Mathematics

AIMS Mathematics, 7(3): 3290–3302.
DOI: 10.3934/math.2022183
Received: 08 October 2021
Accepted: 25 November 2021
Published: 29 November 2021

http://www.aimspress.com/journal/Math

Research article

Hahn-Banach type theorems and the separation of convex sets for fuzzy quasi-normed spaces

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Abstract: In this paper, we first study continuous linear functionals on a fuzzy quasi-normed space, obtain a characterization of continuous linear functionals, and point out that the set of all continuous linear functionals forms a convex cone and can be equipped with a weak fuzzy quasi-norm. Next, we prove a theorem of Hahn-Banach type and two separation theorems for convex subsets of fuzzy quasi-normed spaces.

Keywords: fuzzy quasi-normed space; continuous linear functional; Hahn-Banach extension theorem; separation theorem
Mathematics Subject Classification: 46S40, 46B10

1. Introduction

With the exception of symmetry of fuzzy norm in [5], Alegre and Romaguera [2] introduced the concept of fuzzy quasi-norm with general t-norm, and obtained characterizations of those paratopological vector spaces [1] that are quasi-metrizable, locally bounded, quasi-metrizable locally convex, and quasi-normable, respectively, in terms of appropriate kinds of fuzzy quasi-norms. After that, in [4], Alegre and Romaguera proved some results, such as the uniform boundedness theorem, in fuzzy quasi-normed spaces. Recently, Gao et al. [7] gave the decomposition theorem for a fuzzy quasi-norm. Hussein and Al-Basri [9] introduced quasi-fuzzy normed algebra over a fuzzy field and studied its completion.

Just as pointed in [2], a fuzzy quasi-normed space provides a suitable framework for the complexity analysis, it seems of important roles in discussing some questions in approximation theory and in theoretical computer science. Therefore, it is worthy of further study on fuzzy quasi-normed space.

It is well known that one of the fundamental principles of functional analysis is the Hahn-Banach extension theorem for a linear functional dominated by a sublinear functional. In 2014, Alegre and Romaguera [3] prove an extension theorem for continuous linear functionals on a fuzzy normed space. So, the following question arises in a natural way: Is it possible to give a theorem of Hahn-Banach type in the frame of fuzzy quasi-normed spaces? In this article, we shall give a positive answer to this question. Based on this theorem we shall prove some separation results for convex subsets of fuzzy quasi-normed spaces. These separation theorems will be very efficient tools in the treatment of optimization problems in the framework of fuzzy quasi-normed spaces.

The organization of the paper is as follows. Section 2 comprises the basic notions on fuzzy quasi-normed spaces and some preliminary results. In Section 3, we study continuous linear functionals on a fuzzy quasi-normed space. First, we obtain a characterization of continuous linear functionals, and point out that the set of all continuous linear functionals forms a convex cone which is called the quasi dual. Next, we equip the quasi dual with a weak fuzzy quasi-norm. Finally, in Section 4, we prove a theorem of Hahn-Banach type and two separation theorems for convex subsets of fuzzy quasi-normed spaces. Besides, we obtain some properties of the Minkowski gauge functional defined on a paratopological vector space.

In this paper, $\mathbb{R}$ and $\mathbb{N}$ stand for the set of all real numbers and the set of all positive integers, respectively; X denotes a real vector space, θ is the null element of X.

2. Preliminaries

First, let us recall the concept of a continuous t-norm.

Definition 2.1 ([10]). A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions: $\forall a,b,c,d\in[0,1]$,

(1) $a*b=b*a$ (commutativity);

(2) $(a*b)*c=a*(b*c)$ (associativity);

(3) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ (monotonicity);

(4) $a*1=a$ (boundary condition);

(5) $*$ is continuous on $[0,1]\times[0,1]$ (continuity).

Three paradigmatic examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by

$$a\wedge b=\min\{a,b\},\quad a\cdot b=ab \quad\text{and}\quad a*_L b=\max\{a+b-1,0\}, \text{ respectively.}$$

Definition 2.2 ([2]). A fuzzy quasi-norm on a real vector space X is a pair $(N,*)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X\times[0,+\infty)$ satisfying the following conditions: for every $x,y\in X$,

(FQN1) $N(x,0)=0$;

(FQN2) $N(x,t)=N(-x,t)=1$ for all $t>0\Leftrightarrow x=\theta$;

(FQN3) $N(\lambda x,t)=N(x,t/\lambda)$ for all $\lambda>0$;

(FQN4) $N(x,t)*N(y,s)\le N(x+y,t+s)$ for all $t,s\ge 0$;

(FQN5) $N(x,\_): [0,+\infty)\to[0,1]$ is left continuous;

(FQN6) $\lim_{t\to\infty}N(x,t)=1$.

Obviously, the function $N(x,\_)$ is increasing for each $x \in X$.

By a fuzzy quasi-normed space we mean a triple $(X,N,*)$ such that X is a real vector space and $(N,*)$ is a fuzzy quasi-norm on X.

If condition (FQN6) is omitted we say that $(N,*)$ is a weak fuzzy quasi-norm on X.

Each fuzzy quasi-norm $(N,*)$ on X induces a T_0 topology τ_N on X which has a base given by the family of open balls

$$\mathcal{B}(x)=\{B_N(x,r,t): r\in(0,1), t>0\}$$

at $x \in X$, where

$$B_N(x,r,t)=\{y\in X:\ N(y-x,t)>1-r\}. \tag{2.1}$$

We denote by $\mathrm{cl}_N A$ the closure of A and by $\mathrm{int}_N A$ the interior of A in the topological space (X,τ_N).

A subset A of a real vector space X is

(1) semibalanced [8] provided that for each $x\in A$, $rx\in A$ whenever $0\le r\le 1$;

(2) absorbing provided that for each $x\in X$, there is $\lambda_0>0$ such that $\lambda_0 x\in A$.

Remark 2.1. Obviously, we have

(1) if A is semibalanced, then A is absorbing if and only if for each $x\in X$, there is $\lambda_0>0$ such that $\lambda x\in A$ whenever $0<\lambda<\lambda_0$;

(2) if $\theta\in A$ and A is convex, then A is semibalanced.

Proposition 2.1([2]). Let $(X,N,*)$ be a fuzzy quasi-normed space and let $\mathcal{B}(\theta)$ the family of open balls with center in the origin θ. Then:

(1) $B_N(\theta,r,t)$ is absorbing for all $t>0$ and $r\in(0,1)$.

(2) $B_N(\theta,r,t)$ is semibalanced for all $t>0$ and $r\in(0,1)$.

(3) $\lambda B_N(\theta,r,t)=B_N(\theta,r,\lambda t)$ for every $\lambda>0$, $t>0$ and $r\in(0,1)$.

(4) If $U\in\mathcal{B}(\theta)$, there is $V\in\mathcal{B}(\theta)$, such that $V+V\subseteq U$.

(5) If $U,V\in\mathcal{B}(\theta)$, there is $W\in\mathcal{B}(\theta)$ such that $W\subseteq U\cap V$.

(6) $\forall\ x\in X$, $x+B_N(\theta,r,t)=B_N(x,r,t)$.

Remark 2.2. If the continuous t-norm $*$ is chosen as "$\wedge$", then each element of $\mathcal{B}(\theta)$ is convex.

Remark 2.3. By Proposition 2.1, the mappings: $(x,y)\to x+y$ and $(\lambda,x)\to\lambda x$ are continuous on $X\times X$ and $[0,\infty)\times X$, respectively, and the topology τ_N is translation invariant.

Proposition 2.2 ([2]). If $(X,N,*)$ is a fuzzy quasi-normed space, then $(X,\tau_N,*)$ is a quasi-metrizable paratopological vector space.

Proposition 2.3 ([7]). Let $(X,N,*)$ be a fuzzy quasi- normed space, and let $\alpha\in(0,1)$. The function $\|\cdot\|_\alpha:\ X\to[0,\infty)$ is given by

$$\|x\|_\alpha=\inf\{t>0: N(x,t)\ge\alpha\}. \tag{2.2}$$

Then, for all $x\in X$ and $t>0$:

(1) $\| x \|_\alpha$ is increasing with respect to $\alpha \in (0,1)$;

(2) $\| x \|_\alpha = \sup\{t > 0 : N(x,t) < \alpha\}$;

(3) $N(x,t) \geq \alpha$ implies that $\| x \|_\alpha \leq t$;

(4) $N(x,t) < \alpha$ implies that $\| x \|_\alpha \geq t$.

The set $\{\| \cdot \|_\alpha : \alpha \in (0,1)\}$ is denoted by P_N.

Definition 2.3. ([7]). Let X be a vector space, $*$ be a continuous t-norm. $P = \{p_\alpha : p_\alpha \text{ is a function from } X \text{ to } [0,\infty), \alpha \in (0,1)\}$ is called a family of star quasi-seminorms if it satisfies the following conditions: for all $x, y \in X$, $\alpha, \beta \in (0,1)$ and $c \in [0,\infty)$,

(*QN1) $p_\alpha(cx) = cp_\alpha(x)$,

(*QN2) $p_{\alpha*\beta}(x+y) \leq p_\alpha(x) + p_\beta(y)$.

If P satisfies the following condition:

(*QN3) $p_\alpha(x) = p_\alpha(-x) = 0$ for every $\alpha \in (0,1)$ implies $x = \theta$,

then, P is said to be separating.

Remark 2.4. From (*QN1), we know $p_\alpha(\theta) = 0$ for every $\alpha \in (0,1)$.

Remark 2.5. In [7], P is defined to be separating, if $p_\alpha(x) = 0$ for every $p_\alpha \in P$ implies that $x = \theta$. Obviously, this condition is strictly stronger than (*QN3). We have to point out that definition in [7] is imperfect. In fact, the proofs of the related results in [7] are not true. However, they are true under the condition that P is separating in the sense of (*QN3).

In this paper, a separating family of star quasi-seminorms is always in the sense of (*QN3).

Proposition 2.4. Let $P = \{p_\alpha : p_\alpha \text{ is a function from } X \text{ to } [0,\infty), \alpha \in (0,1)\}$ be a family of star quasi-seminorms. For each $x \in X$, let

$$U_P(x) = \{U(x; \alpha_1, \alpha_2, \cdots, \alpha_n; \varepsilon) : \varepsilon > 0; \alpha_1, \alpha_2, \cdots, \alpha_n \in (0,1), n \in \mathbb{N}\},$$

where

$$\begin{aligned} U(x; \alpha_1, \alpha_2, \cdots, \alpha_n; \varepsilon) &= \{y \in X : p_{\alpha_i}(y - x) < \varepsilon, \alpha_i \in (0,1), i = 1, 2, \cdots, n\} \\ &= \bigcap_{i=1}^n \{y \in X : p_{\alpha_i}(y - x) < \varepsilon, \alpha_i \in (0,1)\} \\ &= \{y \in X : p_{\max\{\alpha_i : 1 \leq i \leq n\}}(y - x) < \varepsilon\}. \end{aligned}$$

Then, $U_P(x)$ is a basis of neighborhoods of x.

The topology taking $U_P(x)$ as a basis of neighborhoods of x is said to be the topology induced by P and denoted by τ_P.

Proposition 2.5 ([7]). Let $(X, N, *)$ be a fuzzy quasi-normed space. $\mathscr{P}_N = \{\| \cdot \|_\alpha : \alpha \in (0,1)\}$ where $\| \cdot \|_\alpha$ is defined by (2.2) for all $\alpha \in (0,1)$. Then,

(1) $\mathscr{P}_N$ is a separating family of star quasi-seminorms;

(2) the topology $\tau_{\mathscr{P}_N}$ induced by $\mathscr{P}_N$ coincides the topology τ_N.

$\mathscr{P}_N$ is called a family of star quasi-seminorms induced by $(N, *)$.

Let X be a real vector space. Recall that a functional $p : X \to \mathbb{R}$ is called sublinear if

(S1) $p(cx) = cp(x)$, $\forall x \in X, \forall c \in [0,\infty)$;

(S2) $p(x+y) \le p(x)+p(y), \forall x, y \in X$.

A positive sublinear functional is a quasi seminorm. A sublinear functional p is called a seminorm if instead of (S1) it satisfies

(S1)′ $p(cx)=|c|\,p(x), \ \forall x \in X, \forall c \in \mathbb{R}$.

Remark 2.6. If the continuous t-norm $*$ is chosen as "$\wedge$", then a family of star quasi-seminorms is a family of quasi seminorms.

3. Continuous linear functionals on a fuzzy quasi-normed space

On the field $\mathbb{R}$ of real numbers, consider the quasi norm $u(x)=\max\{x,0\}$. The topology $\tau(u)$ generated by u is called the upper topology of $\mathbb{R}$. A basis of open $\tau(u)$-neighborhoods of a point $x \in \mathbb{R}$ is formed of the intervals $(-\infty, x+\varepsilon)$, $\varepsilon>0$. While, the intervals $(-\infty, a)$, $a \in \mathbb{R}$, forms a basis of $\tau(u)$.

The quasi dual $(X,N,*)^{\#}$ of a fuzzy quasi-normed space $(X,N,*)$ is formed by all continuous linear functionals from (X,τ_N) to $(\mathbb{R},\tau(u))$, or equivalently, by all upper semi-continuous linear functionals from (X,τ_N) to ($\mathbb{R}$, $|\cdot|$). In the sequel, $(X,N,*)^{\#}$ will be simply denoted by $X^{\#}$ if no confusion arises.

Theorem 3.1. Let $(X,N,*)$ be a fuzzy quasi-normed space. $f \in X^{\#}$ if and only if there are $\alpha \in (0,1)$ and $M>0$ such that $f(x) \le M\|x\|_\alpha$ for all $x \in X$.

Proof. Necessity. Suppose that f is continuous, then f is continuous at θ. Applying Proposition 2.5, it follows that there exist $\alpha \in (0,1)$ and $\varepsilon>0$ such that $f(U(\alpha,\varepsilon)) \subseteq (-\infty,1)$, where $U(\alpha,\varepsilon)=\{x \in X : \|x\|_\alpha < \varepsilon\}$. Take $x \in X$ arbitrarily.

(i) If $\|x\|_\alpha \neq 0$, then $\varepsilon x/\|x\|_\alpha \in U(\alpha,\varepsilon)$, so that

$$f\left(\varepsilon x/\|x\|_\alpha\right)<1, \text{ that is, } \quad f(x) \le (1/\varepsilon)\|x\|_\alpha .$$

(ii) If $\|x\|_\alpha = 0$, then $\|\lambda x\|_\alpha = \lambda\|x\|_\alpha = 0$ for any $\lambda>0$, so that $\lambda x \in U(\alpha,\varepsilon)$, implying $\lambda f(x) = f(\lambda x) < 1$, that is, $f(x) < 1/\lambda$. It follows from the arbitrariness of λ that $f(x) \le 0$.

Combining (i) and (ii), the Necessity is proved.

Sufficiency. Suppose there are $\alpha \in (0,1)$ and $M>0$ such that $f(x) \le M\|x\|_\alpha$ for all $x \in X$. Then $f(U(\alpha,\varepsilon/M)) \subseteq (-\infty,a)$ for any $a>0$. Since $U(\alpha,\varepsilon/M)$ is a τ_N open ball at θ, f is continuous at θ. By Remark 2.3, f is continuous on X. □

Corollary 3.1. Let $(X,N,*)$ be a fuzzy quasi-normed space. $(X,N,*)^{\#}$ is a convex cone.

In the rest of this section, we shall equip $(X,N,*)^{\#}$ with a weak fuzzy quasi-norm.

First, we introduce the concept of a family of star extended quasi-seminorms.

Definition 3.1. Let X be a vector space and let $p_i : X \to [0,\infty]$ be an extended functional for every $i \in I$. If $\{p_i : i \in I\}$ satisfies the conditions of star quasi-seminorms, then it is called a family of star extended quasi-seminorms.

If $\{p_i : i \in I\}$ is a family of star extended quasi-seminorms on X, we say that the family is separating if it satisfies the following condition:

(*QN3)$'$ for all $x \in X$, $x \neq \theta$, there are $j, k \in I$ such that (i) $p_j(x) \neq 0$ or $p_j(-x) \neq 0$, (ii) $p_k(x) \neq \infty$.

Theorem 3.2. Let $P = \{\|\cdot\|_\alpha : \alpha \in (0,1)\}$ be an increasing family of separating star extended quasi-seminorms on a real vector space X, and let $\|\cdot\|_0$ be given by $\|x\|_0 = 0$ for all $x \in X$. The function $N_P(x,t)$: $X \times [0,\infty) \to [0,1]$ is given by

$$N_P(x,t) = \begin{cases} 0, & t = 0 \\ \sup\{\alpha \in (0,1) : \|x\|_\alpha < t\}, & t > 0 \end{cases}. \tag{3.1}$$

Then, the pair $(N_P, *)$ is a weak fuzzy quasi-norm on X.

Proof. (1) (FQN1) is obvious.

(FQN2). If $N_P(x,t) = N_P(-x,t) = 1$ for all $t > 0$, then $\|x\|_\alpha < t$ and $\|-x\|_\alpha < t$ for all $\alpha \in (0,1)$ from (3.1). Therefore, $\|x\|_\alpha = \|-x\|_\alpha = 0$ for all $\alpha \in (0,1)$. Since P is separating, $x = \theta$. Conversely, if $x = \theta$, then it follows from Remark 2.4 that $\|x\|_\alpha = \|-x\|_\alpha = 0 < t$ for all $t > 0$. By (3.1), $N_P(x,t) = N_P(-x,t) = 1$.

(FQN3). Let $c > 0$. From (*QN1), we have

$$N_P(cx,t) = \sup\{\alpha \in (0,1) : \|cx\|_\alpha < t\} = \sup\{\alpha \in (0,1) : \|x\|_\alpha < t/c\} = N_P(x, t/c) \cdot$$

(FQN4). Let $x, y \in X$ and $s, t > 0$, and let $N_P(x,t) = \beta$, $N_P(y,s) = \gamma$. Without loss of generality, we suppose that $0 < \min\{\beta, \gamma\}$. For any $0 < \varepsilon < \min\{\beta, \gamma\}$, there exist $\alpha', \alpha'' \in (0,1)$ such that $\alpha' > \beta - \varepsilon$, $\alpha'' > \gamma - \varepsilon$, $\|x\|_{\alpha'} < t$ and $\|y\|_{\alpha''} < s$. Thus, $\|x\|_{\beta-\varepsilon} < t$ and $\|y\|_{\gamma-\varepsilon} < s$. And hence, $\|x+y\|_{(\beta-\varepsilon)*(\gamma-\varepsilon)} \le \|x\|_{\beta-\varepsilon} + \|y\|_{\gamma-\varepsilon} < t + s$. By (3.1), $N_P(x+y, t+s) \ge (\beta-\varepsilon)*(\gamma-\varepsilon)$. By the arbitrariness of $\varepsilon > 0$ and the continuity of $*$, we know

$$N_P(x+y, t+s) \ge \beta * \gamma = N_P(x,t) * N_P(y,s).$$

(FQN5). Obviously, $N_P(\theta, \_) = 1$, and hence, $N_P(\theta, \_)$ is continuous. Now, take $x_0 \in X \setminus \{\theta\}$ and $t_0 > 0$ arbitrarily. If $N_P(x, t_0) = 0$, then $N_P(x,t) = N_P(x,t_0) = 0$ for all $t < t_0$. So, $N_P(x, \_)$ is left continuous at t_0. Now, we suppose $0 < N_P(x,t_0) \le 1$. Take $\varepsilon > 0$ arbitrarily, from (3.1), there exists $\alpha_0 \in (0,1)$ such that $\|x\|_{\alpha_0} < t_0$ and $N_P(x,t_0) - \varepsilon < \alpha_0$. So, we have $N_P(x,t) \ge \alpha_0$ for any t with $\|x\|_{\alpha_0} < t < t_0$. Hence, $N_P(x,t_0) - N_P(x,t) \le N_P(x,t_0) - \alpha_0 < \varepsilon$. Therefore, $N_P(x, \_)$ is left continuous at t_0. □

Now, for each $f \in X^\#$, we define $\| f \|_0^\# = 0$ and

$$\| f \|_\alpha^\# = \sup\{ f(x) : \| x \|_{1-\alpha} \le 1\}, \quad \forall \alpha \in (0,1). \tag{3.2}$$

Theorem 3.3. Let $(X, N, *)$ be a fuzzy quasi-normed space, $f \in X^\#$, $\alpha \in (0,1)$.

(1) If $f \neq 0$, then $\| f \|_{\alpha}^{\#} > 0$.

(2) $\| f \|_{\alpha}^{\#} = \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\}$.

(3) $\| f \|_{\alpha}^{\#} = \sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\}$.

(4) If $N(x,\_)$ is increasing strictly, then $\| f \|_{\alpha}^{\#} = \sup\left\{ f\left(x\right) : N(x,1) > 1-\alpha \right\}$.

Proof. (1) If $f \neq 0$, then there exists an $x_0 \in X$ such that $f(x_0) \neq 0$. Suppose that $f(x_0) > 0$ (otherwise, replace x_0 with $-x_0$). If $\| x_0 \|_{1-\alpha} =0<1$, then $\| f \|_{\alpha}^{\#} \geq f(x_0) > 0$. If $\| x_0 \|_{1-\alpha} > 0$, then $\| f \|_{\alpha}^{\#} \geq f(x_0 / \| x_0 \|_{1-\alpha}) = f(x_0)/ \| x_0 \|_{1-\alpha} > 0$. In a word, $\| f \|_{\alpha}^{\#} > 0$.

(2) It remains to show that $\| f \|_{\alpha}^{\#} \leq \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\}$. Indeed, if $x \in X$ is such that $\| x \|_{1-\alpha} = 1$, then $\| n(n+1)^{-1} x \|_{1-\alpha} = n(n+1)^{-1} \| x \|_{1-\alpha} < 1$. So that $n(n+1)^{-1} f(x) = f(n(n+1)^{-1} x) \leq \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\}$, for all $n \in \mathbb{N}$. Letting $n \to \infty$, it follows that $f(x) \leq \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\}$, implying $\| f \|_{\alpha}^{\#} \leq \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\}$.

(3) By Proposition 2.3, we obtain

$$\sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\} \leq \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} \leq 1 \right\},$$

$$\sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} < 1 \right\} \leq \sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\}.$$

Which together with (2) implies that

$$\| f \|_{\alpha}^{\#} = \sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\}.$$

(4) It remains to show that

$$\sup\left\{ f\left(x\right) : N(x,1) > 1-\alpha \right\} \geq \sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\}.$$

Indeed, if $x_0 \in X$ is such that $N(x_0,1) = 1-\alpha$, then it follows $N(n(n+1)^{-1} x_0, 1) = N(x_0, (n+1)n^{-1}) > 1-\alpha$ from the hypothesis. So that

$$\sup\left\{ f\left(x\right) : N(x,1) > 1-\alpha \right\} \geq f(n(n+1)^{-1} x_0) = n(n+1)^{-1} f(x_0), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, it follows that

$$\sup\left\{ f\left(x\right) : N(x,1) > 1-\alpha \right\} \geq f(x_0),$$

implying $\sup\left\{ f\left(x\right) : N(x,1) > 1-\alpha \right\} \geq \sup\left\{ f\left(x\right) : N(x,1) \geq 1-\alpha \right\}$. □

Theorem 3.4. Let $\left(X, N, *\right)$ be a fuzzy quasi-normed space. Then

(1) $\left\{ \| \cdot \|_{\alpha}^{\#} : \alpha \in (0,1) \right\}$ is a family of separating star extended quasi-seminorms on $X^{\#}$;

(2) $\left\{ \| \cdot \|_{\alpha}^{\#} : \alpha \in (0,1) \right\}$ is increasing with respect to $\alpha \in (0,1)$.

Proof. (1) Let $f, g \in X^{\#}$, $\alpha, \beta \in (0,1)$ and $c \in [0, \infty)$.

(*QN1). $\| cf \|_{\alpha}^{\#} = \sup\left\{ cf\left(x\right) : \| x \|_{1-\alpha} \leq 1 \right\} = c \cdot \sup\left\{ f\left(x\right) : \| x \|_{1-\alpha} \leq 1 \right\} = c \| f \|_{\alpha}^{\#}$.

(*QN2). Since $\alpha * \beta \leq \alpha$ and $\alpha * \beta \leq \beta$, then

$$\| f + g \|_{\alpha*\beta}^{\#} = \sup\left\{ f(x) + g(x) : \| x \|_{1-\alpha*\beta} \leq 1 \right\}$$

$$\leq \sup\left\{ f(x): \| x\|_{1-\alpha*\beta} \leq 1\right\} + \sup\left\{ g(x): \| x\|_{1-\alpha*\beta} \leq 1\right\}$$

$$\leq \sup\left\{ f(x): \| x\|_{1-\alpha} \leq 1\right\} + \sup\left\{ g(x): \| x\|_{1-\beta} \leq 1\right\}$$

$$= \| f \|_{\alpha}^{\#} + \| g \|_{\beta}^{\#}.$$

(*QN3)$'$. (i) Suppose that $\| f \|_{\alpha}^{\#} = \| -f \|_{\alpha}^{\#} = 0$ and for all $\alpha \in (0,1)$. For each $x \in X \setminus \{\theta\}$, it follows from Proposition 2.5 that $\left\{\| \cdot \|_{\alpha}^{\#} : \alpha \in (0,1)\right\}$ is separating, implying that there is a $\beta \in (0,1)$ such that $\|x\|_{1-\beta} > 0$ or $\|-x\|_{1-\beta} > 0$.

If $\|x\|_{1-\beta} > 0$, then $\left\| x / \|x\|_{1-\beta} \right\|_{1-\beta} = 1$. By the hypothesis, we know that $\| f \|_{\beta}^{\#} = \| -f \|_{\beta}^{\#} = 0$, so that $f\left(x / \|x\|_{1-\beta}\right) \leq 0$ and - $f\left(x / \|x\|_{1-\beta}\right) \leq 0$, implying that $f(x) = 0$. If $\|-x\|_{1-\beta} > 0$, then a similar treatment leads to $f(x) = 0$. By the arbitrariness of x and $f(\theta) = 0$, we get $f(x) = 0$ for all $x \in X$, that is, $f = 0$.

(ii) For any $g \in X^{\#}$ with $g \neq 0$, by Theorem 3.1, there are $\beta \in (0,1)$ and $M > 0$ such that $g(x) \leq M \| x \|_{\beta}$ for all $x \in X$. It follows that $\|g\|_{1-\beta}^{\#} \leq M < +\infty$.

(2) If $\alpha, \beta \in (0,1)$ with $\alpha < \beta$, then $\| x \|_{1-\alpha} \geq \| x \|_{1-\beta}$ for all $x \in X$. By (3.2), we have $\| f \|_{\alpha}^{\#} \leq \| f \|_{\beta}^{\#}$. □

Remark 3.1. $\| f \|_{\alpha}^{\#}$ can be infinity even in symmetrical situations (see Example 19 in [3]).

The following theorem is obvious by Theorem 3.2 and Theorem 3.4.

Theorem 3.5. Let $(X, N, *)$ be a fuzzy quasi-normed space. For each $f \in X^{\#}$, let

$$N_X^{\#}(f,t) = \begin{cases} 0, t = 0 \\ \sup\{\alpha \in [0,1) : \| f \|_{\alpha}^{\#} < t\}, t > 0 \end{cases}, \tag{3.3}$$

Then, $\left(N_X^{\#}, *\right)$ is a weak fuzzy quasi-norme on $X^{\#}$.

$N_X^{\#}$ will be simply denoted by $N^{\#}$ if no confusion arises.

Theorem 3.6. Let $(X, N, *)$ be a fuzzy quasi-normed space, $f \in X^{\#}$, $t > 0$. Let

$$\delta_f = \sup\left\{\alpha \in (0,1) : \|f\|_{\alpha}^{\#} < \infty\right\}, \tag{3.4}$$

then,

$$N^{\#}(f,t) = \sup\left\{\alpha \in [0, \delta_f) : \| f \|_{\alpha}^{\#} < t\right\}. \tag{3.5}$$

Proof. If $f = 0$, then $\|f\|_{\alpha}^{\#} = 0 < \infty$ for all $\alpha \in (0,1)$, so that δ_f =1, implying (3.5).

If $f \neq 0$, by Theorem 3.1, there is an $\alpha \in (0,1)$ such that $\|f\|_{\alpha}^{\#} < \infty$. It follows that $\delta_f > 0$. In this case, it is obvious that $N^{\#}(f,t) \geq \sup\left\{\alpha \in [0, \delta_f) : \| f \|_{\alpha}^{\#} < t\right\}$, For any $\gamma > 0$ with $\gamma < N^{\#}(f,t)$, there exists an $\alpha \in (0,1)$ such that $\alpha > \gamma$ and $\| f \|_{\alpha}^{\#} < t$. By (3.4), $\delta_f \geq \alpha > \gamma$. It follows from Theorem 3.4(2) that $\| f \|_{\gamma}^{\#} \leq \| f \|_{\alpha}^{\#} < t$, so that $\sup\left\{\alpha \in [0, \delta_f) : \| f \|_{\alpha}^{\#} < t\right\} \geq \gamma$. By the arbitrariness of

γ, we get $\sup\left\{\alpha\in[0,\delta_f):\|f\|_\alpha^{\#}<t\right\}\geq N^{\#}(f,t)$. Which implies that (3.5).

4. Hahn-Banach type theorems and the separation of convex sets

In this section, we shall give the Hahn-Banach type theorem and the separation of convex sets in the frame of the fuzzy quasi-normed space in which the continuous t-norm $*$ is chosen as "$\wedge$". First, we recall the classical Hahn-Banach extension theorem for a linear functional dominated by a sublinear functional (see, e.g, Theorem 2.2.1 in [6]).

Lemma 4.1. Let X be a real vector space and p be a sublinear functional on X. If X_0 is a subspace of X and f_0 is a linear functional dominated by p on X_0, then there exists a linear functional f dominated by p on X such that $f|_{X_0}=f_0$.

Theorem 4.1. (Hahn-Banach Extension Theorem) Let $(X,N,\wedge)$ be a fuzzy quasi-normed space, and let f_0 be a continuous linear functional on a subspace $\left(X_0,N|_{X_0},\wedge\right)$ of $(X,N,\wedge)$. Then, there exists $\delta\in(0,1]$ for which the following two conditions are satisfied:

(1) for all $\alpha\in(0,\delta)$, there is $f_\alpha\in(X,N,\wedge)^{\#}$ such that $f_\alpha|_{X_0}=f_0$ and $\|f_\alpha\|_\alpha^{\#}=\|f_0\|_{\alpha,X_0}^{\#}$, where $\|f_0\|_{\alpha,X_0}^{\#}=\sup\left\{f_0(x):x\in X_0,\|x\|_{1-\alpha}\leq 1\right\}$;

(2) $N_{X_0}^{\#}(f_0,t)=\sup\left\{N^{\#}(f_\alpha,t):\alpha\in(0,\delta)\right\}$, $\forall t>0$.

Proof. Put

$$\delta=\sup\left\{\alpha\in(0,1):\|f_0\|_{\alpha,X_0}^{\#}<\infty\right\}. \tag{4.1}$$

Since $f_0\in\left(X_0,N|_{X_0},\wedge\right)^{\#}$, we get $\delta\in(0,1]$ by using the similar proof in Theorem 3.6.

(1) For any $\alpha\in(0,\delta)$, (4.1) implies that $\|f_0\|_{\alpha,X_0}^{\#}<\infty$. Define a functional p_α on X as: $p_\alpha(x)=\|f_0\|_{\alpha,X_0}^{\#}\|x\|_{1-\alpha}$, $\forall x\in X$. It follows from Remark 2.6 that $\|\cdot\|_{1-\alpha}$ is a quasi seminorm, implying that p_α is a sublinear functional on X.

Let $x\in X_0$. If $\|x\|_{1-\alpha}>0$, then $f_0\left(x/\|x\|_{1-\alpha}\right)\leq\|f_0\|_{\alpha,X_0}^{\#}$, so that $f_0(x)\leq p_\alpha(x)$. If $\|x\|_{1-\alpha}=0$, then $\|\lambda x\|_{1-\alpha}=\lambda\|x\|_{1-\alpha}=0$ for all $\lambda>0$. By the definition of $\|f_0\|_{\alpha,X_0}^{\#}$, we get $\|f_0\|_{\alpha,X_0}^{\#}\geq f_0(\lambda x)$, that is, $f_0(x)\leq\|f_0\|_{\alpha,X_0}^{\#}/\lambda$, which together with the arbitrariness of λ implies that $f_0(x)\leq 0=p_\alpha(x)$. Thus, f_0 is dominated by p_α on X_0.

By Lemma 4.1, there is a linear functional f_α on X such that $f_\alpha|_{X_0}=f_0$ and $f_\alpha(x)\leq\|f_0\|_{\alpha,X_0}^{\#}\|x\|_{1-\alpha}$ for all $x\in X$. Now, we prove that $\|f_\alpha\|_\alpha^{\#}=\|f_0\|_{\alpha,X_0}^{\#}$. Since $f_\alpha|_{X_0}=f_0$, it is obvious that $\|f_\alpha\|_\alpha^{\#}\geq\|f_0\|_{\alpha,X_0}^{\#}$. On the other hand, by $f_\alpha(x)\leq\|f_0\|_{\alpha,X_0}^{\#}\|x\|_{1-\alpha}$, we know that $f_\alpha(x)\leq\|f_0\|_{\alpha,X_0}^{\#}$ whenever $\|x\|_{1-\alpha}\leq 1$, which means that

$$\|f_\alpha\|_\alpha^{\#}=\sup\left\{f_\alpha(x):x\in X,\|x\|_{1-\alpha}\leq 1\right\}\leq\|f_0\|_{\alpha,X_0}^{\#}.$$

Thus, $\|f_\alpha\|_\alpha^{\#}=\|f_0\|_{\alpha,X_0}^{\#}$.

(2) For any $\alpha\in(0,\delta)$ and $\gamma\in[0,1)$, since $f_\alpha|_{X_0}=f_0$, it is obvious that $\|f_\alpha\|_\gamma^{\#}\geq\|f_0\|_{\gamma,X_0}^{\#}$, it

follows that

$$N_{X_0}^{*}\left(f_0,t\right)=\sup\left\{\gamma\in[0,1):\| f_0\|_{\gamma,X_0}^{\#}<t\right\}\geq\sup\left\{\gamma\in[0,1):\| f_\alpha\|_{\gamma}^{\#}<t\right\}=N^{\#}\left(f_\alpha,t\right).$$

By the arbitrariness of α, we get $N_{X_0}^{\#}\left(f_0,t\right)\geq\sup\left\{N^{\#}\left(f_\alpha,t\right):\alpha\in(0,\delta)\right\}$.

On the other side, for each $t>0$ and $\varepsilon>0$, by the definition of $N_{X_0}^{\#}\left(f_0,t\right)$, there exists $\beta\in(0,1)$ such that $\beta>N_{X_0}^{\#}\left(f_0,t\right)-\varepsilon\triangleq\beta'$ and $\| f_0\|_{\beta,X_0}^{\#}<t$. It follows from the definition of δ that $\delta\geq\beta>\beta'$, which together with (1) implies that $\| f_{\beta'}\|_{\beta'}^{\#}=\| f_0\|_{\beta',X_0}^{\#}\leq\| f_0\|_{\beta,X_0}^{\#}<t$, so that $N^{\#}\left(f_{\beta'},t\right)\geq\beta'$, hence $\sup\left\{N^{\#}\left(f_\alpha,t\right):\alpha\in(0,\delta)\right\}\geq\beta'=N_{X_0}^{\#}\left(f_0,t\right)-\varepsilon$. By the arbitrariness of ε, we have $\sup\left\{N^{\#}\left(f_\alpha,t\right):\alpha\in(0,\delta)\right\}\geq N_{X_0}^{\#}\left(f_0,t\right)$. □

To prove the separation of convex sets in the frame of the fuzzy quasi-normed space, we first give some properties of the Minkowski gauge functional defined on a paratopological vector space.

Lemma 4.2. Let A be a semibalanced and absorbing subset of a paratopological vector space (X,τ). μ_A is the Minkowski functional of the set A, that is, $\mu_A(x)=\inf\{\lambda>0:x\in\lambda A\}$, $\forall x\in X$. Put

$$B=\{x:\mu_A(x)<1\},\quad C=\{x:\mu_A(x)\leq1\}.$$

(1) $\mu_A(\lambda x)=\lambda\,\mu_A(x)$, $\forall\lambda>0$, $\forall x\in X$.

(2) If A is convex, then $\mu_A(x+y)\leq\mu_A(x)+\mu_A(y)$, $\forall x,y\in X$.

(3) $\mathrm{int}_\tau A\subseteq B\subseteq A\subseteq C\subseteq\mathrm{cl}_\tau A$.

(4) The followings are equivalent:

(i) μ_A: $(X,\tau)\to(R,\tau(u))$ is continuous at θ,

(ii) $\mathrm{int}_\tau A=B$,

(iii) $\theta\in\mathrm{int}_\tau A$.

(5) If A is convex, then μ_A: $(X,\tau)\to(R,\tau(u))$ is continuous at θ if and only if μ_A is continuous on X.

Proof. (1) and (2) are two general properties.

(3) Take an $x\in\mathrm{int}_\tau A$ arbitrarily. Since (X,τ) is a paratopological vector space, there exist a neighborhood V of x and an $\varepsilon\in(0,1)$ such that $[1,1+\varepsilon)V\subseteq\mathrm{int}_\tau A\subseteq A$. So that $(1+\varepsilon/2)x\in A$, implying $\mu_A(x)\leq(1+\varepsilon/2)^{-1}<1$. Hence, $x\in B$. By the arbitrariness of x, we have $\mathrm{int}_\tau A\subseteq B$.

Conversely, take an $x\in B$ arbitrarily, then $\mu_A(x)<1$. By the definition of μ_A, there exists $0<\lambda'<1$ such that $x\in\lambda'A$. Noting that A is semibalanced, we get $x\in\lambda'A\subseteq A$. By the arbitrariness of x, we have $B\subseteq A$.

The fact that $A\subseteq C$ is obvious. It remains to verify that $C\subseteq\mathrm{cl}_\tau A$. Take an $x\in C$ arbitrarily, then $\mu_A(x)\leq1$. Then, $\mu_A\left(n(n+1)^{-1}x\right)=n(n+1)^{-1}\mu_A(x)<1$ for all $n\in\mathbb{N}$, and hence $n(n+1)^{-1}x\in B\subseteq A$. Letting $n\to\infty$, it follows that $x=\lim_{n\to\infty}n(n+1)^{-1}x\in\mathrm{cl}_\tau A$. By the arbitrariness of x, we have $C\subseteq\mathrm{cl}_\tau A$.

(4) (i)⇒(ii). Suppose μ_A is continuous at θ. It follows that $B\in\tau$ from $(-\infty,1)\in\tau(u)$. By

$B \subseteq A$, we get $B \subseteq \text{int}_\tau A$. Which together with (3) implies that $B = \text{int}_\tau A$.

The implication (ii)⇒(iii) follows from $\theta \in B$ immediately.

(iii)⇒(i). Suppose that $\theta \in \text{int}_\tau A$. Take an $\varepsilon > 0$ arbitrarily. Since $\varepsilon \text{int}_\tau A$ is an open neighborhood of θ, for each $x \in \varepsilon \text{int}_\tau A$, there is a $y \in \text{int}_\tau A \subseteq B$ such that $x = \varepsilon y$, so that $0 \le \mu_A(x) = \mu_A(\varepsilon y) = \varepsilon \mu_A(y) < \varepsilon$. Thus, μ_A is continuous at θ.

(5) If A is convex, it follows from (2) that $|\mu_A(x) - \mu_A(y)| \le \mu_A(x-y),\ \forall x, y \in X$. Noting that the topology τ is translation invariant, we know that (5) is true. □

Theorem 4.2. Let $(X, N, \wedge)$ be a fuzzy quasi-normed space and A, B two disjoint nonempty convex subsets of X with A open. Then, there exists a $\delta \in (0,1]$ such that: for each $\alpha \in (0, \delta)$, there is an $f_\alpha \in X^\#$ such that $f_\alpha(x) < f_\alpha(y)$, $\forall x \in A$, $\forall y \in B$.

Proof. Let $\xi_0 \in A$, $\eta_0 \in B$, and let $\zeta = \eta_0 - \xi_0$. Since A is open and the topology τ_N is translation invariant, the set $C = A - B + \zeta$ is open too. It is obvious that C is also convex and $\theta \in C$. By Lemma 4.2, the Minkowski functional μ_C of C is sublinear, $\tau(u)$-continuous. Since $A \cap B = \phi$, then $\zeta \notin C$. By Lemma 4.2(3), $\mu_C(\zeta) \ge 1$.

Let X_0 be the one-dimensional subspace generated by ζ. Define a linear functional f_0: $X_0 \to \mathbb{R}$ by $f_0(t\zeta) = t$, $\forall t \in \mathbb{R}$. Since $f_0(t\zeta) = t \le t\mu_C(\zeta) = \mu_C(t\zeta)$ for $t \ge 0$, and $f_0(t\zeta) = t < 0 \le \mu_C(t\zeta)$ for $t < 0$, it follows that $f_0(x) \le \mu_C(x)$, $\forall x \in X_0$. By $\tau(u)$-continuity of μ_C, f_0 is $\tau(u)$-continuous.

By Theorem 4.1, there exists $\delta \in (0,1]$ such that: for each $\alpha \in (0, \delta)$, there is an $f_\alpha \in X^\#$ such that $f_\alpha|_{X_0} = f_0$. For each $x \in A$ and $y \in B$, since $f(\zeta) = 1$, $x - y + \zeta \in C$ and C is open, it follows from Lemma 4.2 that

$$f_\alpha(x) - f_\alpha(y) + 1 = f_\alpha(x - y + \zeta) \le \mu_C(x - y + \zeta) < 1,$$

implying $f_\alpha(x) < f_\alpha(y)$. □

We prove now another separation theorem.

Theorem 4.3. Let $(X, N, \wedge)$ be a fuzzy quasi-normed space and A, B two disjoint nonempty convex subsets of X, with A compact and B closed. Then, there exists a $\delta \in (0,1]$ such that: for each $\alpha \in (0, \delta)$, there is an $f_\alpha \in X^\#$ such that $\sup_{x \in A} f_\alpha(x) < \inf_{y \in B} f_\alpha(y)$.

Proof. Let $B_r = B_N(\theta, r, 1)$, $r \in (0,1)$. Since $A \cap B = \phi$ and B is closed, for each $x \in A$ there exist $r_x \in (0,1)$ and $t_x > 0$ such that

$$\left(x + 2t_x B_{r_x}\right) \cap B = \phi. \tag{4.2}$$

The open cover $\{x + t_x B_{r_x} : x \in A\}$ of the compact set A, contains a finite subcover $\{x_i + t_i B_{r_i} : i = 1, 2, ..., n\}$, where $t_i = t_{x_i}$ and $r_i = r_{x_i}$ for $i = 1, 2, ..., n$. Put $t = \min\{t_i : i = 1, 2, ..., n\}$, $r = \max\{r_i : i = 1, 2, ..., n\}$. Then, $tB_r \subseteq t_i B_{r_i}$ for $i = 1, 2, ..., n$. Now, we show that

$$\left(A + tB_r\right) \cap B = \phi. \tag{4.3}$$

Indeed, if $x' = x + tb \in B$ for some $x \in A$, $b \in B_r$, choosing $k \in \{i = 1, 2, ..., n\}$ such that $x \in x_k + t_k B_{r_k}$. We have

$$x' = x + tb \in x_k + t_k B_{r_k} + tB_r \subseteq x_k + t_k B_{r_k} + t_k B_{r_k} \subseteq x_k + 2t_k B_{r_k},$$

in contradiction to (4.2). So that, (4.3) holds.

The set $C = A + tB_r$ is convex, open and disjoint from B. By Theorem 4.2, there exists a $\delta \in (0,1)$ such that, for any $\alpha \in (0, \delta)$, there is an $f_\alpha \in (X, N, *)^\#$ such that

$$f_\alpha(x) + tf_\alpha(z) < f_\alpha(y), \ \forall x \in A, \ \forall z \in B_r, \ \forall y \in B. \tag{4.4}$$

Let $0 < r' < r < 1$, then $B_r = \{x : N(x,1) > 1 - r\} \supseteq \{x : N(x,1) \geq 1 - r'\} \triangleq \overline{B}_{r'}$.

By (4.4), $f_\alpha \neq 0$, so that by Theorem 3.3(3) $\| f \|_{1-r'}^\# = \sup f(\overline{B}_{r'}) > 0$. Passing in (4.4) to supremum with respect to $z \in \overline{B}_{r'}$, we get

$$f_\alpha(x) + t \| f \|_{1-r'}^\# \leq f_\alpha(y), \ \forall x \in A, \ \forall y \in B,$$

implying $\sup_{x \in A} f_\alpha(x) + t \| f \|_{1-r'}^\# \leq \inf_{y \in B} f_\alpha(y)$. It follows that $\sup_{x \in A} f_\alpha(x) < \inf_{y \in B} f_\alpha(y)$. □

5. Conclusions

In this paper, the Hahn-Banach extension theorem and the separation of convex sets are generalized to fuzzy quasi-normed spaces. With the help of these important results, the question of extending the related results to fuzzy quasi-normed spaces, deserves attention in a further work. On the other hand, the main results in the paper are obtained under the condition that the continuous t-norm $*$ is chosen as "$\wedge$". Therefore, the following question is also worthy of further study: Is it possible to give types of these theorems in the framework of fuzzy quasi-normed spaces with the general continuous t-norms?

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11971343, 12071225). We are grateful to the anonymous reviewers and editors for their valuable comments which have enabled to improve the original version of this paper.

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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