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Research article

Hermite-Hadamard like inequalities for fractional integral operator via convexity and quasi-convexity with their applications

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Abstract: Since the supposed Hermite-Hadamard inequality for a convex function was discussed, its expansions, refinements, and variations, which are called Hermite-Hadamard type inequalities, have been widely explored. The main objective of this article is to acquire new Hermite-Hadamard type inequalities employing the Riemann-Liouville fractional operator for functions whose third derivatives of absolute values are convex and quasi-convex in nature. Some special cases of the newly presented results are discussed as well. As applications, several estimates concerning Bessel functions and special means of real numbers are illustrated.

Keywords: convexity; quasi-convexity; Hermite-Hadamard inequality; Hölder's Rogers inequality; Young's inequality; power mean inequality
Mathematics Subject Classification: Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15

1. Introduction

The idea of convexity is all around recognized in the theory of inequality, which is widely used in mathematical analysis, probability theory, operations research, finance, decision making, and numerical analysis. They are not only strictly related to continuity and differentiability but also play significant roles in inequalities. The study of integral inequality is an interesting area for research in mathematical analysis [1, 2]. The fundamental integral inequalities can be instrumental in cultivating the subjective properties of convexity. The existence of massive literature surrounding integral inequalities for convex functions [3–7] depicts the importance of this topic. The most beautiful fact about convex function is that, it has a very elegant representation based on an inequality presented when the functional value of a linear combination of two points in its domain does not exceed the linear combination of the functional values at those two points.

A few researchers have studied the concept of convex functions in different directions with the help of some innovative ideas in the field of mathematical analysis, for example (see [8–10]). In recent times, many generalization have been introduced in the framework of convexity such as s-convex function [11], quasi-convex function [12], strongly-convex function [13], m-convex function [14], and (α, m)-convex function [15].

Now, we recall the notion of the convexity.

Definition 1.1. A mapping $\Phi : [\sigma_1, \sigma_2] \subset \Re \rightarrow \Re$ is said to be convex, if

$$\Phi(\mu x + (1-\mu)y) \leq \mu\Phi(u) + (1-\mu)\Phi(y) \tag{1.1}$$

holds for all $x, y \in [\sigma_1, \sigma_2]$, and $\mu \in [0, 1]$.

Now we recall the basic definition, so is said to be quasi-convex function.

Definition 1.2. [12] A mapping $\Phi : [\sigma_1, \sigma_2] \subset \Re \rightarrow \Re$ is called to be quasi-convex on $[\sigma_1, \sigma_2]$, if

$$\Phi(\mu x + (1-\mu)y) \leq max\{\Phi(x), \Phi(y)\} \tag{1.2}$$

holds for any $x, y \in [\sigma_1, \sigma_2]$, $\mu \in [0, 1]$.

It is essential to note that, any convexity is a quasi-convexity but the reverse is not true in general case. In the following example, we describe the reverse case:

Example 1.1. *[16] A mapping* $\Phi : [-2, 2] \rightarrow \Re$, *defined by*

$$\Phi(u) = \begin{cases} 1 & for\ \ u \in [-2, -1], \\ u^2 & for\ \ u \in (-1, 2], \end{cases}$$

is not convex on interval $[-2, 2]$ *but it is easy to note that the function is quasi-convex on* $[-2, 2]$.

It is noted that Φ is quasi-convex if and only if all the level sets of Φ, are intervals (convex sets of the line).

It's a fact that the convexity theory may be the most key and significant theory in the speculation of mathematical inequalities, it has many applications in pure and applied mathematics, statistics, economics, and many more. Lately, the theories, expansions, varieties, and refinements for convexity have attracted the attention of several researchers. Numerous fundamental inequalities so far are created by various analysts in due time, the Hermite-Hadamard integral inequality plays an exceptionally indispensable in the field of pure and applied Mathematics.

The Hermite-Hadamard (H-H) inequality emphasizes that, if a mapping $\Phi : J \subset \Re \rightarrow \Re$ is convex in J for $\sigma_1, \sigma_2 \in J$ and $\sigma_1 < \sigma_2$, then

$$\Phi\left(\frac{\sigma_1 + \sigma_2}{2}\right) \leq \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \Phi(\mu) d\mu \leq \frac{\Phi(\sigma_1) + \Phi(\sigma_2)}{2}. \tag{1.3}$$

Fractional calculus owes its starting point to whether or not the importance of a derivative to an integer order could be generalized to a fractional order which is not an integer. Following this unique conversation between L'Hopital and Leibniz, the concept of fractional calculus grabbed the eye of some extraordinary researchers like Euler, Laplace, Fourier, Lacroix, Abel, Riemann, and Liouville. Over time, fractional operators have been differentiated with their singularity, locality and having general forms with the improvements made in their kernel structures. In this sense, based on the basic concepts of fractional analysis, Riemann-Liouville (R-L) and Caputo operators, various new trends have been successful. Fractional integral inequalities are marvelous tools for building up the qualitative and quantitative properties of convex functions. There has been a ceaseless development of intrigue in such a region of research so as to address the issues of different utilizations of these variants. Interested readers can refer to [9, 19, 23, 26].

Suppose $\Phi \in \mathcal{L}[\sigma_1, \sigma_2]$. Then the Riemann-Liouville (R-L) fractional integrals of order $\omega > 0$ with $\sigma_1 \geq 0$ are defined as follows:

$$J^{\omega}_{\sigma_1^+}\Phi(z) = \frac{1}{\Gamma(\omega)}\int_{\sigma_1}^{z}(z-\mu)^{\omega-1}\ \Phi(\mu)\ d\mu\,, \quad z > \sigma_1$$

and

$$J^{\omega}_{\sigma_2^-}\Phi(z) = \frac{1}{\Gamma(\omega)}\int_{z}^{\sigma_2}(\mu-z)^{\omega-1}\ \Phi(\mu)\ d\mu \quad z < \sigma_2.$$

In [25], Sarikaya et al. described the following fractional integral version of Hermite-Hadamard inequality.

Theorem 1.1. *Suppose a mapping $\Phi : [\sigma_1, \sigma_2] \to \Re$ is positive with $0 \leq \sigma_1 < \sigma_2$ and $\Phi \in \mathcal{L}[\sigma_1, \sigma_2]$. If the convexity of Φ on $[\sigma_1, \sigma_2]$, then the following inequality*

$$\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) \leq \frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}[J^{\omega}_{\sigma_1^+}\Phi(\sigma_2) + J^{\omega}_{\sigma_2^-}\Phi(\sigma_1)] \leq \frac{\Phi(\sigma_1)+\Phi(\sigma_2)}{2},$$

satisfies with $\omega > 0$.

Owing to the aforementioned trend and inspired by the ongoing activities, the rest of this paper is organized as follows: First, in Section 1, we discuss some preliminary concepts about convexity and theory of inequality. Next, Sections 2 and 3 deal with our main results, where we have presented a new fractional identity and employing this new identity, we have derived several results for convexity and quasi convexity. In Sections 4 and 5 , we present some applications of our established result in the form of special means and modified Bessel functions. Finally, in Section 6, we present the conclusion of the paper.

In the sense of above indices, we generalize the results proved in [17] to develop some new modifications Hermite-Hadamard (H–H) type inequalities for thrice differentiable convexity and quasi-convexity. Throughout the article we assume that $\omega > 0$.

2. Main results

In order to establish our main results, firstly we need to prove the following equality.

Lemma 2.1. *Suppose a mapping $\Phi : I \subseteq \Re \to \Re$ is thrice differentiable on I^o (the interior of I) with $\sigma_2 > \sigma_1$ and $\Phi''' \in \mathcal{L}[\sigma_1, \sigma_2]$, then the following equality for fractional integral satisfies:*

$$\begin{aligned} &\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) \\ &-\frac{2^{\omega}\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^{\omega}}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\} \\ &=\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left\{\int_0^1\mu^{\omega+2}\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)d\mu\right. \\ &\left.-\int_0^1(1-\mu)^{\omega+2}\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)d\mu\right\}. \end{aligned} \tag{2.1}$$

Proof. It is easy to write that

$$\begin{aligned} &\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left\{\int_0^1\mu^{\omega+2}\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)d\mu\right. \\ &\left.-\int_0^1(1-\mu)^{\omega+2}\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)d\mu\right\} \\ &=\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\{I_1-I_2\}. \end{aligned} \tag{2.2}$$

Where

$$I_1=\int_0^1\mu^{\omega+2}\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)d\mu.$$

Using integrating by parts

$$\begin{aligned} I_1 &=\mu^{\omega+2}\frac{\Phi''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)}{\frac{\sigma_2-\sigma_1}{2}}\Bigg|_0^1-\int_0^1\frac{\Phi''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)}{\frac{\sigma_2-\sigma_1}{2}}(\omega+2)\mu^{\omega+1}d\mu \\ &=\frac{2}{\sigma_2-\sigma_1}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{2(\omega+2)}{\sigma_2-\sigma_1}\int_0^1\mu^{\omega+1}\Phi''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)d\mu \\ &=\frac{2}{\sigma_2-\sigma_1}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{2(\omega+2)}{\sigma_2-\sigma_1}\left\{\frac{2}{\sigma_2-\sigma_1}\Phi'\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{4(\omega+1)}{(\sigma_2-\sigma_1)^2}\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)\right. \\ &\left.+\frac{2^{\omega+2}(\omega+2)}{(\sigma_2-\sigma_1)^{\omega+2}}J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)\right\} \\ &=\frac{2}{\sigma_2-\sigma_1}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{4(\omega+2)}{(\sigma_2-\sigma_1)^2}\Phi'\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{8(\omega+1)(\omega+2)}{(\sigma_2-\sigma_1)^3}\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) \\ &-\frac{2^{\omega+3}\Gamma(\omega+3)}{(\sigma_2-\sigma_1)^{\omega+3}}J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1) \end{aligned}$$

and similarly, we can get

$$
\begin{aligned}
I_2 &= \int_0^1 (1-\mu)^{\omega+2}\, \Phi'''\left(\mu\sigma_2 + (1-\mu)\,\frac{\sigma_1+\sigma_2}{2}\right) d\mu \\
&= -\,\frac{2}{\sigma_2-\sigma_1}\,\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{4\,(\omega+2)}{(\sigma_2-\sigma_1)^2}\Phi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&\quad - \frac{8\,(\omega+1)\,(\omega+2)}{(\sigma_2-\sigma_1)^3}\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&\quad + \frac{2^{\omega+3}\Gamma\,(\omega+3)}{(\sigma_2-\sigma_1)^{\omega+3}}\, J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\,\Phi\,(\sigma_2)\,.
\end{aligned}
$$

By using the value of I_1 and I_2 in (2.2), we can get (2.1).

This completes the proof. □

Remark 2.1. If we set $\omega = 1$, in Lemma 2.1, then we get Lemma 2.1 in [21].

Theorem 2.1. *Suppose a mapping $\Phi : I \subseteq \Re \to \Re$ is thrice differentiable on I^o (the interior of I) with $\sigma_2 > \sigma_1$ and $\Phi''' \in \mathcal{L}[\sigma_1, \sigma_2]$ such that $|\Phi'''|$ is convex function on $[\sigma_1, \sigma_2]$, then the following inequality satisfies:*

$$
\begin{aligned}
&\Bigg|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{4\,(\omega+1)\,(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&- \frac{2^{\omega}\Gamma\,(\omega+3)}{2\,(\omega+1)\,(\omega+2)\,(\sigma_2-\sigma_1)^{\omega}}\left\{ J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\,\Phi\,(\sigma_1) \;+\; J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\,\Phi\,(\sigma_2)\right\}\Bigg| \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)}\Bigg[\,|\Phi'''\,(\sigma_1)|\,\frac{1}{(\omega+3)\,(\omega+4)} + \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\frac{2}{(\omega+4)} \\
&+ \;|\Phi'''\,(\sigma_2)|\,\frac{1}{(\omega+3)\,(\omega+4)}\Bigg]. \qquad (2.3)
\end{aligned}
$$

Proof. From (2.1), with the convexity of $|\Phi'''|$, we obtain

$$
\begin{aligned}
&\Bigg|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{4\,(\omega+1)\,(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&- \frac{2^{\omega}\Gamma\,(\omega+3)}{2\,(\omega+1)\,(\omega+2)\,(\sigma_2-\sigma_1)^{\omega}}\left\{ J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\,\Phi\,(\sigma_1) \;+\; J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\,\Phi\,(\sigma_2)\right\}\Bigg| \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)} \\
&\times\left\{\int_0^1 \mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2} + (1-\mu)\,\sigma_1\right)\right| d\mu\right. \\
&+ \left.\int_0^1 (1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2 + (1-\mu)\,\frac{\sigma_1+\sigma_2}{2}\right)\right| d\mu\right\} \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)}\left\{\int_0^1 \mu^{\omega+2}\left\{\mu\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right| + (1-\mu)\,|\Phi'''\,(\sigma_1)|\right\} d\mu\right. \\
&+ \left.\int_0^1 (1-\mu)^{\omega+2}\left\{\mu\,|\Phi'''\,(\sigma_2)| + (1-\mu)\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right\} d\mu\right\}
\end{aligned}
$$

$$\leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\Big[|\Phi'''(\sigma_1)|\frac{1}{(\omega+3)(\omega+4)}+\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\frac{2}{(\omega+4)}$$
$$+ \ |\Phi'''(\sigma_2)|\frac{1}{(\omega+3)(\omega+4)}\Big].$$

This completes the proof. □

Corollary 2.1. *Under the same suppositions as defined in Theorem 2.1, if we choose* $\omega = 1$, *we get*

$$\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{24}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Phi(u)\,du\right|$$
$$\leq \frac{(\sigma_2-\sigma_1)^3}{96}\left\{\frac{|\Phi'''(\sigma_1)|}{20}+\frac{2}{5}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|+\frac{|\Phi'''(\sigma_2)|}{20}\right\}.$$

Theorem 2.2. *Suppose a mapping* $\Phi: I\subseteq\Re\rightarrow\Re$ *is thrice differentiable on* I^o *(the interior of I) with* $\sigma_2>\sigma_1$ *and* $\Phi'''\in\mathcal{L}[\sigma_1,\sigma_2]$ *such that* $|\Phi'''|^r$ *is convex function on* $[\sigma_1,\sigma_2]$ *for* $r>1$, *then the following inequality satisfies:*

$$\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.$$
$$\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{2}|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}+\left(\frac{1}{2}|\Phi'''(\sigma_2)|^r+\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\right\}, \tag{2.4}$$

where $p^{-1}=1-r^{-1}$.

Proof. From (2.1), with the convexity of $|\Phi'''|^r$ and the well-known Hölder's Rogers integral inequality, we obtain

$$\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.$$
$$\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}$$
$$\times\left\{\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.$$
$$\left.+\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right\}$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}$$

$$\times\left\{\left(\int_0^1 \mu^{(\omega+2)p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\,\sigma_1\right)\right|^r d\mu\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\int_0^1 (1-\mu)^{(\omega+2)p}\,d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\,\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right)^{\frac{1}{r}}\right\}$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)}$$
$$\times\left\{\left(\int_0^1 \mu^{(\omega+2)p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left\{\mu\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+(1-\mu)\,|\Phi'''(\sigma_1)|^r\right\}d\mu\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\int_0^1 (1-\mu)^{(\omega+2)p}\,d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left\{\mu\,|\Phi'''(\sigma_2)|^r+(1-\mu)\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}d\mu\right)^{\frac{1}{r}}\right\}$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)}\left(\frac{1}{(\omega+2)\,p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{2}\,|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}+\left(\frac{1}{2}\,|\Phi'''(\sigma_2)|^r+\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\right\}.$$

This completes the proof. □

Corollary 2.2. *Under the same supposition as defined in Theorem 2.2, if we choose* $\omega = 1$, *we get*

$$\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{24}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Phi(u)\,du\right|$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{2}\,|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}+\left(\frac{1}{2}\,|\Phi'''(\sigma_2)|^r+\frac{1}{2}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\right].$$

Theorem 2.3. *Suppose a mapping* $\Phi: I\subseteq\Re\to\Re$ *is thrice differentiable on* I^o *(the interior of* I*) with* $\sigma_2>\sigma_1$ *and* $\Phi'''\in\mathcal{L}[\sigma_1,\sigma_2]$ *such that* $|\Phi'''|^r$ *is convex function on* $[\sigma_1,\sigma_2]$ *for* $r\geq 1$, *then the following inequality satisfies:*

$$\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4\,(\omega+1)\,(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.$$
$$\left.-\frac{2^\omega\Gamma(\omega+3)}{2\,(\omega+1)\,(\omega+2)\,(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|$$
$$\leq\frac{(\sigma_2-\sigma_1)^3}{16\,(\omega+1)\,(\omega+2)}\left(\frac{1}{\omega+3}\right)^{1-\frac{1}{r}}$$
$$\times\left[\left(\frac{1}{\omega+4}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{(\omega+3)\,(\omega+4)}\,|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{1}{(\omega+3)\,(\omega+4)}\,|\Phi'''(\sigma_2)|^r+\frac{1}{\omega+4}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\right],\qquad(2.5)$$

where $p^{-1} = 1 - r^{-1}$.

Proof. From (2.1), with the convexity of $|\Phi'''|^r$ and the well-known power mean integral inequality, we obtain

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^{\omega}\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^{\omega}}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)\;+\;J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.\\
&\left.+\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right]\\
&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left(\int_0^1\mu^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\left(\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1(1-\mu)^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\left(\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right)^{\frac{1}{r}}\right]\\
&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left(\int_0^1\mu^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\right.\\
&\times\left(\int_0^1\mu^{\omega+2}\left\{\mu\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+(1-\mu)|\Phi'''(\sigma_1)|^r\right\}d\mu\right)^{\frac{1}{r}}\\
&+\left(\int_0^1(1-\mu)^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\\
&\left.\times\left(\int_0^1(1-\mu)^{\omega+2}\left\{\mu|\Phi'''(\sigma_2)|^r+(1-\mu)\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}d\mu\right)^{\frac{1}{r}}\right]\\
&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{\omega+3}\right)^{1-\frac{1}{r}}\\
&\times\left[\left(\frac{1}{\omega+4}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{(\omega+3)(\omega+4)}|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\frac{1}{(\omega+3)(\omega+4)}|\Phi'''(\sigma_2)|^r+\frac{1}{\omega+4}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.3. *Under the same suppositions as defined in Theorem 2.3, if we choose* $\omega = 1$, *we get*

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{24}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Phi(u)\,du\right|\\
&\le\frac{(\sigma_2-\sigma_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{r}}\left[\left(\frac{1}{5}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r+\frac{1}{20}|\Phi'''(\sigma_1)|^r\right)^{\frac{1}{r}}\right.
\end{aligned}$$

$$+\left(\frac{1}{20}\left|\Phi'''\left(\sigma_2\right)\right|^r+\frac{1}{5}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)^{\frac{1}{r}}\Bigg].$$

Theorem 2.4. *Suppose a mapping $\Phi: I\subseteq\Re\rightarrow\Re$ is thrice differentiable on I^o (the interior of I) with $\sigma_2>\sigma_1$ and $\Phi'''\in\mathcal{L}[\sigma_1,\sigma_2]$ such that $|\Phi'''|^r$ is concave function on $[\sigma_1,\sigma_2]$ with $p\in\Re$, $p>1$, then the following inequality satisfies:*

$$\begin{aligned}&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left[\left|\Phi'''\left(\frac{3\sigma_1+\sigma_2}{4}\right)\right|+\left|\Phi'''\left(\frac{\sigma_1+3\sigma_2}{4}\right)\right|\right],\end{aligned}$$

where $p^{-1}=1-r^{-1}$.

Proof. From (2.1), with the concavity of $|\Phi'''|^r$ and the well-known Hölder's Rogers integral inequality, we obtain

$$\begin{aligned}&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.\\&\left.+\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right]\end{aligned}$$

$$\begin{aligned}&\le\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left(\int_0^1\mu^{(\omega+2)p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu\right)^{\frac{1}{r}}\right.\\&\left.+\left(\int_0^1(1-\mu)^{(\omega+2)p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right)^{\frac{1}{r}}\right].\end{aligned}\tag{2.6}$$

Since $|\Phi'''|^r$ is concave on $[\sigma_1,\sigma_2]$, we can use the Jensen's integral inequality to find that

$$\begin{aligned}&\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu=\int_0^1\mu^o\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu\\&\le\left(\int_0^1\mu^o d\mu\right)\left|\Phi'''\left(\frac{\int_0^1\mu^o\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)d\mu}{\int_0^1\mu^o d\mu}\right)\right|^r\\&\le\left|\Phi'''\left(\frac{3\sigma_1+\sigma_2}{4}\right)\right|^r\end{aligned}$$

and similarly

$$\int_0^1 \left|\Phi'''\left(\mu\sigma_2 + (1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu \le \left|\Phi'''\left(\frac{\sigma_1+3\sigma_2}{4}\right)\right|^r,$$

then Eq (2.6), becomes

$$\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left[\left|\Phi'''\left(\frac{3\sigma_1+\sigma_2}{4}\right)\right| + \left|\Phi'''\left(\frac{\sigma_1+3\sigma_2}{4}\right)\right|\right].$$

This completes the proof. □

Corollary 2.4. *Under the same suppositions as defined in Theorem 2.4, if we choose* $\omega = 1$, *we get*

$$\begin{aligned}&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{24}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Phi(u)\,du\right|\\ &\le \frac{(\sigma_2-\sigma_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\left|\Phi'''\left(\frac{3\sigma_1+\sigma_2}{4}\right)\right| + \left|\Phi'''\left(\frac{\sigma_1+3\sigma_2}{4}\right)\right|\right].\end{aligned}$$

Theorem 2.5. *Suppose a mapping* $\Phi: I \subseteq \Re \to \Re$ *is thrice differentiable on* I^o *(the interior of I) with* $\sigma_2 > \sigma_1$ *and* $\Phi''' \in \mathcal{L}[\sigma_1, \sigma_2]$ *such that* $|\Phi'''|^r$ *is convex function on* $[\sigma_1, \sigma_2]$ *for* $r > 1$, *then the following inequality satisfies:*

$$\begin{aligned}&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\ &\left.- \frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1) + J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\ &\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left\{\frac{1}{p((\omega+2)p+1)} + \frac{1}{2r}\left(\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r + |\Phi'''(\sigma_1)|^r\right)\right\}\right.\\ &\left.+ \left\{\frac{1}{p((\omega+2)p+1)} + \frac{1}{2r}\left(|\Phi'''(\sigma_2)|^r + \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)\right\}\right],\end{aligned}$$

where $p^{-1} = 1 - r^{-1}$.

Proof. From (2.1), we obtain

$$\begin{aligned}&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\ &\left.- \frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1) + J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\ &\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\\ &\times\left[\int_0^1 \mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2} + (1-\mu)\sigma_1\right)\right|d\mu\right.\\ &\left.+ \int_0^1 (1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2 + (1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right].\end{aligned}$$

By using the Young's inequality as

$$uv < \frac{1}{p}u^p + \frac{1}{r}v^r$$

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right. \\
&\left. - \frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{ J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1) + J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right| \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left\{\frac{1}{p}\int_0^1 \mu^{(\omega+2)p}d\mu + \frac{1}{r}\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu\right\}\right. \\
&\left. + \left\{\frac{1}{p}\int_0^1 (1-\mu)^{(\omega+2)p}d\mu + \frac{1}{r}\int_0^1\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right\}\right] \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)} \\
&\times\left[\left\{\frac{1}{p}\int_0^1 \mu^{(\omega+2)p}d\mu + \frac{1}{r}\int_0^1\left\{\mu\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r + (1-\mu)\left|\Phi'''(\sigma_1)\right|^r\right\}d\mu\right\}\right. \\
&\left. + \left\{\frac{1}{p}\int_0^1 (1-\mu)^{(\omega+2)p}d\mu + \frac{1}{r}\int_0^1\left\{\mu\left|\Phi'''(\sigma_2)\right|^r + (1-\mu)\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}d\mu\right\}\right] \\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\left\{\frac{1}{p((\omega+2)p+1)} + \frac{1}{2r}\left(\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r + \left|\Phi'''(\sigma_1)\right|^r\right)\right\}\right. \\
&\left. + \left\{\frac{1}{p((\omega+2)p+1)} + \frac{1}{2r}\left(\left|\Phi'''(\sigma_2)\right|^r + \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)\right\}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.5. *Under the same supposition as defined in Theorem 2.5, if we choose* $\omega = 1$, *we get*

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_2-\sigma_1)^2}{24}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Phi(u)\,du\right| \\
&\le \frac{(\sigma_2-\sigma_1)^3}{96}\left[\left\{\frac{1}{p(3p+1)} + \frac{1}{2r}\left(\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r + \left|\Phi'''(\sigma_1)\right|^r\right)\right\}\right. \\
&\left. + \left\{\frac{1}{p(3p+1)} + \frac{1}{2r}\left(\left|\Phi'''(\sigma_2)\right|^r + \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right)\right\}\right].
\end{aligned}$$

3. Results related to quasi-convexity

In this section, we will present some results related to quasi-convex function.

Theorem 3.1. *Suppose a mapping* $\Phi \subseteq [0, +\infty) \to \Re$ *is thrice differentiable on* I^o *(the interior of* I*) with* $\sigma_2 > \sigma_1$ *and* $\Phi''' \in \mathcal{L}[\sigma_1, \sigma_2]$ *such that* $|\Phi'''|$ *is quasi-convex function on* $[\sigma_1, \sigma_2]$*, then the following inequality satisfies:*

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^{\omega}\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^{\omega}}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[\max\left\{|\Phi'''(\sigma_1)|,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right\}\right.\\
&\left.+\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|,|\Phi'''(\sigma_2)|\right\}\right].
\end{aligned}\tag{3.1}$$

Proof. From (2.1) and the quasi-convexity of $|\Phi'''|$, we obtain

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^{\omega}\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^{\omega}}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\\
&\times\left[\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.\\
&\left.+\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right]\\
&\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left[\int_0^1\mu^{\omega+2}\max\left\{|\Phi'''(\sigma_1)|,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right\}d\mu\right.\\
&\left.+\int_0^1(1-\mu)^{\omega+2}\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|,|\Phi'''(\sigma_2)|\right\}d\mu\right]\\
&\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[\max\left\{|\Phi'''(\sigma_1)|,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right\}\right.\\
&\left.+\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|,|\Phi'''(\sigma_2)|\right\}\right].
\end{aligned}$$

This completes the proof. □

Remark 3.1. If we prefer $\omega=1$, in Theorem 3.1, then we obtain the Theorem 2.1 in [21].

Corollary 3.1. *Let us consider the assumptions as defined in Theorem 3.1, be valid and let*

$$\begin{aligned}
K:=&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^{\omega}\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^{\omega}}\left\{J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^{\omega}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|.
\end{aligned}$$

(i) If $|\Phi'''|$ is increasing, then we have

$$K\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|+|\Phi'''(\sigma_2)|\right].$$

(ii) If $|\Phi'''|$ is decreasing, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[|\Phi'''(\sigma_1)| + \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right].$$

(iii) If $\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)=0$, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[|\Phi'''(\sigma_1)| + |\Phi'''(\sigma_2)|\right].$$

(iv) If $\Phi'''(\sigma_1)=\Phi'''(\sigma_2)=0$, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|.$$

Theorem 3.2. *Suppose a mapping $\Phi \subseteq [0,+\infty) \to \Re$ is thrice differentiable on I^o (the interior of I) with $\sigma_2>\sigma_1$ and $\Phi''' \in \mathcal{L}[\sigma_1,\sigma_2]$ such that $|\Phi'''|^r$ is quasi-convex function on $[\sigma_1,\sigma_2]$ and $r>1$ with $r^{-1}=1-p^{-1}$, then the following inequality satisfies:*

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\\
&\times\left[\left(\max\left\{|\Phi'''(\sigma_1)|^r, \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r, |\Phi'''(\sigma_2)|^r\right\}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Proof. Suppose $p>1$. Then from (2.1), with the quasi-convexity of $|\Phi'''|^r$ and using the Hölder's Rogers inequality, we obtain

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\\
&\times\left[\int_0^1 \mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.\\
&\left.+\int_0^1 (1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right]\\
&\leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}
\end{aligned}$$

$$\times\Bigg[\left(\int_0^1 \mu^{(\omega+2)p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\,\sigma_1\right)\right|^r d\mu\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1 (1-\mu)^{(\omega+2)p}\,d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\,\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right)^{\frac{1}{r}}\Bigg].$$

The quasi-convexity of $|\Phi'''|^r$ on $[\sigma_1,\sigma_2]$ implies that

$$\int_0^1\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\,\sigma_1\right)\right|^r d\mu\leq\max\left\{|\Phi'''(\sigma_1)|^r,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\},$$

and

$$\int_0^1\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\,\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\leq\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r,|\Phi'''(\sigma_2)|^r\right\},$$

then we obtain

$$\Bigg|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)$$

$$-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)\;+\;J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\Bigg|$$

$$\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\max\left\{|\Phi'''(\sigma_1)|^r,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}\right)^{\frac{1}{r}}$$

$$+\left(\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r,|\Phi'''(\sigma_2)|^r\right\}\right)^{\frac{1}{r}}\Bigg].$$

This completes the proof. □

Remark 3.2. If we prefer $\omega = 1$, in Theorem 3.2, then we obtain the Theorem 2.2 in [21].

Corollary 3.2. *Let us consider the assumptions as defined in Theorem 3.2, be valid and let*

$$K:=\Bigg|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)$$

$$-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)\;+\;J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\Bigg|.$$

(i) If $|\Phi'''|$ is increasing, then we have

$$K\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left[\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|+|\Phi'''(\sigma_2)|\right].$$

(ii) If $|\Phi'''|$ is decreasing, then we have

$$K\leq\frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left[|\Phi'''(\sigma_1)|+\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right].$$

(iii) If $\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right) = 0$ *, then we have*

$$K \le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left[|\Phi'''(\sigma_1)|+|\Phi'''(\sigma_2)|\right].$$

(iv) If $\Phi'''(\sigma_1) = \Phi'''(\sigma_2) = 0$ *, then we have*

$$K \le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\left(\frac{1}{(\omega+2)p+1}\right)^{\frac{1}{p}}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|.$$

Theorem 3.3. *Suppose a mapping* $\Phi \subseteq [0,+\infty) \to \Re$ *is thrice differentiable on* I^o *(the interior of I) with* $\sigma_2 > \sigma_1$ *and* $\Phi''' \in \mathcal{L}[\sigma_1,\sigma_2]$ *such that* $|\Phi'''|^r$ *is quasi-convex function on* $[\sigma_1,\sigma_2]$ *and* $r \ge 1$ *with* $r^{-1} = 1 - p^{-1}$*, then the following inequality satisfies:*

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\\
&\times\left[\left(\max\left\{|\Phi'''(\sigma_1)|^r,\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r,|\Phi'''(\sigma_2)|^r\right\}\right)^{\frac{1}{r}}\right].
\end{aligned} \tag{3.2}$$

Proof. From (2.1), with the quasi-convexity of $|\Phi'''|^r$ and using the power-mean inequality, we obtain

$$\begin{aligned}
&\left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right.\\
&\left.-\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1)+J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|\\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\\
&\times\left[\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|d\mu\right.\\
&\left.+\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|d\mu\right]\\
&\le \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)}\\
&\times\left[\left(\int_0^1\mu^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\left(\int_0^1\mu^{\omega+2}\left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1(1-\mu)^{\omega+2}d\mu\right)^{1-\frac{1}{r}}\left(\int_0^1(1-\mu)^{\omega+2}\left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu\right)^{\frac{1}{r}}\right].
\end{aligned} \tag{3.3}$$

The quasi-convexity of $|\Phi'''|^r$ on $[\sigma_1, \sigma_2]$ implies that

$$\int_0^1 \mu^{\omega+2} \left|\Phi'''\left(\mu\frac{\sigma_1+\sigma_2}{2}+(1-\mu)\sigma_1\right)\right|^r d\mu \\ \leq \frac{1}{\omega+3}\max\left\{|\Phi'''(\sigma_1)|^r, \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}$$

and

$$\int_0^1 (1-\mu)^{\omega+2} \left|\Phi'''\left(\mu\sigma_2+(1-\mu)\frac{\sigma_1+\sigma_2}{2}\right)\right|^r d\mu \\ \leq \frac{1}{\omega+3}\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r, |\Phi'''(\sigma_2)|^r\right\}.$$

Therefore, (3.3) becomes

$$\leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)} \\ \times\left[\left(\max\left\{|\Phi'''(\sigma_1)|^r, \left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r\right\}\right)^{\frac{1}{r}} \right. \\ \left. +\left(\max\left\{\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^r, |\Phi'''(\sigma_2)|^r\right\}\right)^{\frac{1}{r}}\right].$$

This completes the proof. □

Remark 3.3. If we set $\omega = 1$, in Theorem 3.3, then we get the Theorem 2.3 in [21].

Corollary 3.3. *Let us consider the assumptions as defined in Theorem 3.3, be valid and let*

$$K := \left|\Phi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{4(\omega+1)(\omega+2)}\Phi''\left(\frac{\sigma_1+\sigma_2}{2}\right) \right. \\ \left. -\frac{2^\omega\Gamma(\omega+3)}{2(\omega+1)(\omega+2)(\sigma_2-\sigma_1)^\omega}\left\{J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Phi(\sigma_1) + J^\omega_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}\Phi(\sigma_2)\right\}\right|.$$

(i) If $|\Phi'''|$ is increasing, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|+|\Phi'''(\sigma_2)|\right].$$

(ii) If $|\Phi'''|$ is decreasing, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[|\Phi'''(\sigma_1)|+\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\right].$$

(iii) If $\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)=0$, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left[|\Phi'''(\sigma_1)|+|\Phi'''(\sigma_2)|\right].$$

(iv) If $\Phi'''(\sigma_1)=\Phi'''(\sigma_2)=0$, then we have

$$K \leq \frac{(\sigma_2-\sigma_1)^3}{16(\omega+1)(\omega+2)(\omega+3)}\left|\Phi'''\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|.$$

4. Applications

Let's consider the following special means for real numbers σ_1, σ_2 such that $\sigma_1 \neq \sigma_2$.
The arithmetic mean:

$$A(\sigma_1, \sigma_2) = \frac{\sigma_1 + \sigma_2}{2}.$$

The logarithmic mean:

$$L(\sigma_1, \sigma_2) = \frac{\sigma_2 - \sigma_1}{log\sigma_2 - log\sigma_1}.$$

The generalized logarithmic mean:

$$L_c(\sigma_1, \sigma_2) = \left[\frac{\sigma_2^{c+1} - \sigma_1^{c+1}}{(c+1)(\sigma_2 - \sigma_1)}\right]^{\frac{1}{c}}; \quad c \in \Re \setminus \{-1, 0\}.$$

The identric mean:

$$I(\sigma_1, \sigma_2) = \begin{cases} \sigma_1 & if \quad \sigma_1 = \sigma_2 \\ \frac{1}{e}\left(\frac{\sigma_2^{\sigma_2}}{\sigma_1^{\sigma_1}}\right)^{\frac{1}{\sigma_2 - \sigma_1}} & if \quad \sigma_1 \neq \sigma_2 \end{cases} \quad \sigma_1, \sigma_2 \in \Re, \sigma_1, \sigma_2 > 0.$$

Proposition 4.1. *Suppose $k \in Z\backslash\{-1, 0\}$ and $\sigma_1, \sigma_2 \in \Re$ such that $0 < \sigma_1 < \sigma_2$, then the following inequality*

$$\begin{aligned} &\left|A^k(\sigma_1, \sigma_2) + \frac{k(k-1)}{24}(\sigma_2 - \sigma_1)^2 A^{k-2}(\sigma_1, \sigma_2) - L_k^k(\sigma_1, \sigma_2)\right| \\ \leq\ & \frac{k(k-1)(k-2)}{960}(\sigma_2 - \sigma_1)^3 \left[A\left(|\sigma_1|^{k-3}, |\sigma_2|^{k-3}\right) + 4A^{k-3}(\sigma_1, \sigma_2)\right], \end{aligned}$$

satisfies.

Proof. The assertion follows from Corollary 2.1 for the function $\Phi(x) = x^k$ and k as specified above. □

Proposition 4.2. *Suppose $r \geq 1$ and $\sigma_1, \sigma_2 \in \Re$ such that $0 < \sigma_1 < \sigma_2$, then the following inequality*

$$\begin{aligned} &\left|A^{-1}(\sigma_1, \sigma_2) + \frac{(\sigma_2 - \sigma_1)^2}{24} A^{-2}(\sigma_1, \sigma_2) - L^{-1}(\sigma_1, \sigma_2)\right| \\ \leq\ & \frac{(\sigma_2 - \sigma_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{r}} \\ &\times \left[\left(\frac{2}{5}|A(\sigma_1, \sigma_2)|^{-3r} + \frac{1}{10}|\sigma_1|^{-3r}\right)^{\frac{1}{r}} + \left(\frac{1}{10}|\sigma_2|^{-3r} + \frac{2}{5}|A(\sigma_1, \sigma_2)|^{-3r}\right)^{\frac{1}{r}}\right], \end{aligned}$$

satisfies.

Proof. The assertion follows from Corollary 2.3 for the function $\Phi(x) = \frac{1}{x}$. □

Proposition 4.3. *Suppose $r \geq 1$ and $\sigma_1, \sigma_2 \in \Re$ such that $0 < \sigma_1 < \sigma_2$, then the following inequality*

$$\left|\frac{(\sigma_2-\sigma_1)^2}{24}A^{-2}(\sigma_1,\sigma_2) - \ln[A(\sigma_1,\sigma_2)\times I(\sigma_1,\sigma_2)]\right|$$
$$\leq \frac{(\sigma_2-\sigma_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{r}}$$
$$\times\left[\left(\frac{2}{5}|A(\sigma_1,\sigma_2)|^{-3r}+\frac{1}{10}|\sigma_1|^{-3r}\right)^{\frac{1}{r}}+\left(\frac{1}{10}|\sigma_2|^{-3r}+\frac{2}{5}|A(\sigma_1,\sigma_2)|^{-3r}\right)^{\frac{1}{r}}\right],$$

satisfies.

Proof. The assertion follows from Corollary 2.3 for the function $\Phi(x) = -lnx$. □

5. Modified Bessel function

We recall the first kind modified Bessel function $\Im_m$, which has the series representation (see [30], p.77)

$$\Im_m(\zeta) = \Sigma_{n\geq 0}\frac{\left(\frac{\zeta}{2}\right)^{m+2n}}{n!\Gamma(m+n+1)},$$

where $\zeta \in \Re$ and $m > -1$, while the second kind modified Bessel function Φ_m (see [30], p.78) is usually defined as

$$\Phi_m(\zeta) = \frac{\pi}{2}\frac{\Im_{-m}(\zeta) - \Im_m(\zeta)}{\sin m\pi}.$$

Consider the function $\Omega_m(\zeta) : \Re \to [1,\infty)$ defined by

$$\Omega_m(\zeta) = 2^m\Gamma(m+1)\zeta^{-m}\Phi_m(\zeta),$$

where Γ is the gamma function.

The first order derivative formula of $\Omega_m(\zeta)$ is given by [30]:

$$\Omega_m'(\zeta) = \frac{\zeta}{2(m+1)}\Omega_{m+1}(\zeta) \tag{5.1}$$

and the second derivative can be easily calculated from (5.1) as

$$\Omega_m''(\zeta) = \frac{\zeta^2}{4(m+1)(m+2)}\Omega_{m+2}(\zeta) + \frac{1}{2(m+1)}\Omega_{m+1}(\zeta), \tag{5.2}$$

and the third derivative can be easily calculated from (5.2) as

$$\Omega_m'''(\zeta) = \frac{\zeta^3}{4(m+1)(m+2)(m+3)}\Omega_{m+3}(\zeta) + \frac{3\zeta}{4(m+1)(m+2)}\Omega_{m+2}(\zeta). \tag{5.3}$$

Proposition 5.1. *Suppose that $m > -1$ and $0 < \sigma_1 < \sigma_2$. Then we have*

$$\begin{aligned}
&\Bigg|\Omega_m\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{24}\Bigg\{\frac{(\sigma_1+\sigma_2)^2}{16(m+1)(m+2)}\Omega_{m+2}\left(\frac{\sigma_1+\sigma_2}{2}\right)\\
&+\frac{1}{2(m+1)}\ \Omega_{m+1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\Bigg\}-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Omega_m(\zeta)\,d\zeta\Bigg|\\
&\le\frac{(\sigma_2-\sigma_1)^3}{96}\Bigg[\frac{1}{20}\Bigg\{\frac{\sigma_1^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\sigma_1)+\frac{3\sigma_1}{4(m+1)(m+2)}\Omega_{m+2}(\sigma_1)\Bigg\}\\
&+\frac{2}{5}\Bigg\{\frac{(\sigma_1+\sigma_2)^3}{64(m+1)(m+2)(m+3)}\Omega_{m+3}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{3(\sigma_1+\sigma_2)}{8(m+1)(m+2)}\Omega_{m+2}\left(\frac{\sigma_1+\sigma_2}{2}\right)\Bigg\}\\
&+\frac{1}{20}\Bigg\{\frac{\sigma_2^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\sigma_2)+\frac{3\sigma_2}{4(m+1)(m+2)}\Omega_{m+2}(\sigma_2)\Bigg\}\Bigg].
\end{aligned}$$

Proof. The assertion follows immediately from Corollary 2.1 using $\Phi(\zeta)=\Omega_m(\zeta)$, $\zeta > 0$ and the identities (5.2) and (5.3). □

Proposition 5.2. *Suppose that $m > -1$ and $0 < \sigma_1 < \sigma_2, r > 1$. Then we have*

$$\begin{aligned}
&\Bigg|\Omega_m\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{(\sigma_2-\sigma_1)^2}{24}\Bigg\{\frac{(\sigma_1+\sigma_2)^2}{16(m+1)(m+2)}\Omega_{m+2}\left(\frac{\sigma_1+\sigma_2}{2}\right)\\
&+\frac{1}{2(m+1)}\ \Omega_{m+1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\Bigg\}-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Omega_m(\zeta)\,d\zeta\Bigg|\\
&\le\frac{(\sigma_2-\sigma_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{r}}\\
&\times\Bigg[\Bigg\{\Bigg\{\frac{(\sigma_1+\sigma_2)^3}{64(m+1)(m+2)(m+3)}\Omega_{m+3}\left(\frac{\sigma_1+\sigma_2}{2}\right)\\
&+\frac{3(\sigma_1+\sigma_2)}{8(m+1)(m+2)}\Omega_{m+2}\left(\frac{\sigma_1+\sigma_2}{2}\right)\Bigg\}^r\\
&+\Bigg\{\frac{\sigma_1^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\sigma_1)+\frac{3\sigma_1}{4(m+1)(m+2)}\Omega_{m+2}(\sigma_1)\Bigg\}^r\Bigg\}^{\frac{1}{r}}\\
&+\Bigg\{\Bigg\{\frac{\sigma_2^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\sigma_2)+\frac{3\sigma_2}{4(m+1)(m+2)}\Omega_{m+2}(\sigma_2)\Bigg\}^r\\
&+\Bigg\{\frac{(\sigma_1+\sigma_2)^3}{64(m+1)(m+2)(m+3)}\Omega_{m+3}\left(\frac{\sigma_1+\sigma_2}{2}\right)\\
&+\frac{3(\sigma_1+\sigma_2)}{8(m+1)(m+2)}\Omega_{m+2}\left(\frac{\sigma_1+\sigma_2}{2}\right)\Bigg\}\Bigg\}^{\frac{1}{r}}\Bigg].
\end{aligned}$$

Proof. The assertion follows immediately from Corollary 2.2 using $\Phi(\zeta)=\Omega_m(\zeta)$, $\zeta > 0$ and the identities (5.2) and (5.3). □

6. Conclusions

In this article, we have presented some basic Hermite-Hadamard type integral inequlaities. Moreover, a new integral equality for Riemann-Liouville fractional integral operator have been established. Employing this equality, some related generalizations of Hermite-Hadamard inequality for convex function and quasi-convex function with $\Phi \in C^3([\sigma_1, \sigma_2])$ such that $\Phi''' \in \mathcal{L}([\sigma_1, \sigma_2])$ via Riemann-Liouville fractional integrals are deduced. Finally, we give some applications for special mean and modified bessel function. The results, which we have presented in this article, will potentially motivate researchers to study analogous and more general integral inequalities for various other kinds of fractional integral operators.

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Conflict of interest

There is no any conflict to the authors.

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