Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 3303–3320.
DOI: 10.3934/math.2022184
Received: 21 September 2021
Accepted: 17 November 2021
Published: 29 November 2021

Research article

Some Ostrowski type inequalities for mappings whose second derivatives are preinvex function via fractional integral operator

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Abstract: The comprehension of inequalities in preinvexity is very important for studying fractional calculus and its effectiveness in many applied sciences. In this article, we develop and study of fractional integral inequalities whose second derivatives are preinvex functions. We investigate and prove new lemma for twice differentiable functions involving Riemann-Liouville(R-L) fractional integral operator. On the basis of this newly developed lemma, we make some new results regarding of this identity. These new results yield us some generalizations of the prior results. This study builds upon on a novel new auxiliary result which enables us to develop new variants of Ostrowski type inequalities for twice differentiable preinvex mappings. As an application, several estimates concerning Bessel functions of real numbers are also illustrated.

Keywords: convex function; Hermite-Hadamard inequality; Hölder inequality; power mean inequality; Young's inequality; Hölder-İşcan; improved power means inequality; Riemann-Liouville fractional integral operator
Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15

1. Introduction

The study of integral inequality is an interesting area for research in mathematical analysis [1, 2]. The fundamental integral inequalities can be instrumental in cultivating the subjective properties of convexity. The existence of massive literature surrounding integral inequalities for convex functions [3–7] depicts the importance of this topic. The most beautiful fact about convex function is that, it has a very elegant representation based on an inequality presented when the functional value of a linear combination of two points in its domain does not exceed the linear combination of the functional values at those two points. Fractional calculus owes its starting point to whether or not the importance of a derivative to an integer order could be generalized to a fractional order which is not an integer. Following this unique conversation between L'Hopital and Leibniz, the concept of fractional calculus grabbed the eye of some extraordinary researchers like Euler, Laplace, Fourier, Lacroix, Abel, Riemann, and Liouville. Over time, fractional operators have been differentiated with their singularity, locality and having general forms with the improvements made in their kernel structures. In this sense, based on the basic concepts of fractional analysis, Riemann-Liouville(R-L) and Caputo operators, various new trends have been successful. Fractional integral inequalities are marvelous tools for building up the qualitative and quantitative properties of preinvex functions. There has been a ceaseless development of intrigue in such a region of research so as to address the issues of different utilizations of these variants. In 1938, Ostrowski inequality established the following useful and interesting integral inequality, (see [12] and [13]). This review assumed a vital part in growing and getting varieties of well-known integral inequalities with the assistance of fractional integral operators. Then again, by characterizing various forms of Riemann-Liouville(R-L) fractional operator somewhat recently, new forms and refinements of integral inequalities involving differentiable functions have been presented. Studies in the field of fractional calculus have carried another point of view and direction in different fields of applied sciences. It has revealed insight into numerous real-life issues with the utilizations of recently characterized fractional operators.

For recent result and their related some generalizations, variants and extensions concerning Ostrowski inequality (see [9, 10, 14–17]).

The aim of this paper is to establish some integral inequalities for functions whose derivatives in absolute value are preinvex. Now we recall some notions in invexity analysis which will be used through the paper (see [20, 21, 24, 26, 28]) and references therein.

Let $g : \mathcal{K} \to \Re$ and $\eta : \mathcal{K} \times \mathcal{K} \to \Re$, where $\mathcal{K}$ is a nonempty set in $\Re^n$, be continuous functions.

Definition [19] A function $g : \mathcal{K} \subseteq \Re = (-\infty, \infty) \to \Re$ is said to be convex, if we have

$$g\left(v\mathfrak{c} + (1-v)\,\mathfrak{e}\right) \leq v g\left(\mathfrak{c}\right) + (1-v)\, g\left(\mathfrak{e}\right).$$

for all $\mathfrak{c}, \mathfrak{e} \in \mathcal{K}$ and $v \in [0, 1]$.

Definition [25] The set $\mathcal{K} \subset \Re^n$ is said to be invex with respect to $\eta(.,.)$, if for every $\mathfrak{c}, \mathfrak{e} \in \mathcal{K}$ and $v \in [0, 1]$

$$\mathfrak{c} + v\eta(\mathfrak{e}, \mathfrak{c}) \in \mathcal{K}.$$

The above set $\mathcal{K}$ is also called η-connected set.

It is obvious that every convex set is invex with respect to $\eta(\mathfrak{e}, \mathfrak{c}) = \mathfrak{e} - \mathfrak{c}$ but there exist invex sets which are not convex [20].

Definition The function g on the invex set $\mathcal{K}$ is said to be preinvex with respect to η if

$$g\left(\mathfrak{c} + v\eta\left(\mathfrak{e}, \mathfrak{c}\right)\right) \leq (1-v)\ g(\mathfrak{c}) + v\, g\left(\mathfrak{e}\right), \qquad \forall \mathfrak{c},\ \mathfrak{e} \in \mathcal{K},\ \ v \in [0, 1].$$

The function $-g$ is said to be preconcave if and only if g is preinvex .

The important note that every convex function is a preinvex function but the converse is not true [21]. For example $g(v) = -|v|$, $\forall\, v \in \Re$, is not convex function but it is preinvex function with respect to

$$\eta\left(\mathfrak{e}, \mathfrak{c}\right) = \begin{cases} \mathfrak{e} - \mathfrak{c} & if\ \ \mathfrak{c}\mathfrak{e} \geq 0, \\ \mathfrak{c} - \mathfrak{e} & if\ \ \mathfrak{c}\mathfrak{e} < 0. \end{cases}$$

We also want the following hypothesis regarding the function η which is due to Mohan et al. [22].
Condition-C: Let $\mathcal{K} \subset \Re^n$ be an open invex subset with respect to $\eta : \mathcal{K} \times \mathcal{K} \to \Re$. For any $\mathfrak{c}, \mathfrak{e} \in \mathcal{K}$ and $v \in [0, 1]$

$$\begin{aligned} \eta(\mathfrak{e}, \mathfrak{e} + v\,\eta(\mathfrak{c}, \mathfrak{e})) &= -v\,\eta(\mathfrak{c}, \mathfrak{e}), \\ \eta(\mathfrak{c}, \mathfrak{e} + v\,\eta(\mathfrak{c}, \mathfrak{e})) &= (1 - v)\;\eta(\mathfrak{c}, \mathfrak{e}). \end{aligned} \tag{1.1}$$

For any $\mathfrak{c}, \mathfrak{e} \in \mathcal{K}$ and $v_1, v_2 \in [0, 1]$ from condition C, we have

$$\eta(\mathfrak{e} + v_2\,\eta(\mathfrak{c}, \mathfrak{e})\;,\;\mathfrak{e} + v_1\,\eta(\mathfrak{c}, \mathfrak{e})) = (v_2 - v_1)\,\eta(\mathfrak{c}, \mathfrak{e}).$$

If g is a preinvex function on $[\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})]$ and the mapping η satisfies condition C, then for every $v \in [0, 1]$, from Eq (1.1), it yields that

$$\begin{aligned} |g(\mathfrak{c} + v\eta(\mathfrak{e}, \mathfrak{c}))| &= |g(\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})) + (1 - v)\,\eta(\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c}))| \\ &\le v\;|g(\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c}))| + (1 - v)\,|g(\mathfrak{c})|, \end{aligned}$$

and

$$\begin{aligned} |g(\mathfrak{c} + (1 - v)\,\eta(\mathfrak{e}, \mathfrak{c}))| &= |g(\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})) + v\eta(\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c}))| \\ &\le (1 - v)\;|g(\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c}))| + v\,|g(\mathfrak{c})|. \end{aligned}$$

There are many vector functions that satisfy the condition C in [25], which trivial case $\eta(\mathfrak{c}, \mathfrak{e}) = \mathfrak{c} - \mathfrak{e}$. For example suppose $\mathcal{K} = \Re \backslash \{0\}$ and

$$\eta(\mathfrak{e}, \mathfrak{c}) = \begin{cases} \mathfrak{e} - \mathfrak{c} & if \;\; \mathfrak{c} > 0, \mathfrak{e} > 0 \\ \mathfrak{e} - \mathfrak{c} & if \;\; \mathfrak{c} < 0, \mathfrak{e} < 0 \\ -\mathfrak{e}, & \quad otherwise \end{cases}$$

The set $\mathcal{K}$ is invex set and η satisfies the condition C.
Noor et al. [23], proved the following Hermite-Hadamard type inequalities.

Theorem 1.1. *Let $g : \mathcal{K} = [\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})] \to (0, \infty)$ be a preinvex function on the interval of real numbers $\mathcal{K}^0$ with $\eta(\mathfrak{e}, \mathfrak{c}) > 0$, then the following inequalities hold:*

$$g\left(\frac{2\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})}{2}\right) \le \frac{1}{\eta(\mathfrak{e}, \mathfrak{c})} \int_{\mathfrak{c}}^{\mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})} g(x)\;dx \le \frac{g(\mathfrak{c}) + g(\mathfrak{e})}{2}.$$

Then Riemann-Liouville(R-L) fractional integrals of order $\varepsilon > 0$ with $\mathfrak{c} \ge 0$ are defined as follows:

$$J^{\varepsilon}_{\mathfrak{c}+} g(\mathfrak{z}) = \frac{1}{\Gamma(\varepsilon)} \int_{\mathfrak{c}}^{\mathfrak{z}} (\mathfrak{z} - v)^{\varepsilon - 1}\; g(v)\;dv, \quad \mathfrak{z} > \mathfrak{c},$$

and

$$J^{\varepsilon}_{\mathfrak{e}-} g(\mathfrak{z}) = \frac{1}{\Gamma(\varepsilon)} \int_{\mathfrak{z}}^{\mathfrak{e}} (v - \mathfrak{z})^{\varepsilon - 1}\; g(v)\;dv, \quad \mathfrak{z} < \mathfrak{e}.$$

In [30], Sarikaya et al. also described the inequality in fractional integral version. In this study, considering the above mentioned theoretical framework, firstly, an integral identity which is candidate to produce Ostrowski type inequalities has been proved. With the help of such identity like Hölder, Power mean, Young's inequalities, Hölder-İşcan, Improved power means inequality and convexity, a new type of inequality, Ostrowski type inequalities, has been obtained.

2. Main results

In this section, we give Ostrowski inequalities for Riemann-Liouville(R-L) fractional integrals operator are obtained for a differentiable functions on $(\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c}))$. For this purpose, we give a new identity that involve Riemann-Liouville(R-L) fractional integrals operator whose second derivatives are preinvex functions.

Lemma 2.1. *Suppose that a mapping* $g : [\mathfrak{c}, \mathfrak{c}+\eta(\mathfrak{e}, \mathfrak{c})] \rightarrow \mathfrak{R}$ *is twice differentiable with* $\mathfrak{c} < \mathfrak{c}+\eta(\mathfrak{e}, \mathfrak{c})$. *If* $g'' \in \mathcal{L}_1[\mathfrak{c}, \mathfrak{c}+\eta(\mathfrak{e}, \mathfrak{c})]$, *then for all* $\mathfrak{z} \in [\mathfrak{c}, \mathfrak{c}+\eta(\mathfrak{e}, \mathfrak{c})]$ *and* $\varepsilon > 0$, *the following equality*

$$\begin{aligned}
&\frac{\eta^{\varepsilon+1}(\mathfrak{z}, \mathfrak{c}) - \eta^{\varepsilon+1}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z}, \mathfrak{c}) + \eta^{\varepsilon}(\mathfrak{e}, \mathfrak{z})}{\eta(\mathfrak{e}, \mathfrak{c})} g(\mathfrak{z}) \\
&+ \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e}, \mathfrak{c})}\left\{ J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-} g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+} g(\mathfrak{e}) \right\} \\
&= \frac{\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} \int_0^1 v^{\varepsilon+1} g''(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))\,dv \\
&+ \frac{\eta^{\varepsilon+2}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} \int_0^1 v^{\varepsilon+1} g''(\mathfrak{e}+v\eta(\mathfrak{z}, \mathfrak{e}))\,dv,
\end{aligned} \tag{2.1}$$

satisfies for $v \in [0, 1]$.

Proof. Let us assume that

$$\begin{aligned}
&\frac{\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} \int_0^1 v^{\varepsilon+1} g''(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))\,dv \\
&+ \frac{\eta^{\varepsilon+2}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} \int_0^1 v^{\varepsilon+1} g''(\mathfrak{e}+v\eta(\mathfrak{z}, \mathfrak{e}))\,dv, \\
I &= \frac{\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} I_1 + \frac{\eta^{\varepsilon+2}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e}, \mathfrak{c})} I_2,
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 v^{\varepsilon+1} g''(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))\,dv \\
&= \frac{v^{\varepsilon+1} g'(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))}{\eta(\mathfrak{z}, \mathfrak{c})}\Bigg|_0^1 - \int_0^1 \frac{(\varepsilon+1) v^{\varepsilon} g'(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))}{\eta(\mathfrak{z}, \mathfrak{c})} dv \\
&= \frac{g'(\mathfrak{z})}{\eta(\mathfrak{z}, \mathfrak{c})} - \frac{\varepsilon+1}{\eta(\mathfrak{z}, \mathfrak{c})} \int_0^1 v^{\varepsilon} g'(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))\,dv \\
&= \frac{g'(\mathfrak{z})}{\eta(\mathfrak{z}, \mathfrak{c})} - \frac{\varepsilon+1}{\eta^2(\mathfrak{z}, \mathfrak{c})} g(\mathfrak{z}) + \frac{\varepsilon(\varepsilon+1)}{\eta^2(\mathfrak{z}, \mathfrak{c})} \int_0^1 v^{\varepsilon-1} g(\mathfrak{c}+v\eta(\mathfrak{z}, \mathfrak{c}))\,dv \\
&= \frac{g'(\mathfrak{z})}{\eta(\mathfrak{z}, \mathfrak{c})} - \frac{\varepsilon+1}{\eta^2(\mathfrak{z}, \mathfrak{c})} g(\mathfrak{z}) + \frac{\Gamma(\varepsilon+2)}{\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c})} J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-} g(\mathfrak{c}),
\end{aligned}$$

and similarly

$$I_2 = \int_0^1 v^{\varepsilon+1} g''(\mathfrak{e}+v\eta(\mathfrak{z}, \mathfrak{e}))\,dv$$

$$
\begin{aligned}
&= \left.\frac{v^{\varepsilon+1} g'\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)}{\eta\left(\mathfrak{z},\mathfrak{e}\right)}\right|_0^1 - \int_0^1 \frac{\left(\varepsilon+1\right) v^{\varepsilon}\, g'\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)}{\eta(\mathfrak{z},\mathfrak{e})} dv \\
&= \frac{g'\left(\mathfrak{z}\right)}{\eta\left(\mathfrak{z},\mathfrak{e}\right)} - \frac{\varepsilon+1}{\eta\left(\mathfrak{z},\mathfrak{c}\right)} \int_0^1 v^{\varepsilon}\, g'\left(\mathfrak{c}+v\eta\left(\mathfrak{z},\mathfrak{c}\right)\right) dv \\
&= -\frac{g'\left(\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{z}\right)} - \frac{\varepsilon+1}{\eta^2\left(\mathfrak{z},\mathfrak{e}\right)} g\left(\mathfrak{z}\right) + \frac{\varepsilon\left(\varepsilon+1\right)}{\eta^2\left(\mathfrak{z},\mathfrak{e}\right)} \int_0^1 v^{\varepsilon-1}\, g\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right) dv \\
&= -\frac{g'\left(\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{z}\right)} - \frac{\varepsilon+1}{\eta^2\left(\mathfrak{e},\mathfrak{z}\right)} g\left(\mathfrak{z}\right) + \frac{\Gamma\left(\varepsilon+2\right)}{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}\, J^{\varepsilon}_{\left[\mathfrak{e}+\eta\left(\mathfrak{z},\mathfrak{e}\right)\right]^{+}} g\left(\mathfrak{e}\right),
\end{aligned}
$$

Combining I_1 and I_2 with (2.2), we get (2.3). □

Remark 2.1. *If we set* $\varepsilon = 1$ *and* $\eta\left(\mathfrak{c},\mathfrak{e}\right) = \mathfrak{c}-\mathfrak{e}$ *in Lemma 2.1, we get* (*Lemma 1 in [11]*).

Theorem 2.1. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|$ *is preinvex function on* $[\mathfrak{c},\mathfrak{c}+\eta\left(\mathfrak{e},\mathfrak{c}\right)]$, *then for all* $\varepsilon > 0$, *the following inequality*

$$
\begin{aligned}
&\left|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} g'\left(\mathfrak{z}\right) - \frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)} g\left(\mathfrak{z}\right)\right. \\
&\left. + \frac{\Gamma\left(\varepsilon+1\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{\left[\mathfrak{c}+\eta\left(\mathfrak{z},\mathfrak{c}\right)\right]^{-}} g\left(\mathfrak{c}\right) + J^{\varepsilon}_{\left[\mathfrak{e}+\eta\left(\mathfrak{z},\mathfrak{e}\right)\right]^{+}} g\left(\mathfrak{e}\right)\right\}\right| \\
&\le \frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\left(\varepsilon+3\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{|g''\left(\mathfrak{z}\right)| + |g''\left(\mathfrak{c}\right)| \frac{1}{\varepsilon+2}\right\} \\
&+ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\left(\varepsilon+3\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{|g''\left(\mathfrak{z}\right)| + |g''\left(\mathfrak{e}\right)| \frac{1}{\varepsilon+2}\right\}.
\end{aligned}
\tag{2.3}
$$

satisfies for $v \in [0,1]$.

Proof. From Lemma 2.1 and since $|g''|$ is preinvex function on $[\mathfrak{c},\mathfrak{c}+\eta\left(\mathfrak{e},\mathfrak{c}\right)]$, we obtain

$$
\begin{aligned}
&\left|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} g'\left(\mathfrak{z}\right) - \frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)} g\left(\mathfrak{z}\right)\right. \\
&\left. + \frac{\Gamma\left(\varepsilon+1\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{\left[\mathfrak{c}+\eta\left(\mathfrak{z},\mathfrak{c}\right)\right]^{-}} g\left(\mathfrak{c}\right) + J^{\varepsilon}_{\left[\mathfrak{e}+\eta\left(\mathfrak{z},\mathfrak{e}\right)\right]^{+}} g\left(\mathfrak{e}\right)\right\}\right| \\
&\le \frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} \int_0^1 v^{\varepsilon+1} |g''\left(\mathfrak{c}+v\eta\left(\mathfrak{z},\mathfrak{c}\right)\right)| dv \\
&+ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} \int_0^1 v^{\varepsilon+1} |g''\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)| dv \\
&\le \frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} \int_0^1 v^{\varepsilon+1}\left\{v\,|g''\left(\mathfrak{z}\right)| + \left(1-v\right)|g''\left(\mathfrak{c}\right)|\right\} dv \\
&+ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)} \int_0^1 v^{\varepsilon+1}\left\{v\,|g''\left(\mathfrak{z}\right)| + \left(1-v\right)|g''\left(\mathfrak{e}\right)|\right\} dv \\
&\le \frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\left(\varepsilon+3\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{|g''\left(\mathfrak{z}\right)| + |g''\left(\mathfrak{c}\right)| \frac{1}{\varepsilon+2}\right\} \\
&+ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\left(\varepsilon+3\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{|g''\left(\mathfrak{z}\right)| + |g''\left(\mathfrak{e}\right)| \frac{1}{\varepsilon+2}\right\}.
\end{aligned}
$$

This completes the proof. □

Remark 2.2. *If we set* $\varepsilon = 1$ *and* $\eta(\mathfrak{c}, \mathfrak{e}) = \mathfrak{c} - \mathfrak{e}$, *then from Theorem 2.1, we get* (*Theorem 4 in [11]*) *with* $s = 1$.

Corollary 2.1. *By using Theorem 2.1 with* $|g''| \leq \mathcal{M}$, *we get the following inequality*

$$\begin{aligned}&\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z}, \mathfrak{c}) - \eta^{\varepsilon+1}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e}, \mathfrak{c})} g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z}, \mathfrak{c}) + \eta^{\varepsilon}(\mathfrak{e}, \mathfrak{z})}{\eta(\mathfrak{e}, \mathfrak{c})} g(\mathfrak{z})\right.\\ &\left.+ \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e}, \mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-} g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+} g(\mathfrak{e})\right\}\right|\\ &\leq \mathcal{M}\left(\frac{1}{(\varepsilon+1)(\varepsilon+2)\eta(\mathfrak{e}, \mathfrak{c})}\right)\left[\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c}) + \eta^{\varepsilon+2}(\mathfrak{e}, \mathfrak{z})\right].\end{aligned}$$

Remark 2.3. *If we set* $\varepsilon = 1$ *and* $\eta(\mathfrak{c}, \mathfrak{e}) = \mathfrak{c} - \mathfrak{e}$, *then from Corollary 2.1, we recapture* (*Theorem 2.1, [32]*).

Corollary 2.2. *If we set* $\eta(\mathfrak{c}, \mathfrak{e}) = \mathfrak{c} - \mathfrak{e}$ *and* $\mathfrak{z} = \frac{\mathfrak{c}+\mathfrak{e}}{2}$, *in Corollary 2.1, we get the mid-point inequality*

$$\begin{aligned}&\left|\frac{\Gamma(\varepsilon+1)}{(\mathfrak{e}-\mathfrak{c})}\left\{J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^-} g(\mathfrak{c}) + J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^+} g(\mathfrak{e})\right\} - \left(\frac{\mathfrak{e}-\mathfrak{c}}{2}\right)^{\varepsilon-1} g(\mathfrak{z})\right|\\ \leq\ & \mathcal{M}\,\frac{(\mathfrak{e}-\mathfrak{c})^{\varepsilon+1}}{2^{\varepsilon+1}}\left(\frac{1}{(\varepsilon+1)(\varepsilon+2)}\right).\end{aligned}$$

Theorem 2.2. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|^q$, $q > 1$ *is preinvex function on* $[\mathfrak{c}, \mathfrak{c} + \eta(\mathfrak{e}, \mathfrak{c})]$, *then for all* $\varepsilon > 0$, *the following inequality*

$$\begin{aligned}&\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z}, \mathfrak{c}) - \eta^{\varepsilon+1}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e}, \mathfrak{c})} g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z}, \mathfrak{c}) + \eta^{\varepsilon}(\mathfrak{e}, \mathfrak{z})}{\eta(\mathfrak{e}, \mathfrak{c})} g(\mathfrak{z})\right.\\ &\left.+ \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e}, \mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-} g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+} g(\mathfrak{e})\right\}\right|\\ &\leq \left(\frac{1}{(\varepsilon+1)\,p+1}\right)^{\frac{1}{p}}\\ &\times\left[\frac{\eta^{\varepsilon+2}(\mathfrak{z}, \mathfrak{c})}{(\varepsilon+1)\,\eta(\mathfrak{e}, \mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q + |g''(\mathfrak{c})|^q}{2}\right)^{\frac{1}{q}} + \frac{\eta^{\varepsilon+2}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e}, \mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q + |g''(\mathfrak{e})|^q}{2}\right)^{\frac{1}{q}}\right],\end{aligned} \tag{2.4}$$

satisfies for $v \in [0, 1]$, *where* $q^{-1} + p^{-1} = 1$.

Proof. Suppose that $p > 1$. From Lemma 2.1, by using the well-known Hölder integral inequality and the preinvexity of $|g''|^q$, we obtain

$$\begin{aligned}&\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z}, \mathfrak{c}) - \eta^{\varepsilon+1}(\mathfrak{e}, \mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e}, \mathfrak{c})} g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z}, \mathfrak{c}) + \eta^{\varepsilon}(\mathfrak{e}, \mathfrak{z})}{\eta(\mathfrak{e}, \mathfrak{c})} g(\mathfrak{z})\right.\\ &\left.+ \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e}, \mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-} g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+} g(\mathfrak{e})\right\}\right|\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})} \int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|dv \\
&+ \frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})} \int_0^1 v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|dv \qquad (2.6)\\
&\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})} \left(\int_0^1 v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}} \left(\int_0^1 |g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q\,dv\right)^{\frac{1}{q}} \\
&+ \frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})} \left(\int_0^1 v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}} \left(\int_0^1 |g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q\,dv\right)^{\frac{1}{q}}. \qquad (2.7)
\end{aligned}
$$

Since $|g''|^q$ is preinvexity on $[\mathfrak{c},\mathfrak{c}+\eta(\mathfrak{e},\mathfrak{c})]$, we obtain

$$
\begin{aligned}
\int_0^1 |g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q\,dv &\leq \int_0^1 \Big\{v\,|g''(\mathfrak{z})|^q+(1-v)\,|g''(\mathfrak{c})|^q\Big\}dv \qquad (2.8)\\
&= \frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{c})|^q}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 |g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q\,dv &\leq \int_0^1 \Big\{v\,|g''(\mathfrak{z})|^q+(1-v)\,|g''(\mathfrak{e})|^q\Big\}dv \qquad (2.9)\\
&= \frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{e})|^q}{2}.
\end{aligned}
$$

By using (2.8) and (2.9) with (2.7), we obtain

$$
\begin{aligned}
&\Bigg|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z}) \\
&+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\Big\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\Big\}\Bigg| \\
&\leq \left(\frac{1}{(\varepsilon+1)\,p+1}\right)^{\frac{1}{p}} \\
&\times\left[\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{c})|^q}{2}\right)^{\frac{1}{q}}+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{e})|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. □

Remark 2.4. *If we set $\varepsilon = 1$ and $\eta(\mathfrak{c},\mathfrak{e}) = \mathfrak{c}-\mathfrak{e}$, then from Theorem 2.2, we get (Theorem 5, [11]) with $s = 1$.*

Corollary 2.3. *Using Theorem 2.2 with $|g''| \leq \mathcal{M}$, we get*

$$
\begin{aligned}
&\Bigg|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z}) \\
&+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\Big\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\Big\}\Bigg| \\
&\leq \mathcal{M}\left(\frac{1}{(\varepsilon+1)\,p+1}\right)^{\frac{1}{p}}\left[\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\,\eta(\mathfrak{e},\mathfrak{c})}\right].
\end{aligned}
$$

Corollary 2.4. *If in Corollary 2.3, we set* $\eta(\mathfrak{c},\mathfrak{e}) = \mathfrak{c}-\mathfrak{e}$ *and* $\mathfrak{z} = \frac{\mathfrak{c}+\mathfrak{e}}{2}$, *then we get the mid-point inequality*

$$\left|\frac{\Gamma(\varepsilon+1)}{(\mathfrak{e}-\mathfrak{c})}\left\{J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{-}}g(\mathfrak{c}) + J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^{+}}g(\mathfrak{e})\right\} - \left(\frac{\mathfrak{e}-\mathfrak{c}}{2}\right)^{\varepsilon-1} g(\mathfrak{z})\right|$$
$$\leq \mathcal{M}\frac{(\mathfrak{e}-\mathfrak{c})^{\varepsilon+1}}{(\varepsilon+1)2^{\varepsilon+1}}\left(\frac{1}{(\varepsilon+1)p+1}\right)^{\frac{1}{p}}.$$

Theorem 2.3. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|^q$, $q \geq 1$ *is preinvex function on* $[\mathfrak{c},\mathfrak{c}+\eta(\mathfrak{e},\mathfrak{c})]$, *then for all* $\varepsilon > 0$, *the following inequality*

$$\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z}) + \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^{-}}g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^{+}}g(\mathfrak{e})\right\}\right|$$
$$\leq \left(\frac{1}{\varepsilon+2}\right)^{1-\frac{1}{q}}\left[\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+3} + \frac{|g''(\mathfrak{c})|^q}{(\varepsilon+2)(\varepsilon+3)}\right)^{\frac{1}{q}} + \frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+3} + \frac{|g''(\mathfrak{e})|^q}{(\varepsilon+2)(\varepsilon+3)}\right)^{\frac{1}{q}}\right], \tag{2.10}$$

satisfies for $v \in [0,1]$.

Proof. Suppose that $q \geq 1$. From Lemma 2.1, by using the power-mean integral inequality and preinvexity of $|g''|^q$, we obtain

$$\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z}) - \frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z}) + \frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^{-}}g(\mathfrak{c}) + J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^{+}}g(\mathfrak{e})\right\}\right|$$
$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|dv + \frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|dv$$

$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left(\int_0^1 v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 |g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv\right)^{\frac{1}{q}} + \frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left(\int_0^1 v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 |g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q dv\right)^{\frac{1}{q}}. \tag{2.11}$$

Since $|g''|^q$ is preinvexity on $[\mathfrak{c},\mathfrak{c}+\eta(\mathfrak{e},\mathfrak{c})]$, we obtain

$$\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv \leq \int_0^1 v^{\varepsilon+1}\left\{v|g''(\mathfrak{z})|^q + (1-v)|g''(\mathfrak{c})|^q\right\}dv \tag{2.12}$$
$$= \frac{|g''(\mathfrak{z})|^q}{\varepsilon+3} + \frac{|g''(\mathfrak{c})|^q}{(\varepsilon+2)(\varepsilon+3)}$$

and

$$\int_0^1 v^{\varepsilon+1}\left|g''\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)\right|^q dv \leq \int_0^1 v^{\varepsilon+1}\left\{v\left|g''\left(\mathfrak{z}\right)\right|^q+(1-v)\left|g''\left(\mathfrak{e}\right)\right|^q\right\}dv \tag{2.13}$$
$$= \frac{\left|g''\left(\mathfrak{z}\right)\right|^q}{\varepsilon+3}+\frac{\left|g''\left(\mathfrak{e}\right)\right|^q}{(\varepsilon+2)(\varepsilon+3)}.$$

By using (2.12) and (2.13) with (2.11), we obtain

$$\left|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{(\varepsilon+1)\,\eta\left(\mathfrak{e},\mathfrak{c}\right)}g'\left(\mathfrak{z}\right)-\frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}g\left(\mathfrak{z}\right)\right.$$
$$\left.+\frac{\Gamma(\varepsilon+1)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g\left(\mathfrak{e}\right)\right\}\right|$$
$$\leq \left(\frac{1}{\varepsilon+2}\right)^{1-\frac{1}{q}}\left[\frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{(\varepsilon+1)\,\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left(\frac{\left|g''\left(\mathfrak{z}\right)\right|^q}{\varepsilon+3}+\frac{\left|g''\left(\mathfrak{c}\right)\right|^q}{(\varepsilon+2)(\varepsilon+3)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{(\varepsilon+1)\,\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left(\frac{\left|g''\left(\mathfrak{z}\right)\right|^q}{\varepsilon+3}+\frac{\left|g''\left(\mathfrak{e}\right)\right|^q}{(\varepsilon+2)(\varepsilon+3)}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Remark 2.5. *If we set* $\varepsilon=1$ *and* $\eta\left(\mathfrak{c},\mathfrak{e}\right)=\mathfrak{c}-\mathfrak{e}$, *then from Theorem 2.3, we get* (*Theorem 6, [11]*)*with* $s=1$.

Corollary 2.5. *Under the same assumptions in Theorem 2.3 with* $|g''|\leq\mathcal{M}$, *we get the following inequality*

$$\left|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{(\varepsilon+1)\,\eta\left(\mathfrak{e},\mathfrak{c}\right)}g'\left(\mathfrak{z}\right)-\frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}g\left(\mathfrak{z}\right)\right.$$
$$\left.+\frac{\Gamma(\varepsilon+1)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g\left(\mathfrak{e}\right)\right\}\right|$$
$$\leq \mathcal{M}\left(\frac{1}{(\varepsilon+1)(\varepsilon+2)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\right)\left[\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)\right].$$

Corollary 2.6. *If in Corollary 2.5, we set* $\eta\left(\mathfrak{c},\mathfrak{e}\right)=\mathfrak{c}-\mathfrak{e}$ *and* $\mathfrak{z}=\frac{\mathfrak{c}+\mathfrak{e}}{2}$, *then we get the mid-point inequality*

$$\left|\frac{\Gamma(\varepsilon+1)}{(\mathfrak{e}-\mathfrak{c})}\left\{J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^+}g\left(\mathfrak{e}\right)\right\}-\left(\frac{\mathfrak{e}-\mathfrak{c}}{2}\right)^{\varepsilon-1}g\left(\mathfrak{z}\right)\right|$$
$$\leq \mathcal{M}\,\frac{(\mathfrak{e}-\mathfrak{c})^{\varepsilon+1}}{(\varepsilon+1)(\varepsilon+2)2^{\varepsilon+1}}.$$

Theorem 2.4. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|^q$, $q>1$ *is preinvex function on* $[\mathfrak{c},\mathfrak{c}+\eta\left(\mathfrak{e},\mathfrak{c}\right)]$, *then for all* $\varepsilon>0$, *the following inequality*

$$\left|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{(\varepsilon+1)\,\eta\left(\mathfrak{e},\mathfrak{c}\right)}g'\left(\mathfrak{z}\right)-\frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}g\left(\mathfrak{z}\right)\right. \tag{2.14}$$
$$\left.+\frac{\Gamma(\varepsilon+1)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g\left(\mathfrak{e}\right)\right\}\right|$$

$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{((\varepsilon+1)p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{c})|^q}{2q}\right\}$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{((\varepsilon+1)p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{e})|^q}{2q}\right\},$$

satisfies for $v\in[0,1]$.

Proof. From Lemma 2.1, we obtain

$$\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})\right.$$
$$\left.+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\right|$$
$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|dv$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|dv.$$

By using the Young's inequality as

$$xy<\frac{1}{p}x^p+\frac{1}{q}y^q.$$

$$\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})\right.$$
$$\left.+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\right|$$
$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\varepsilon+1)p}dv+\frac{1}{q}\int_0^1|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv\right\}$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\varepsilon+1)p}dv+\frac{1}{q}\int_0^1|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q dv\right\}$$
$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\varepsilon+1)p}dv+\frac{1}{q}\int_0^1\left\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{c})|^q\right\}\right\}$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\varepsilon+1)p}dv+\frac{1}{q}\int_0^1\left\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{e})|^q\right\}\right\}$$
$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\alpha+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{((\varepsilon+1)p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{c})|^q}{2q}\right\}$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left\{\frac{1}{((\varepsilon+1)p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{e})|^q}{2q}\right\}.$$

This completes the proof. □

Corollary 2.7. *If we set* $\eta(\mathfrak{c},\mathfrak{e})=\mathfrak{c}-\mathfrak{e}$ *and* $\varepsilon=1$ *in Theorem 2.4, we get*

$$\left|\frac{1}{\mathfrak{e}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{e}}g(u)\,du-g(\mathfrak{z})+\left(\mathfrak{z}-\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)g'(\mathfrak{z})\right|$$

$$\leq \frac{(\mathfrak{z}-\mathfrak{c})^3}{2(\mathfrak{e}-\mathfrak{c})}\left[\frac{1}{(2p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{c})|^q}{2q}\right]+\frac{(\mathfrak{e}-\mathfrak{z})^3}{2(\mathfrak{e}-\mathfrak{c})}\left[\frac{1}{(2p+1)p}+\frac{|g''(\mathfrak{z})|^q+|g''(\mathfrak{e})|^q}{2q}\right].$$

Corollary 2.8. *If in Theorem 2.4, we set* $\eta(\mathfrak{c},\mathfrak{e})=\mathfrak{c}-\mathfrak{e}$ *and* $\mathfrak{z}=\frac{\mathfrak{c}+\mathfrak{e}}{2}$, *then we get the mid-point inequality*

$$\begin{aligned}&\left|\frac{\Gamma(\varepsilon+1)}{(\mathfrak{e}-\mathfrak{c})}\left\{J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^-}g(\mathfrak{c})+J^{\varepsilon}_{\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)^+}g(\mathfrak{e})\right\}-\left(\frac{\mathfrak{e}-\mathfrak{c}}{2}\right)^{\varepsilon-1}g(\mathfrak{z})\right|\\&\leq\frac{(\mathfrak{e}-\mathfrak{c})^{\varepsilon+1}}{2^{\varepsilon+2}(\varepsilon+1)}\left\{\frac{2}{((\varepsilon+1)p+1)p}+\frac{\left|g''\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)\right|^q+|g''(\mathfrak{c})|^q}{2q}+\frac{\left|g''\left(\frac{\mathfrak{c}+\mathfrak{e}}{2}\right)\right|^q+|g''(\mathfrak{e})|^q}{2q}\right\}.\end{aligned}$$

Theorem 2.5. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|^q$, $q>1$ *is preinvex function on* $[\mathfrak{c},\mathfrak{c}+\eta(\mathfrak{e},\mathfrak{c})]$, *then for all* $\varepsilon>0$, *the following inequality*

$$\begin{aligned}&\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})\right.\\&\left.+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\right|\end{aligned}\tag{2.15}$$

$$\begin{aligned}&\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left[\left(\frac{1}{(\varepsilon p+p+1)(\varepsilon p+p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{6}|g''(\mathfrak{z})|^q+\frac{1}{3}|g''(\mathfrak{c})|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{(\varepsilon+1)p+2}\right)^{\frac{1}{p}}\left(\frac{1}{3}|g''(\mathfrak{z})|^q+\frac{1}{6}|g''(\mathfrak{c})|^q\right)^{\frac{1}{q}}\right]\\&+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left[\left(\frac{1}{(\varepsilon p+p+1)(\varepsilon p+p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{6}|g''(\mathfrak{z})|^q+\frac{1}{3}|g''(\mathfrak{e})|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{(\varepsilon+1)p+2}\right)^{\frac{1}{p}}\left(\frac{1}{3}|g''(\mathfrak{z})|^q+\frac{1}{6}|g''(\mathfrak{e})|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

satisfies for $v\in[0,1]$, *where* $q^{-1}+p^{-1}=1$.

Proof. From Lemma 2.1, by using the Hölder-İşcan integral inequality (see in [33]) and the preinvexity of $|g''|^q$, we obtain

$$\begin{aligned}&\left|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})\right.\\&\left.+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\right|\\&\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|dv\\&+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|dv\end{aligned}$$

$$\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\left[\left(\int_0^1(1-v)v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1(1-v)|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 v^{(\varepsilon+1)p+1}dv\right)^{\frac{1}{p}}\left(\int_0^1 v\left|g''\left(\mathfrak{c}+v\eta\left(\mathfrak{z},\mathfrak{c}\right)\right)\right|^q dv\right)^{\frac{1}{q}}\Bigg]$$

$$+\ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\Bigg[\left(\int_0^1\left(1-v\right)v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1\left(1-v\right)\left|g''\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 v^{(\varepsilon+1)p+1}dv\right)^{\frac{1}{p}}\left(\int_0^1 v\left|g''\left(\mathfrak{e}+v\eta\left(\mathfrak{z},\mathfrak{e}\right)\right)\right|^q dv\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\ \frac{\eta^{\alpha+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\Bigg[\left(\int_0^1\left(1-v\right)v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1\left(1-v\right)\left\{v\left|g''\left(\mathfrak{z}\right)\right|^q+\left(1-v\right)\left|g''\left(\mathfrak{c}\right)\right|^q\right\}dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 v^{(\varepsilon+1)p+1}dv\right)^{\frac{1}{p}}\left(\int_0^1 v\{v\left|g''\left(\mathfrak{z}\right)\right|^q+\left(1-v\right)\left|g''\left(\mathfrak{c}\right)\right|^q\}dv\right)^{\frac{1}{q}}\Bigg]$$

$$+\ \frac{\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\Bigg[\left(\int_0^1\left(1-v\right)v^{(\varepsilon+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1\left(1-v\right)\left\{v\left|g''\left(\mathfrak{z}\right)\right|^q+\left(1-v\right)\left|g''\left(\mathfrak{e}\right)\right|^q\right\}dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 v^{(\varepsilon+1)p+1}dv\right)^{\frac{1}{p}}\left(\int_0^1 v\{v\left|g''\left(\mathfrak{z}\right)\right|^q+\left(1-v\right)\left|g''\left(\mathfrak{e}\right)\right|^q\}dv\right)^{\frac{1}{q}}\Bigg].$$

After simplification, we get (2.15). This completes the proof. □

Corollary 2.9. *Using the same assumptions in Theorem 2.5 with* $|g''|\leq\mathcal{M}$, *we get*

$$\Bigg|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}g'\left(\mathfrak{z}\right)-\ \frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}g\left(\mathfrak{z}\right)$$
$$+\ \frac{\Gamma\left(\varepsilon+1\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g\left(\mathfrak{e}\right)\right\}\Bigg|$$
$$\leq\frac{\mathcal{M}}{2^{\frac{1}{q}}\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\Bigg[\left(\frac{1}{\left(\varepsilon p+p+1\right)\left(\varepsilon p+p+2\right)}\right)^{\frac{1}{p}}+\left(\frac{1}{\left(\varepsilon+1\right)p+2}\right)^{\frac{1}{p}}\Bigg]$$
$$\times\left[\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon+2}\left(\mathfrak{e},\mathfrak{z}\right)\right].$$

Theorem 2.6. *Assume that all the assumptions as defined in Lemma 2.1 and* $|g''|^q$, $q\geq1$ *is preinvex function on* $[\mathfrak{c},\mathfrak{c}+\eta\left(\mathfrak{e},\mathfrak{c}\right)]$, *then for all* $\varepsilon>0$, *the following inequality*

$$\Bigg|\frac{\eta^{\varepsilon+1}\left(\mathfrak{z},\mathfrak{c}\right)-\eta^{\varepsilon+1}\left(\mathfrak{e},\mathfrak{z}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}g'\left(\mathfrak{z}\right)-\ \frac{\eta^{\varepsilon}\left(\mathfrak{z},\mathfrak{c}\right)+\eta^{\varepsilon}\left(\mathfrak{e},\mathfrak{z}\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}g\left(\mathfrak{z}\right)\tag{2.16}$$
$$+\ \frac{\Gamma\left(\varepsilon+1\right)}{\eta\left(\mathfrak{e},\mathfrak{c}\right)}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g\left(\mathfrak{c}\right)+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g\left(\mathfrak{e}\right)\right\}\Bigg|$$
$$\leq\ \frac{\eta^{\varepsilon+2}\left(\mathfrak{z},\mathfrak{c}\right)}{\left(\varepsilon+1\right)\eta\left(\mathfrak{e},\mathfrak{c}\right)}\Bigg[\ \left(\frac{1}{\left(\varepsilon+2\right)\left(\varepsilon+3\right)}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''\left(\mathfrak{z}\right)\right|^q}{\left(\varepsilon+3\right)\left(\varepsilon+4\right)}+\frac{2\left|g''\left(\mathfrak{c}\right)\right|^q}{\left(\varepsilon+2\right)\left(\varepsilon+3\right)\left(\varepsilon+4\right)}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\varepsilon+3}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+4}+\frac{|g''(\mathfrak{c})|^q}{(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}\Bigg]$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\frac{1}{(\varepsilon+2)(\varepsilon+3)}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{(\varepsilon+3)(\varepsilon+4)}+\frac{2|g''(\mathfrak{e})|^q}{(\varepsilon+2)(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\varepsilon+3}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+4}+\frac{|g''(\mathfrak{e})|^q}{(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}\Bigg],$$

satisfies for $v\in[0,1]$, *where* $q^{-1}+p^{-1}=1$.

Proof. From Lemma 2.1, improved power-mean integral inequality(see in [33]) and the preinvexity of $|g''|^q$, we obtain

$$\Bigg|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})$$
$$+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\Bigg|$$
$$\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|dv$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\int_0^1 v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|dv$$
$$\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(1-v)v^{\varepsilon+1}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 v^{\varepsilon+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\varepsilon+2}|g''(\mathfrak{c}+v\eta(\mathfrak{z},\mathfrak{c}))|^q dv\right)^{\frac{1}{q}}\Bigg]$$

$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(1-v)v^{\varepsilon+1}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q dv\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 v^{\varepsilon+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\varepsilon+2}|g''(\mathfrak{e}+v\eta(\mathfrak{z},\mathfrak{e}))|^q dv\right)^{\frac{1}{q}}\Bigg]$$
$$\leq\frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-v)v^{\varepsilon+1}\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{c})|^q\}dv\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 v^{\varepsilon+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\varepsilon+2}\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{c})|^q\}dv\right)^{\frac{1}{q}}\Bigg]$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\varepsilon+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(1-v)v^{\varepsilon+1}\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{e})|^q\}dv\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 v^{\varepsilon+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\varepsilon+2}\{v|g''(\mathfrak{z})|^q+(1-v)|g''(\mathfrak{e})|^q\}dv\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\frac{1}{(\varepsilon+2)(\varepsilon+3)}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{(\varepsilon+3)(\varepsilon+4)}+\frac{2|g''(a)|^q}{(\varepsilon+2)(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\varepsilon+3}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+4}+\frac{|g''(\mathfrak{c})|^q}{(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}\Bigg]$$
$$+\frac{\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}\Bigg[\left(\frac{1}{(\varepsilon+2)(\varepsilon+3)}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{(\varepsilon+3)(\varepsilon+4)}+\frac{2|g''(\mathfrak{e})|^q}{(\varepsilon+2)(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\varepsilon+3}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{z})|^q}{\varepsilon+4}+\frac{|g''(\mathfrak{e})|^q}{(\varepsilon+3)(\varepsilon+4)}\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Corollary 2.10. *Using the same assumption of Theorem 2.6 with $|g''| \leq \mathcal{M}$, we get*

$$\Bigg|\frac{\eta^{\varepsilon+1}(\mathfrak{z},\mathfrak{c})-\eta^{\varepsilon+1}(\mathfrak{e},\mathfrak{z})}{(\varepsilon+1)\eta(\mathfrak{e},\mathfrak{c})}g'(\mathfrak{z})-\frac{\eta^{\varepsilon}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon}(\mathfrak{e},\mathfrak{z})}{\eta(\mathfrak{e},\mathfrak{c})}g(\mathfrak{z})$$
$$+\frac{\Gamma(\varepsilon+1)}{\eta(\mathfrak{e},\mathfrak{c})}\left\{J^{\varepsilon}_{[\mathfrak{c}+\eta(\mathfrak{z},\mathfrak{c})]^-}g(\mathfrak{c})+J^{\varepsilon}_{[\mathfrak{e}+\eta(\mathfrak{z},\mathfrak{e})]^+}g(\mathfrak{e})\right\}\Bigg|$$
$$\leq \frac{\mathcal{M}}{(\varepsilon+1)(\varepsilon+2)\eta(\mathfrak{e},\mathfrak{c})}\left[\eta^{\varepsilon+2}(\mathfrak{z},\mathfrak{c})+\eta^{\varepsilon+2}(\mathfrak{e},\mathfrak{z})\right].$$

3. Modified bessel function

We recall the first kind modified Bessel function $\mathfrak{I}_m$, which has the series representation (see [42], p.77)

$$\mathfrak{I}_m(\zeta)=\Sigma_{n\geq 0}\frac{\left(\frac{\zeta}{2}\right)^{m+2n}}{n!\Gamma(m+n+1)}.$$

where $\zeta \in \mathfrak{R}$ and $m > -1$, while the second kind modified Bessel function g_m (see [42], p.78) is usually defined as

$$g_m(\zeta)=\frac{\pi}{2}\,\frac{\mathfrak{I}_{-m}(\zeta)-\mathfrak{I}_m(\zeta)}{\sin m\pi}.$$

Consider the function $\Omega_m(\zeta):\mathfrak{R}\to[1,\infty)$ defined by

$$\Omega_m(\zeta)=2^m\Gamma(m+1)\zeta^{-m}\hbar_m(\zeta),$$

where Γ is the gamma function.
The first order derivative formula of $\Omega_m(\zeta)$ is given by [42]:

$$\Omega'_m(\zeta)=\frac{\zeta}{2(m+1)}\Omega_{m+1}(\zeta) \tag{3.1}$$

and the second derivative can be easily calculated from (3.1) as

$$\Omega''_m(\zeta)=\frac{\zeta^2}{4(m+1)(m+2)}\Omega_{m+2}(\zeta)+\frac{1}{2(m+1)}\Omega_{m+1}(\zeta). \tag{3.2}$$

and the third derivative can be easily calculated from (3.2) as

$$\Omega_m'''(\zeta) = \frac{\zeta^3}{4(m+1)(m+2)(m+3)}\Omega_{m+3}(\zeta) + \frac{3\zeta}{4(m+1)(m+2)}\Omega_{m+2}(\zeta). \tag{3.3}$$

Proposition 3.1. *Suppose that $m > -1$ and $0 < \mathfrak{c} < \mathfrak{e}$. Then we get the inequality*

$$\begin{aligned}
&\Bigg|\frac{\Omega_m(\mathfrak{e}) - \Omega_m(\mathfrak{c})}{\mathfrak{e}-\mathfrak{c}} - \frac{\mathfrak{z}}{2(m+1)}\Omega_{m+1}(\mathfrak{z}) + \left(\mathfrak{z} - \frac{\mathfrak{c}+\mathfrak{e}}{2}\right) \\
&\times\left\{\frac{\mathfrak{z}^2}{4(m+1)(m+2)}\Omega_{m+2}(\mathfrak{z}) + \frac{1}{2(m+1)}\Omega_{m+1}(\mathfrak{z})\right\}\Bigg| \\
&\leq \frac{(\mathfrak{z}-\mathfrak{c})^3}{2(\mathfrak{e}-\mathfrak{c})}\Bigg[\frac{1}{(2p+1)p} + \frac{1}{2q}\Bigg\{\left(\frac{\mathfrak{z}^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\mathfrak{z}) + \frac{3\mathfrak{z}}{4(m+1)(m+2)}\Omega_{m+2}(\mathfrak{z})\right)^q \\
&+ \left(\frac{\mathfrak{c}^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\mathfrak{z}) + \frac{3\mathfrak{c}}{4(m+1)(m+2)}\Omega_{m+2}(\mathfrak{c})\right)^q\Bigg\}\Bigg] \\
&+ \frac{(\mathfrak{e}-\mathfrak{z})^3}{2(\mathfrak{e}-\mathfrak{c})}\Bigg[\frac{1}{(2p+1)p} + \frac{1}{2q}\Bigg\{\left(\frac{\mathfrak{z}^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\mathfrak{z}) + \frac{3\mathfrak{z}}{4(m+1)(m+2)}\Omega_{m+2}(\mathfrak{z})\right)^q \\
&+ \left(\frac{\mathfrak{e}^3}{8(m+1)(m+2)(m+3)}\Omega_{m+3}(\mathfrak{z}) + \frac{3\mathfrak{e}}{4(m+1)(m+2)}\Omega_{m+2}(\mathfrak{e})\right)^q\Bigg\}\Bigg].
\end{aligned}$$

Proof. The assertion follows immediately from Corollary 2.7 using $g(\zeta) = \Omega_m'(\zeta)$, $\zeta > 0$ and the identities (3.2) and (3.3). □

4. Conclusions

In this paper, we have defined an idea of fractional integral inequalities whose second derivatives are preinvex functions. We also investigated and proved a new lemma for the second derivatives of Riemann-Liouville fractional integral operator. Some new special cases are discovered in the form of corollaries. We hope that the strategies of this paper will motivate the researchers working in functional analysis, information theory and statistical theory. It is quite open to think about Ostrowski variants for generalized integral operators having Atangana-Baleanu operator etc. by applying generalized preinvexity. The results, which we have presented in this article, will potentially motivate researchers to study analogous and more general integral inequalities for various other kinds of fractional integral operators.

Conflict of interest

All authors have no conflict of interest.

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