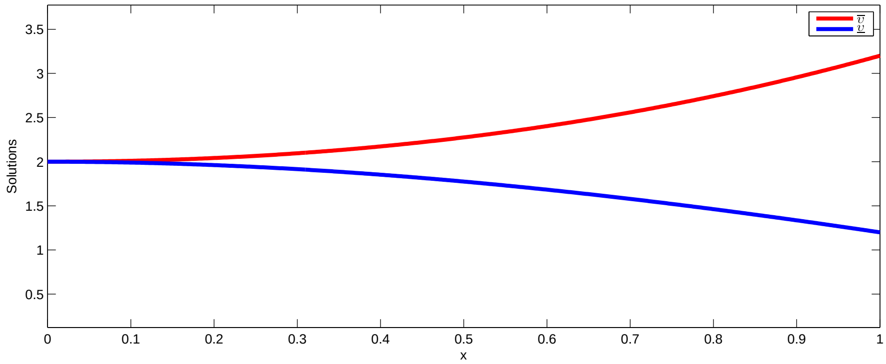

Figure 1. Graphical presentation of $(\underline{\upsilon}, \overline{\upsilon})$.

5. Conclusions

In this work, we have proved successfully the monotone iterative method is an effective method to study FLEs in the frame of ϕ-Hilfer fractional derivative with multi-point boundary conditions. Firstly, the formula of explicit solution of ϕ-Hilfer type FLE (1.1) in the term of Mittag-Leffler function has been derived. Next, we have investigated the lower and upper explicit monotone iterative sequences and proved that converge to the extremal solution of boundary value problems with multi-point boundary conditions. Finally, a numerical example has been given in order to illustrate the validity of our results.

Furthermore, it will be very important to study the present problem in this article regarding the Mittag-Leffler power low [36], the generalized Mittag-Leffler power low with another function [37,38], and the fractal-fractional operators [39].

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Conflict of interest

The authors declare that they have no competing interests.

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