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Research article

Decay result in a problem of a nonlinearly damped wave equation with variable exponent

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Abstract: In this work we study a wave equation with a nonlinear time dependent frictional damping of variable exponent type. The existence and uniqueness results are established using Fadeo-Galerkin approximation method. We also exploit the Komornik lemma to prove the uniform stability result for the energy associated to the solution of the problem under consideration.

Keywords: viscoelasticity; relaxation function; general decay; logarithmic nonlinearity; variable exponent
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1. Introduction

In this work we are concerned with the decay rate of the following problem with nonlinear damping of variable exponent

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \alpha(t)\left[u_t(x,t) + u_t(x,t)|u_t|^{m(x)-2}(x,t)\right] = 0, & \text{in} \quad \Omega\times(0,T), \\ u = 0, & \text{on} \quad \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$. The functions u_0, u_1 are initial data and the variable exponent $m(\cdot) \in C\left(\overline{\Omega}\right)$ is a given functions satisfying

$$1 < m_1 \leq m(x) \leq m_2 < 2^*, \tag{1.2}$$

where

$$m_1 := \inf_{x\in\Omega} m(x), \quad m_2 := \sup_{x\in\Omega} m(x), \quad 2^* = \begin{cases} \frac{2n}{n-2}, & \text{if} \quad n \geq 3, \\ \infty, & \text{if} \quad n < 3, \end{cases}$$

and also satisfies the log-Hölder continuity condition:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \tag{1.3}$$

for $x, y \in \Omega$, with $|x - y| < \delta$, $A > 0$ and $0 < \delta < 1$. The function $\alpha : [0, \infty) \to (0, \infty)$ is a bounded nonincreasing C^1−function and

$$\exists \alpha_0 > 0 \text{ such that } \alpha(t) \geq \alpha_0, \ \forall t \geq 0. \tag{1.4}$$

Problems with variable exponents appear as a direct consequence of the advancement of science and technology. Many physical and engineering models require more sophisticated mathematical functional spaces to be studied and well understood. For example, in fluid dynamics, the electrorheological fluids (smart fluids) have the property that the viscosity changes when exposed to an electrical field. More examples are found in studying models of the image processing and filtration processes through a porous media. The Lebesgue and Sobolev spaces with variable exponents proved to be efficient tools to study such problems. More details on applications of these problems can be found in ([1–3]).

A lot of papers in the literature dealt with stabilization of wave equations with different types of nonlinearities such as linear, polynomial and logarithmic. For instance, the following problem was studied by Nakao [4].

$$u_{tt} - \Delta u + |u_t|^{m-2} u_t + |u|^{p-2} u = 0, \text{ in } \Omega \times (0, \infty),$$

where $m, p > 2$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain. He showed that, with Dirichlet-boundary conditions, the problem has a unique global weak solution if $2 \leq p \leq 2(n-1)/(n-2)$, $n \geq 3$ and a global unique strong solution if $p > 2(n-1)/(n-2)$, $n \geq 3$. In both cases, he proved that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. Benaissa and Messaoudi [5] considered

$$u_{tt} - \Delta u + a\left(1 + |u_t|^{m-2}\right) u_t = |u|^{p-2} u, \text{ in } \Omega \times (0, \infty),$$

where $m, p > 2$ and showed, for small initial data in an appropriate function space, that the problem has a global weak solution which decays exponentially even if $m > 2$. We also mention here the work of Mustafa and Messaoudi [6], where they considered

$$u_{tt} - \Delta u + \alpha(t)\, g(u_t) = 0, \text{ in } \Omega \times (0, \infty),$$

and established an explicit and general decay rate result, without imposing any restrictive growth assumption on the frictional damping term.

As we mentioned earlier, modern technology and engineering required the use of variable exponents nonlinearities and the Lebesgue and Sobolev spaces with variable exponents as well. In this regard, we mention the work of Ghegal et al. [7] where, in a bounded domain, the following equation is considered

$$u_{tt} - \Delta u + |u_t|^{m(\cdot)-2} u_t = |u|^{p(\cdot)-2} u, \text{ in } \Omega \times (0, \infty).$$

Under suitable conditions on the initial data and the variable exponents, the authors used stable-set method to prove a global existence result. Then, by applying an integral inequality due to Komornik, they obtained the stability result. More results can be found in ([8–10]).

Hyperbolic problems involving variable-exponent nonlinearities with delay are also considered. For instance, Kafini and Messaoudi [11] studied the problem

$$u_{tt} - \Delta u + \mu_1 |u_t|^{m(x)-2} u_t + \mu_2 |u_t|^{m(x)-2} (t-\tau) u_t (t-\tau) = bu\, |u|^{p(x)-2} .$$

For $b > 0$, they established a global nonexistence result under suitable conditions on $\mu_1, \mu_2, m(\cdot), p(\cdot)$ and the initial data. While, for $b = 0$, they obtained a decay result which is of either polynomial or exponential type depending on the nature of $m(\cdot)$.

Recently, Messaoudi in [12] considered the problem

$$u_{tt} - div\Big(|\nabla u|^{r(\cdot)-2} \nabla u\Big) - \Delta u_t + |u_t|^{m(\cdot)-2} u_t = 0, \quad \Omega \times (0, T),$$

and established several decay results depending on the nature of variable exponents $r(\cdot)$ and $m(\cdot)$. See [13–17], for more results on the local existence and blow up for some problems with variable exponent nonlinearities.

Fractional derivatives have been also influenced by variable orders. One can see variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications as in [18]. Analyzing a variable-order time-fractional wave equation, which models, e.g., the vibration of a membrane in a viscoelastic environment examined in [19]. See also [20–22] for more details.

In our work, we aim to study the nonlinear wave Eq (1.1) with nonlinear feedback having a variable exponent $m(x)$ and a time-dependent coefficient $\alpha(t)$. We establish a decay result of an exponential and polynomial type under specific conditions on both $m(\cdot)$ and $\alpha(t)$ and the initial data. This paper consists of three sections in addition to the introduction. In Section 2, we recall the basic definitions of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as well as some of their properties. Section 3 is devoted to the existence and uniqueness of a weak global solution. In the last section, we show the decay result.

2. Preliminaries

In this section, we present some materials needed for the statement and the proof of our results. In what follows, we give definitions and properties related to Lebesgue and Sobolev spaces with variable exponents, see [23, 24] for more details.

Let Ω be a domain of $\mathbb{R}^n$ with $n \geq 2$ and $p : \Omega \longrightarrow [1, \infty]$ be a measurable function. The Lebesgue space $L^{p(\cdot)}(\Omega)$ with a variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{ v : \Omega \longrightarrow \mathbb{R};\ \text{measurable such that}\ \varrho_{p(\cdot)}(\lambda v) < +\infty,\ \text{for some}\ \lambda > 0 \Big\},$$

where

$$\varrho_{p(\cdot)}(v) := \int_\Omega |v(x)|^{p(x)}\, dx.$$

The Luxembourg-type norm is given by

$$\|v\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The space $L^{p(\cdot)}(\Omega)$, equipped with the above norm, is a Banach space.

Lemma 2.1. ***(Hölder's inequality)*** Let $p, q, s \ge 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \le 2\, \|f\|_{p(\cdot)}\, \|g\|_{q(\cdot)}\,.$$

Lemma 2.2. If $p : \Omega \longrightarrow [1, \infty)$ is a measurable function and $1 \le p_1 \le p(x) \le p_2 < \infty$, then

$$\min\left\{\|v\|^{p_1}_{p(\cdot)}, \|v\|^{p_2}_{p(\cdot)}\right\} \le \varrho_{p(\cdot)}(v) \le \max\left\{\|v\|^{p_1}_{p(\cdot)}, \|v\|^{p_2}_{p(\cdot)}\right\},$$

for a.e. $x \in \Omega$ and for any $v \in L^{p(\cdot)}(\Omega)$.

Lemma 2.3. *[12]* If $p : \Omega \longrightarrow [1, \infty)$ is a measurable function and $1 \le p_1 \le p(x) \le p_2 < \infty$, then

$$\int_\Omega |v(x)|^{p(x)}\, dx \le \|v\|^{p_1}_{p_1} + \|v\|^{p_2}_{p_2}, \quad \forall v \in L^{p(\cdot)}(\Omega).$$

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{v \in L^{p(\cdot)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{p(\cdot)}(\Omega)\right\}.$$

This space is a Banach space with respect to the norm

$$\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)}\,.$$

Suppose $p(\cdot)$ satisfies (1.3). Then the space $W^{1,p(\cdot)}_0(\Omega)$ is defined to be the closure of $C^\infty_0(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The definition of the space $W^{1,p(\cdot)}_0(\Omega)$ is usually different from the constant exponent case. However, under condition (1.3) both definitions coincide. The dual space of $W^{1,p(\cdot)}_0(\Omega)$ is $W^{-1,p'(\cdot)}_0(\Omega)$ defined in the same way as in the classical Sobolev spaces, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Lemma 2.4. ***(Poincaré's inequality)*** Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies (1.2) and (1.3), then

$$\|v\|_{p(\cdot)} \le C\, \|\nabla v\|_{p(\cdot)}\,, \textit{ for all } v \in W^{1,p(\cdot)}_0(\Omega),$$

where C is a positive constant depends on $p(\cdot)$ and Ω. In particular, the space $W^{1,p(\cdot)}_0(\Omega)$ has an equivalent norm given by

$$\|v\|_{W^{1,p(\cdot)}_0(\Omega)} = \|\nabla v\|_{p(\cdot)}\,.$$

Lemma 2.5. If $p : \overline{\Omega} \longrightarrow [1, \infty)$ is continuous and

$$2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \quad n \geq 3,$$

then the embedding $H^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.6. *[27]* Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a nonincreasing function and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an increasing C^1 -function satisfying

$$\phi(0) = 0 \ \ and \ \ \phi(t) \to +\infty \quad as \quad t \to +\infty.$$

Assume further, that there exist $q \geq 0$, $A > 0$ such that

$$\int_S^\infty E^{q+1}(t)\phi'(t)dt \leq AE(S), \quad \forall S > 0.$$

Then, $\forall t \ \geq \ 0$,

$$\begin{array}{ll} E(t) \leq CE(0)(1 + \phi(t))^{-1/q}, & if \quad q > 0, \\ E(t) \leq CE(0)e^{-\omega\phi(t)}, & if \quad q = 0, \end{array}$$

where C and ω are positive constants independent of the initial energy $E(0)$.

Definition 2.7. Given the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, a function u defined on $\Omega \times (0, T)$ is called a weak solution of problem (1.1) if

$$\begin{array}{rcl} u & \in & L^\infty\left((0,T)\,; H_0^1(\Omega)\right), \\ u_t & \in & L^\infty\left((0,T)\,; L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)) \end{array}$$

and it verifies the variational equation

$$\langle u_{tt}, w\rangle + (\nabla u, \nabla w) + \alpha(t)\left[(u_t, w) + \left(|u_t|^{m(x)-2}u_t, w\right)\right] = 0, \ \ \forall w \in C_0^\infty(\Omega).$$

We introduce the energy functional associated to problem (1.1) as

$$E(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2, \quad t \geq 0. \tag{2.1}$$

Lemma 2.8. Let u be the solution of (1.1) . Then,

$$E'(t) = -\alpha(t)\int_\Omega \left(|u_t|^2 + |u_t|^{m(x)}\right)dx \leq 0, \quad t \geq 0. \tag{2.2}$$

Proof. Multiplying Eq (1.1) by u_t and integrating over Ω, the result follows.

Remark 2.9. *In the sequel, we use C to denote a generic constant which may differ from one place to another.*

3. Existence and uniqueness

The following theorem states our existence and uniqueness results, which are the main focus of this section.

Theorem 3.1. Assume that the variable exponent $m(\cdot)$ satisfies conditions (1.2) and (1.3). Then, for any initial data $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, problem (1.1) admits a unique global weak solution.

Proof. To prove the existence of a weak solution to (1.1) , we make use of the Galerkin approximation method. For that reason we assume $\left\{v_j\right\}_{j\geq 1}$ is an orthogonal basis for $H_0^1(\Omega)$ and orthonormal in $L^2(\Omega)$. We find a solution of the form

$$u^k(x,t) = \sum_{j=1}^{k} a_{jk}(t)\, v_j(x), \quad a_{jk}(t) = \left\langle u^k(t), v_j \right\rangle,$$

to the approximate problem

$$\left(u_{tt}^k, v_j\right) + \left(\nabla u^k, \nabla v_j\right) + \alpha(t)\left[\left(u_t^k, v_j\right) + \left(\left|u_t^k\right|^{m(\cdot)-2} u_t^k, v_j\right)\right] = 0, \tag{3.1}$$

where

$$\begin{aligned} u^k(x,0) &= u_0^k(x) = \sum_{j=1}^{k}\left(u_0^k, v_j\right) v_j \to u_0 \text{ strongly in } H_0^1(\Omega), \\ u_t^k(x,0) &= u_1^k(x) = \sum_{j=1}^{k}\left(u_1^k, v_j\right) v_j \to u_1 \text{ strongly in } L^2(\Omega). \end{aligned} \tag{3.2}$$

This system, by the standard ODE theory has a unique solution guaranteed on $[0, t_k)$, $0 < t_k \leq T$. Next, we need to show that this solution can be extended to the maximal interval $[0, T)$, $\forall k \geq 1$ and for any $T > 0$.

Replace v_j by u_t^k in (3.1) to get

$$\frac{d}{dt}\left[\left\|u_t^k\right\|_2^2 + \left\|\nabla u^k\right\|_2^2\right] + 2\alpha(t)\left[\left\|u_t^k\right\|_2^2 + \int_\Omega \left|u_t^k\right|^{m(\cdot)} dx\right] = 0,$$

and integrate over $(0, t)$ for $t \in (0, t_k)$ to arrive at

$$\begin{aligned} &\left\|u_t^k\right\|_2^2 + \left\|\nabla u^k\right\|_2^2 + 2\int_0^t \alpha(s)\left[\left\|u_t^k(s)\right\|_2^2 + \int_\Omega \left|u_t^k\right|^{m(\cdot)}(s)\, dx\right] ds \\ = &\left\|u_1^k\right\|_2^2 + \left\|\nabla u_0^k\right\|_2^2 \leq C, \quad \forall k \geq 1. \end{aligned} \tag{3.3}$$

Hence, the solution can be extended to $[0, T)$, for any given $T > 0$.

Using (1.4), we arrive at

$$\left\|u_t^k\right\|_2^2 + \left\|\nabla u^k\right\|_2^2 + 2\alpha_0 \int_0^t \left[\left\|u_t^k(s)\right\|_2^2 + \int_\Omega \left|u_t^k\right|^{m(\cdot)}(s)\, dx\right] ds \leq C,$$

where we can conclude that

$$u^k \text{ is bounded in } L^\infty\left((0,T);H_0^1(\Omega)\right)$$
$$u_t^k \text{ is bounded in } L^\infty\left((0,T);L^2(\Omega)\right)$$
$$u_t^k \text{ is bounded in } L^{m(\cdot)}\left(\Omega\times(0,T)\right).$$

Therefore, we can extract subsequences, still denoted by u^k and u_t^k, such that

$$u^k \to u \text{ weakly star in } L^\infty\left((0,T);H_0^1(\Omega)\right)$$

$$u_t^k \to u_t \text{ weakly star in } L^\infty\left((0,T);L^2(\Omega)\right).$$

As u_t^k is bounded in $L^{m(\cdot)}\left(\Omega\times(0,T)\right)$, then $\left|u_t^k\right|^{m(\cdot)-2}u_t^k$ is bounded in $L^{\frac{m(\cdot)}{m(\cdot)-1}}\left(\Omega\times(0,T)\right)$. Hence,

$$\left|u_t^k\right|^{m(\cdot)-2}u_t^k \to \psi \quad \text{weakly in} \quad L^{\frac{m(\cdot)}{m(\cdot)-1}}\left(\Omega\times(0,T)\right).$$

To show that $\psi = |u_t|^{m(\cdot)-2}u_t$, we integrate (3.1) over $(0,t)$ to get, $\forall j = 1,...,k$,

$$\int_\Omega u_t^k v_j dx - \int_\Omega u_1^k v_j dx + \int_0^t \int_\Omega \nabla u^k\cdot\nabla v_j dx + \int_0^t \alpha(s)\int_\Omega\left(u_t^k(s) + \left|u_t^k\right|^{m(\cdot)-2}u_t^k(s)\right)v_j dxds = 0.$$

Now, letting $k\to+\infty$ and differentiating the latter result with respect to t gives

$$\frac{d}{dt}\int_\Omega u_t v dx + \int_\Omega \nabla u\cdot\nabla v dx + \alpha(t)\int_\Omega (u_t+\psi)v dx = 0, \quad \forall v\in H_0^1(\Omega). \tag{3.4}$$

Hence,

$$u_{tt} - \Delta u + \alpha(t)(u_t+\psi) = 0, \quad \text{in } \mathcal{D}'\left(\Omega\times(0,T)\right).$$

If we define

$$\chi^k = 2\int_0^T \alpha(t)\int_\Omega\left(\left|u_t^k\right|^{m(\cdot)-2}u_t^k - |v|^{m(\cdot)-2}v\right)\left(u_t^k - v\right)dxdt,\ \forall v\in L^{m(\cdot)}\left((0,T);H_0^1(\Omega)\right),$$

and

$$A(v) = |v|^{m(\cdot)-2}v,$$

then we have

$$\chi^k = 2\int_0^T \alpha(t)\int_\Omega\left(A\left(u_t^k\right) - A(v)\right)\left(u_t^k - v\right)dxdt \ge 0,\ \forall v\in L^{m(\cdot)}\left((0,T);H_0^1(\Omega)\right).$$

Using Eq (3.3), we get

$$\begin{aligned}\chi^k &= \left\|u_1^k\right\|_2^2 + \left\|\nabla u_0^k\right\|_2^2 - \int_\Omega\left(\left|u_t^k(T)\right|^2 + \left|\nabla u^k(T)\right|^2\right)dx - 2\int_0^T\alpha(t)\int_\Omega\left|u_t^k\right|^2 dxdt\\ &\quad -2\int_0^T\alpha(t)\int_\Omega A\left(u_t^k\right)v dxdt - 2\int_0^T\alpha(t)\int_\Omega A(v)\left(u_t^k - v\right)dxdt.\end{aligned}$$

As $k \to +\infty$,

$$\begin{aligned} 0 &\leq \limsup_k \chi^k \leq \|u_1\|_2^2 + \|\nabla u_0\|_2^2 - \int_\Omega \left(|u_t(T)|^2 + |\nabla u(T)|^2 \right) dx \\ &\quad -2 \int_0^T \alpha(t) \int_\Omega |u_t|^2 \, dxdt - 2 \int_0^T \alpha(t) \int_\Omega \psi v dxdt \\ &\quad -2 \int_0^T \alpha(t) \int_\Omega A(v)(u_t - v) \, dxdt. \end{aligned} \tag{3.5}$$

Integration of (3.4) over $(0, T)$ after replacing v by u_t give

$$\int_\Omega |u_t(T)|^2 \, dx + \int_\Omega |\nabla u(T)|^2 \, dx - \|u_1\|_2^2 - \|\nabla u_0\|_2^2 + 2 \int_0^T \alpha(t) \int_\Omega \left(|u_t|^2 + \psi u_t \right) dxdt = 0. \tag{3.6}$$

Adding (3.5) and (3.6) give

$$\begin{aligned} 0 &\leq \limsup_k \chi^k \leq 2 \int_0^T \alpha(t) \int_\Omega \psi u_t dxdt - 2 \int_0^T \alpha(t) \int_\Omega \psi v dxdt \\ &\quad -2 \int_0^T \alpha(t) \int_\Omega A(v)(u_t - v) \, dxdt \\ &= 2 \int_0^T \alpha(t) \int_\Omega (\psi - A(v))(u_t - v) \, dxdt, \quad \forall v \in L^{m(\cdot)}\left((0, T); H_0^1(\Omega)\right). \end{aligned}$$

Thus, by the density of $H_0^1(\Omega)$ in $L^{m(\cdot)}(\Omega)$ we have

$$\int_0^T \int_\Omega (\psi - A(v))(u_t - v) \, dxdt \geq 0, \quad \forall v \in L^{m(\cdot)}(\Omega \times (0, T)).$$

If we let $v = \lambda w + u_t$ for $w \in L^{m(\cdot)}(\Omega \times (0, T))$ then

$$-\int_0^T \int_\Omega (\psi - A(\lambda w + u_t)) \, w dxdt \geq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0, T)).$$

As $0 < \lambda \to 0$, we have,

$$\int_0^T \int_\Omega (\psi - A(u_t)) \, w dxdt \leq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0, T)).$$

Similarly, if $0 > \lambda \to 0$, we have,

$$\int_0^T \int_\Omega (\psi - A(u_t)) \, w dxdt \geq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0, T)).$$

This implies that $\psi = A(u_t) = |u_t|^{m(\cdot)-2} u_t$.
To handle the initial conditions, we use Lions' Lemma [25], to obtain, up to a subsequence, that

$$u^k \to u \text{ in } C\left([0, T]; L^2(\Omega)\right).$$

Therefore, $u^k(\cdot,0)$ makes sense and $u^k(\cdot,0) \to u(\cdot,0)$ in $L^2(\Omega)$. Also, by density we have

$$u^k(\cdot,0) = u_0^k \to u_0 \ \text{ in } \ H_0^1(\Omega),$$

hence $u(\cdot,0) = u_0$.

For the other condition, as in [26], we obtain from (3.1) and for any $j \le k$ and $\phi \in C_0^\infty(0,T)$,

$$\begin{aligned} & -\int_0^T \int_\Omega u_t^k v_j(x)\, \phi'(t)\, dxdt \\ = & -\int_0^T \int_\Omega \nabla u^k \nabla v_j(x)\, \phi(t)\, dxdt + \int_0^T \alpha(t) \int_\Omega \left(u_t^k + \left| u_t^k \right|^{m(\cdot)-2} u_t^k \right) v_j(x)\, \phi(t)\, dxdt. \end{aligned}$$

As $k \to +\infty$, we obtain that, for all $v \in H_0^1(\Omega)$,

$$-\int_0^T \int_\Omega u_t v(x)\, \phi'(t)\, dxdt = \int_0^T \left\langle \Delta u - \alpha(t) \left(u_t + |u_t|^{m(\cdot)-2} u_t \right), v(x) \right\rangle \phi(t)\, dt.$$

This implies that

$$u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}} \left([0,T); H^{-1}(\Omega) \right),$$

and u solves the equation

$$u_{tt} - \Delta u + \alpha(t) \left(u_t + |u_t|^{m(\cdot)-2} u_t \right) = 0.$$

Therefore,

$$u_t \in C\left([0,T); H^{-1}(\Omega) \right),$$

where $u_t^k(\cdot,0)$ makes sense and $u_t^k(\cdot,0) \to u_t(\cdot,0)$ in $H^{-1}(\Omega)$. But we have

$$u_t^k(\cdot,0) = u_1^k \to u_1 \ \text{ in } \ L^2(\Omega).$$

So $u_t(\cdot,0) = u_1$.

To prove the uniqueness, we assume u and v are two solutions of (3.1). Then $w = u - v$ satisfies the following problem

$$\begin{cases} w_{tt} - \Delta w + \alpha(t) \left(w_t + |u_t|^{m(\cdot)-2} u_t - |v_t|^{m(\cdot)-2} v_t \right) = 0 & \text{in} \quad \Omega \times (0,T), \\ w = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & \text{in} \quad \Omega. \end{cases}$$

Multiply the equation by w_t and integrate over Ω, to obtain

$$\frac{1}{2} \frac{d}{dt} \left[\int_\Omega \left(|w_t|^2 + |\nabla w|^2 \right) dx \right] + \alpha(t) \int_\Omega \left[|w_t|^2 + \left(|u_t|^{m(\cdot)-2} u_t - |v_t|^{m(\cdot)-2} v_t \right) (u_t - v_t) \right] dx = 0.$$

Integration over $(0,t)$, to get

$$\int_\Omega \left(|w_t|^2 + |\nabla w|^2 \right) dx + 2 \int_0^t \alpha(t) \int_\Omega \left[|w_t|^2 + \left(|u_t|^{m(\cdot)-2} u_t - |v_t|^{m(\cdot)-2} v_t \right) (u_t - v_t) \right] dxdt = 0.$$

Using the fact that

$$\left(|a|^{m(\cdot)-2} a - |b|^{m(\cdot)-2} b \right) (a-b) \ge 0, \ \forall a, b \in \mathbb{R} \ \text{ and a.e } x \in \Omega,$$

we obtain

$$\int_\Omega \left(|w_t|^2 + |\nabla w|^2 \right) dx = 0.$$

This implies that $w = C = 0$, since $w = 0$ on $\partial\Omega$. Hence, the uniqueness. This completes the proof of Theorem 3.1.

4. Decay of solutions

Theorem 4.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that $\int_0^\infty \alpha(\tau)\, d\tau = \infty$ and $m(\cdot) \in C\left(\overline{\Omega}\right)$ that satisfies

$$2 \leq m_1 \leq m(x) \leq m_2 < 2^*.$$

Then, the solution energy (2.1) satisfies, for two positive constants k_1, k_2,

$$E(t) \leq k_1 e^{-k_2 \int_0^t \alpha(s)ds}, \quad \forall t \geq 0. \tag{4.1}$$

Proof. Multiply (1.1) by $\alpha u E^q(t)$ and integrate over $\Omega \times (s, T)$, $0 < s < T$, to obtain

$$\int_s^T \alpha E^q(t) \int_\Omega \left(u u_{tt} - u\Delta u + \alpha \left(u u_t + u u_t |u_t|^{m(x)-2}\right)\right) dx dt = 0,$$

which gives

$$\int_s^T \alpha E^q(t) \int_\Omega \left(\frac{d}{dt}(u u_t) - u_t^2 + |\nabla u|^2 + \alpha \left(u u_t + u u_t |u_t|^{m(x)-2}\right)\right) dx dt = 0, \tag{4.2}$$

for $q \geq 0$ to be specified later.

Recalling the fact that $\int_\Omega \left(|\nabla u|^2 + u_t^2\right) dx = 2E(t)$ and using the relation

$$\begin{aligned} & \frac{d}{dt}\left(\alpha E^q(t) \int_\Omega u u_t dx\right) \\ = \; & \alpha' E^q(t) \int_\Omega u u_t dx + q\alpha E^{q-1}(t) E'(t) \int_\Omega u u_t dx + \alpha E^q(t) \frac{d}{dt} \int_\Omega u u_t dx, \end{aligned}$$

equation (4.2) becomes

$$\begin{aligned} & 2\int_s^T \alpha E^{q+1}(t) dt \qquad\qquad (4.3)\\ = \; & -\int_s^T \frac{d}{dt}\left(\alpha E^q(t) \int_\Omega u u_t dx\right) - \int_s^T \alpha^2 E^q(t) \int_\Omega u u_t dx dt \\ & + q\int_s^T \alpha E^{q-1}(t) E'(t) \int_\Omega u u_t dx dt + \int_s^T \alpha' E^q(t) \int_\Omega u u_t dx dt \\ & + 2\int_s^T \alpha E^q(t) \int_\Omega u_t^2 dx dt - \int_s^T \alpha^2 E^q(t) \int_\Omega u u_t(x,t) |u_t|^{m(x)-2} dx dt. \end{aligned}$$

The first term in the right side of (4.3) is estimated, using Poincaré's inequality, (2.2) and the fact that

$$\int_\Omega u u_t dx \leq \frac{1}{2} \int_\Omega \left(|u|^2 + u_t^2\right) dx \leq C \int_\Omega \left(|\nabla u|^2 + u_t^2\right) dx \leq CE(t),$$

to have

$$\left| -\int_s^T \frac{d}{dt}\left(\alpha E^q(t) \int_\Omega u u_t dx\right) dt \right| \tag{4.4}$$

$$\leq C\left[\alpha(s)E^{q+1}(s)+\alpha(T)E^{q+1}(T)\right]\leq C\alpha(0)E^{q}(0)E(s)\leq CE(s).$$

Using Young's inequality, the second term leads to

$$\begin{aligned}\left|-\int_s^T\alpha^2E^q(t)\int_\Omega uu_t dxdt\right| &\leq \int_s^T E^q(t)\left[\delta C\alpha(t)\int_\Omega|\nabla u|^2dx+\frac{C}{4\delta}\alpha(t)\int_\Omega u_t^2dx\right]dt\\ &\leq \delta C\int_s^T\alpha E^{q+1}(t)dt-\frac{C}{4\delta}\int_s^T E^q(t)E'(t)dt,\quad\forall\delta>0.\end{aligned}$$

Taking $\delta=1/2C$, we get

$$\left|-\int_s^T\alpha^2E^q(t)\int_\Omega uu_t dxdt\right|\leq\frac{1}{2}\int_s^T\alpha E^{q+1}(t)dt+CE(s). \tag{4.5}$$

Similar to the first term, we have

$$\begin{aligned}&\left|q\int_s^T\alpha E^{q-1}(t)E'(t)\int_\Omega uu_t dxdt\right| \qquad (4.6)\\ \leq\ &-C\int_s^T E^q(t)E'(t)dt\leq CE^{q+1}(s)\leq CE(s).\end{aligned}$$

The fourth term:

$$\begin{aligned}\left|\int_s^T\alpha'(t)E^q(t)\int_\Omega uu_t dxdt\right| &\leq C\int_s^T|\alpha'(t)|E^{q+1}(t)dt\leq CE^{q+1}(s)\int_s^T|\alpha'(t)|dt\\ &\leq CE^{q+1}(s)\alpha(s)\leq CE(s). \qquad (4.7)\end{aligned}$$

The fifth term:

$$2\int_s^T\alpha E^q(t)\int_\Omega u_t^2dxdt\leq-2\int_s^T E^q(t)E'(t)dt\leq CE^{q+1}(s)\leq CE(s). \tag{4.8}$$

The last term in the right-hand side of (4.3) is handled by using Young's inequality with

$$a(x)=\frac{m(x)}{m(x)-1}\text{ and }a'(x)=m(x).$$

So, for a.e. $x\in\Omega$, $\varepsilon>0$, and

$$c_\varepsilon(x)=\varepsilon^{1-m(x)}(m(x))^{-m(x)}(m(x)-1)^{m(x)-1},$$

we have

$$\begin{aligned}&\left|-\int_s^T\alpha^2E^q(t)\int_\Omega uu_t|u_t|^{m(x)-2}dxdt\right|\\ \leq\ &C\int_s^T\alpha E^q(t)\left[\varepsilon\int_\Omega|u(t)|^{m(x)}dx+\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right]dt\end{aligned}$$

$$
\begin{aligned}
&\le C\int_s^T \alpha E^q(t)\left[\varepsilon\left(\int_\Omega |u(t)|^{m_1}\,dx+\int_\Omega |u(t)|^{m_2}\,dx\right)+\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dx\right]dt\\
&\le C\int_s^T \alpha E^q(t)\left[\varepsilon\left(\|\nabla u(t)\|_2^{m_1}+\|\nabla u(t)\|_2^{m_2}\right)+\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dx\right]dt \qquad (4.9)\\
&\le C\int_s^T \alpha E^q(t)\left[\varepsilon\left(E^{\frac{m_1}{2}}(t)+E^{\frac{m_2}{2}}(t)\right)+\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dx\right]dt\\
&\le \varepsilon C E^{\frac{m_1}{2}-1}(0)\int_s^T \alpha E^{q+1}(t)dt + C\int_s^T \alpha E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt.
\end{aligned}
$$

If we fix $\varepsilon = 1/2CE^{\frac{m_1}{2}-1}(0)$, noting that $c_\varepsilon(x)$ is bounded since $m(x)$ is bounded, then (4.9) becomes

$$
\begin{aligned}
\left|-\int_s^T \alpha^2 E^q(t)\int_\Omega uu_t\,|u_t|^{m(x)-2}\,dxdt\right| &\le \frac{1}{2}\int_s^T \alpha E^{q+1}(t)dt - c\int_s^T E^q(t)E'(t)dt\\
&\le \frac{1}{2}\int_s^T \alpha E^{q+1}(t)dt + CE(s). \qquad (4.10)
\end{aligned}
$$

Combining (4.3)–(4.10) and taking $T\to\infty$ we arrive at

$$
\int_s^\infty \alpha E^{q+1}(t)dt \le CE(s).
$$

Therefore, (4.1) is established by the virtue of Lemma 2.6 for $q=0$ and $\phi(t)=\int_0^t \alpha(s)\,ds$.

Example 1. *If we take $\alpha(t)=1$ and $m(x)=2$ then we have $\phi(t)=t$ and hence, for two positive constants k_1,k_2,*

$$
E(t)\le k_1e^{-k_2t},\ \forall t\ge 0.
$$

The next theorem handles the case: $1<m_1<2$.

Theorem 4.2. *(Polynomial Decay)* Let $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ be given. Assume that $\int_0^\infty \alpha(\tau)\,d\tau=\infty$ and $m(\cdot)\in C\left(\overline{\Omega}\right)$ and satisfies (1.2). Assume further that $m_1<2$. Then, the solution energy (1.1) satisfies, for some positive constant K,

$$
E(t)\le K\left(1+\int_0^t \alpha(\tau)\,d\tau\right)^{\frac{1-m_1}{2-m_1}},\ \forall t\ge 0. \qquad (4.11)
$$

Proof. We follow the same steps in the proof of the previous theorem. But we have to re-estimate the last term in (4.3). For this purpose, we define

$$
\Omega_1=\{x\in\Omega\mid m(x)<2\}\ \text{ and }\ \Omega_2=\{x\in\Omega\mid m(x)\ge 2\}.
$$

Thus,

$$
\int_\Omega uu_t\,|u_t|^{m(x)-2}\,dx=\int_{\Omega_1} uu_t\,|u_t|^{m(x)-2}\,dx+\int_{\Omega_2} uu_t\,|u_t|^{m(x)-2}\,dx.
$$

Then we use Young's inequality and Poincaré's inequality, to get

$$
\left|-\int_{\Omega_1} uu_t\,|u_t|^{m(x)-2}\,dx\right|\le \delta C\int_\Omega |\nabla u|^2\,dx+\frac{1}{4\delta}\int_{\Omega_1}|u_t|^{2m(x)-2}\,dx. \qquad (4.12)
$$

In order to estimate the last term of (4.12), we define

$$m_3 := \sup_{x\in\Omega_1} m(x) \le 2.$$

Then Hölder's inequality and the embedding give

$$\begin{aligned}
&\left|-\alpha \int_{\Omega_1} uu_t \left|u_t\right|^{m(x)-2} dx\right| \\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{\alpha}{4\delta}\left[\int_{\Omega_1} |u_t|^{2m_1-2}\, dx + \int_{\Omega_1} |u_t|^{2m_3-2}\, dx\right] \\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{C\alpha}{4\delta}\left[\left(\int_{\Omega_1} |u_t|^{2}\, dx\right)^{m_1-1} + \left(\int_{\Omega_1} |u_t|^{2}\, dx\right)^{m_3-1}\right] \\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{C\alpha}{4\delta}\left[\left(\int_{\Omega} |u_t|^{2}\, dx\right)^{m_1-1} + \left(\int_{\Omega} |u_t|^{2}\, dx\right)^{m_3-1}\right] \qquad (4.13)\\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{C\alpha}{4\delta}\left[1 + \left(\int_{\Omega} |u_t|^{2}\, dx\right)^{m_3-m_1}\right]\left(\int_{\Omega} |u_t|^{2}\, dx\right)^{m_1-1} \\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{C}{4\delta}\left[1 + (2E(0))^{m_3-m_1}\right](-E'(t))^{m_1-1} \\
\le\ & \delta C \int_{\Omega} |\nabla u|^2 \, dx + \frac{C}{4\delta}(-E'(t))^{m_1-1}.
\end{aligned}$$

Thus,

$$\left|-\int_s^T \alpha^2 E^q(t) \int_{\Omega_1} uu_t \left|u_t\right|^{m(x)-2} dxdt\right| \le \delta C \int_s^T \alpha E^{q+1}(t)dt + c_\delta \int_s^T \alpha E^q(t)\left(-E'(t)\right)^{m_1-1} dt. \qquad (4.14)$$

Using Young's inequality, we obtain for any $\lambda > 0$,

$$E^q(t)\left(-E'(t)\right)^{m_1-1} \le \lambda \left(E(t)\right)^{\frac{q}{2-m_1}} + c_\lambda \left(-E'(t)\right).$$

If we let $q+1 = \frac{q}{2-m_1}$ hence $q = \frac{2-m_1}{m_1-1}$, then (4.14) implies that

$$\begin{aligned}
\left|-\int_s^T \alpha^2 E^q(t) \int_{\Omega_1} uu_t \left|u_t\right|^{m(x)-2} dxdt\right| \le\ & \delta C \int_s^T \alpha E^{q+1}(t)dt + \lambda c_\delta \int_s^T \alpha E^{q+1}(t)dt \\
& + c_\delta c_\lambda \int_s^T \alpha\left(-E'(t)\right) dt.
\end{aligned}$$

Then we choose $\delta = 1/4C$. After δ is fixed, we choose $\lambda = 1/4c_\delta$ to obtain

$$\left|-\int_s^T \alpha^2 E^q(t) \int_{\Omega_1} uu_t \left|u_t\right|^{m(x)-2} dxdt\right| \le \frac{1}{2}\int_s^T \alpha E^{q+1}(t)dt + CE(s). \qquad (4.15)$$

Now over Ω_2, we follow the same steps as in (4.9) to conclude that

$$\left|-\int_s^T \alpha^2 E^q(t) \int_{\Omega_2} uu_t \left|u_t\right|^{m(x)-2} dxdt\right| \le \frac{1}{2}\int_s^T \alpha E^{q+1}(t)dt + CE(s). \qquad (4.16)$$

Combining (4.15) and (4.16), give

$$\left| -\int_s^T \alpha^2 E^q(t) \int_\Omega u u_t |u_t|^{m(x)-2} dxdt \right| \leq \int_s^T \alpha E^{q+1}(t)dt + CE(s). \tag{4.17}$$

Consequently, from (4.3)–(4.8) and (4.17) , we have

$$\int_s^T \alpha E^{q+1}(t)dt \leq CE(s).$$

If we let $T \to \infty$, then from Lemma 2.6 with $\phi(t) = \int_0^t \alpha(\tau) d\tau$ and $q = \frac{2-m_1}{m_1-1} > 0$, we arrive for some $K > 0$,

$$E(t) \leq K\left(1 + \int_0^t \alpha(\tau) d\tau\right)^{\frac{1-m_1}{2-m_1}}.$$

This completes the proof.

Example 2. *If we take* $\Omega = (0, 1)$, $\alpha(t) = \frac{2+t}{1+t}$ *and* $m(x) = 2 - \frac{1}{2+x}$, *then we have* $\phi(t) = t + \ln(1 + t)$, $m_1 = 3/2$ *and*

$$E(t) \leq K(1 + t + \ln(1 + t))^{-1}, \ \forall t \geq 0,$$

for a positive constant K.

5. Conclusions

In this paper, we have shown that the time varying coefficient appears in the problem has a direct effect in well posednesss and the decay rates. In fact, we investigated the nonlinear wave Eq (1.1) with nonlinear feedback having a variable exponent $m(x)$ and a time-dependent coefficient $\alpha(t)$. We established a decay result of an exponential and polynomial type under specific conditions on both $m(\cdot)$ and $\alpha(t)$ and the initial data.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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