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Research article

Torse-forming vector fields on m-spheres

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Abstract: A characterization of an m-sphere $\mathbf{S}^m(a)$ is obtained using a non-trivial torse-forming vector field ζ on an m-dimensional Riemannian manifold.

Keywords: torse-forming vector fields; Ricci operator; Ricci curvature
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1. Introduction

An important space in differential geometry is the m-sphere $\mathbf{S}^m(a)$. It is known for its elegant geometry and topology. Characterizing m-spheres among complete connected m-dimensional Riemannian manifolds is a challenging question in differential geometry. This question has been addressed through several ways. One way is that the manifold admits a nontrivial solution of certain differential equations (cf. [18, 19, 22–24]). Other way is that the manifold admits certain special vector fields with certain additional conditions (cf. [1, 7–10, 12–17, 25–27]). An important vector field among special vector fields is torse-forming vector field introduced by Yano (cf. [28]). These vector fields have immense importance in physics and geometry (cf. [2–6, 11, 16, 20, 21, 24]). However, it is not known whether a torse-forming vector field can be used to characterize an m-sphere $\mathbf{S}^m(a)$. In this short note, we find a characterization of an m-sphere $\mathbf{S}^m(a)$ using a torse-forming vector field on a compact and connected m-dimensional Riemannian manifold (N, g). It should be noted that there are some specific torse-forming vector fields which do not exist on an m-sphere $\mathbf{S}^m(a)$ (cf. [11, 16]). A torse-forming vector field ζ on a Riemannian manifold (N, g) satisfies

$$D_X\zeta = hX + \gamma(X)\zeta, \quad X \in \mathfrak{X}(N), \tag{1.1}$$

where D is the Riemannian connection on (N, g), h a smooth function, γ a 1-form and $\mathfrak{X}(M)$ Lie algebra of smooth vector fields on N. We call h the torsed function and γ the torsed form of the torse-forming vector field ζ. We shall abbreviate the torse-forming vector field ζ by TFVF ζ. Note that if the torsed form $\gamma = 0$, then Eq (1.1) implies that TFVF ζ is a concircular vector field. We say a TFVF ζ on a Riemannian manifold (N, g) a non-trivial TFVF if $\zeta \neq 0$ and the torsed form $\gamma \neq 0$.

Now, we proceed to show that an m-sphere $\mathbf{S}^m(a)$ admits a non-trivial TFVF ζ. Note that the Wiengarten map L of $\mathbf{S}^m(a)$ in the Euclidean space $\mathbf{E}^{m+1}$ is given by $L = -\sqrt{a}I$ and denote the unit normal of $\mathbf{S}^m(a)$ by ξ. Then for Euclidean coordinates $u^1, ..., u^{m+1}$, we define a function $\sigma = \left\langle \frac{\partial}{\partial u^1}, \xi \right\rangle$ on $\mathbf{S}^m(a)$, where $\langle , \rangle$ is the Euclidean metric and we have

$$\frac{\partial}{\partial u^1} = \mathbf{v} + \sigma \xi, \tag{1.2}$$

where $\mathbf{v} \in \mathfrak{X}(\mathbf{S}^m(a))$ is the projection of $\frac{\partial}{\partial u^1}$ to $\mathbf{S}^m(a)$. Taking covariant derivative in (1.2) with respect to Euclidean connection in the direction of $X \in \mathfrak{X}(\mathbf{S}^m(a))$ while using Gauss and Wiengarten formulas on equating tangential and normal components, we conclude

$$D_X \mathbf{v} = -\sqrt{a}\sigma X, \quad X(\sigma) = \sqrt{a} g(\mathbf{v}, X), \tag{1.3}$$

where g is the induced metric and D is the Riemannian connection on $\mathbf{S}^m(a)$. Now, define a vector field $\zeta = e^{-\sigma}\mathbf{v}$ on $(\mathbf{S}^m(a), g)$, then using Eq (1.3) we have

$$D_X \zeta = -\sqrt{a}\sigma e^{-\sigma} X - X(\sigma)\zeta, \tag{1.4}$$

that is,

$$D_X \zeta = hX + \gamma(X)\zeta,$$

where $h = -\sqrt{a}\sigma e^{-\sigma}$ and $\gamma(X) = -\sqrt{a} g(\mathbf{v}, X)$. This proves that ζ is a TFVF on $\mathbf{S}^m(a)$. We claim that ζ is a non-trivial TFVF on $\mathbf{S}^m(a)$. To establish this claim, it is enough to show $\mathbf{v} \neq 0$. We assume on the contrary that $\mathbf{v} = 0$. Then the second equation in (1.3) gives $X(\sigma) = 0$, that is, the function σ is a constant. Moreover, the first equation in (1.3) implies $\sigma = 0$ (as $\mathbf{v} = 0$). Consequently, Eq (1.2) implies the constant unit vector field $\frac{\partial}{\partial u^1} = 0$, a contradiction. Hence, ζ is a non-trivial TFVF on $\mathbf{S}^m(a)$.

2. Preliminaries

Let ζ be a TFVF on a Riemannian manifold (N, g) with torsed function h and torsed form γ. Then using Eq (1.1), for $X, Y \in \mathfrak{X}(N)$, we have

$$D_X D_Y \zeta = X(h)Y + hD_X Y + X(\gamma(Y))\zeta + \gamma(Y)(hX + \gamma(X)\zeta),$$

and using definition of curvature tensor field of (N, g), we get

$$R(X, Y)\zeta = (X(h)Y - Y(h)X) + h(\gamma(Y)X - \gamma(X)Y) + d\gamma(X, Y)\zeta, \tag{2.1}$$

where $d\gamma$ is the differential of γ. Let $\mathbf{w}$ be the dual vector field to torsed form γ, $\gamma(X) = g(\mathbf{w}, X)$, $X \in \mathfrak{X}(M)$. Define a skew-symmetric operator φ and a symmetric operator B associated to torsed form γ by

$$d\gamma(X, Y) = 2g(\varphi X, Y), \quad (\pounds_{\mathbf{w}} g)(X, Y) = 2g(BX, Y), \quad X, Y \in \mathfrak{X}(N). \tag{2.2}$$

We denote by T the Ricci tensor and by S the Ricci operator of (N, g), that is,

$$T(X,Y) = \sum_{i=1}^{m} g\left(R(u_i, X)\,Y, u_i\right), \quad T(X,Y) = g(S\,(X)\,,Y),$$

where $\{u_1, ..., u_m\}$ is a local frame and $\dim N = m$. Then Eqs (2.1) and (2.2) imply

$$T(Y,\zeta) = -(m-1)Y(h) + (m-1)h\gamma\,(Y) + 2g\,(\varphi\zeta, Y)\,, \quad Y \in \mathfrak{X}(N), \tag{2.3}$$

and

$$S\,(\zeta) = -(m-1)\nabla h + (m-1)h\mathbf{w} + 2\varphi\zeta, \tag{2.4}$$

where ∇h is gradient of h.

Lemma 2.1. *Let ζ be a non-trivial TFVF on a connected Riemannian manifold (N, g) with torsed function h and torsed form γ. If ζ annihilates the skew-symmetric operator φ associated to torsed form γ, then there exists a function ρ on N such that $\nabla h = h\mathbf{w} + \rho\zeta$, where $\mathbf{w}$ is vector field dual to γ and $\varphi = 0$. Moreover, in this case*

$$S\,(\zeta) = -(m-1)\rho\zeta.$$

Proof. Suppose ζ annihilates φ. Then Eq (2.1) implies

$$R(X,\zeta)\zeta = (X(h)\zeta - \zeta(h)X) + h\left(\gamma\,(\zeta)\,X - \gamma\,(X)\,\zeta\right), \quad X \in \mathfrak{X}(N).$$

Using symmetry of the operator $R(X,\zeta)\zeta$, above equation implies

$$(X(h) - h\gamma\,(X))\,g\,(\zeta, Y) = (Y(h) - h\gamma\,(Y))\,g\,(\zeta, X)\,, \quad X, Y \in \mathfrak{X}(N),$$

which gives,

$$g\,(\nabla h - h\mathbf{w}, X)\,\zeta = g\,(\zeta, X)\,(\nabla h - h\mathbf{w})\,, \quad X \in \mathfrak{X}(N).$$

Inserting $X = \nabla h - h\mathbf{w}$ in above equation, we have

$$\|\nabla h - h\mathbf{w}\|^2\,\zeta = g\,(\zeta, \nabla h - h\mathbf{w})\,(\nabla h - h\mathbf{w})\,.$$

On taking the inner product with ζ, we conclude

$$\|\nabla h - h\mathbf{w}\|^2\,\|\zeta\|^2 = g\,(\zeta, \nabla h - h\mathbf{w})^2\,,$$

and it implies $\nabla h - h\mathbf{w}$ and ζ are parallel. This guarantees the existence of a smooth function ρ on N with $\nabla h - h\mathbf{w} = \rho\zeta$ and it proves the first part.

Next observe that by virtue of (2.2), we have

$$d\gamma(\zeta, X) = 0, \quad X \in \mathfrak{X}(N). \tag{2.5}$$

Let β be dual 1-form to ζ. Then, we have

$$d\beta\,(X,Y) = g\,(D_X\zeta, Y) - g\,(D_Y\zeta, X)\,,$$

which in view of Eq (1.1), gives

$$d\beta(X,Y)=\gamma(X)\beta(Y)-\gamma(Y)\beta(X),\quad X,Y\in\mathfrak{X}(N),$$

that is, $d\beta=\frac{1}{2}\gamma\wedge\beta$. Taking differential in this last equation we have $d\gamma\wedge\beta=\gamma\wedge d\beta=\frac{1}{2}\gamma\wedge(\gamma\wedge\beta)=0$. Choosing $X,Y\in\mathfrak{X}(N)$ orthogonal to ζ and using (2.5) and $d\gamma\wedge\beta=0$, we get

$$d\gamma(X,Y)\,\|\zeta\|^2=0.$$

Since, ζ is non-trivial TFVF, we have $\zeta\neq0$ and N is connected, through above equation, we have $d\gamma(X,Y)=0$ for $X,Y\in\mathfrak{X}(N)$ orthogonal to ζ. Observe that for any $X\in\mathfrak{X}(N)$ the vector fields $\|\zeta\|^2X-\beta(X)\zeta$ and ζ are orthogonal and we conclude for any $X,Y\in\mathfrak{X}(N)$

$$d\gamma\left(\|\zeta\|^2X-\beta(X)\zeta,\|\zeta\|^2Y-\beta(Y)\zeta\right)=0.$$

and using Eq (2.5), we get

$$\|\zeta\|^4\,d\gamma(X,Y)=0,\quad X,Y\in\mathfrak{X}(N).$$

Using $\zeta\neq0$ and N is connected in above equation to arrive at

$$d\gamma(X,Y)=0,\quad X,Y\in\mathfrak{X}(N),$$

which in view of Eq (2.2), we conclude $\varphi=0$. Finally, using $\varphi=0$ and $\nabla h-h\mathbf{w}=\rho\zeta$ in (2.4), we conclude $S(\zeta)=-(m-1)\rho\zeta$. □

Note that for a non-trivial TFVF ζ on a connected (N,g) with torsed function h and torsed form γ that annihilates the skew-symmetric operator φ associated to γ, using Lemma 2.1, we have $\varphi=0$, that is, $d\gamma=0$ and the vector field $\mathbf{w}$ dual to γ satisfies

$$g(D_X\mathbf{w},Y)=g(D_Y\mathbf{w},X),\quad X,Y\in\mathfrak{X}(N).$$

Using above equation and Eq (2.2), we have

$$2g(BX,Y)=(\pounds_{\mathbf{w}}g)(X,Y)=g(D_X\mathbf{w},Y)+g(D_Y\mathbf{w},X)=2g(D_X\mathbf{w},Y),$$

that is,

$$D_X\mathbf{w}=BX,\quad X\in\mathfrak{X}(N).\tag{2.6}$$

Definition 2.1. If ζ is a non-trivial TFVF with torsed function h and torsed form γ on a Riemannian manifold (N,g) that annihilates the skew-symmetric operator φ associated to the torsed form γ, then the function ρ satisfying $\nabla h=h\mathbf{w}+\rho\zeta$ in the Lemma 2.1 is called the function associated to TFVF ζ.

Definition 2.2. We say that the Ricci operator S is invariant under ζ if S is invariant under the local flow of ζ or equivalently

$$\pounds_\zeta S=0,$$

where $\pounds_\zeta$ stands for the Lie differentiation with respect to ζ.

Lemma 2.2. *Let ζ be a non-trivial TFVF with torsed function h and torsed form γ on a connected Riemannian manifold (N, g) that annihilates the skew-symmetric operator φ associated to the torsed form γ. If the Ricci operator S is invariant under ζ, then the function ρ associated to ζ is a constant c and the vector field $\mathbf{w}$ dual to γ satisfies*

$$S(\mathbf{w}) = -(m-1)c\mathbf{w}.$$

Proof. Suppose ζ annihilates φ and that the Ricci operator S is invariant under ζ. Then we have

$$\nabla h = h\mathbf{w} + \rho\zeta, \tag{2.7}$$

and $\left(\pounds_\zeta S\right)(X) = 0$, that is, in view of Eq (1.1), we get

$$\left(D_\zeta S\right)(X) = \gamma\left(S(X)\right)\zeta - \gamma(X)S(\zeta).$$

Using Lemma 2.1 we get

$$\left(D_\zeta S\right)(X) = \gamma\left(S(X) + (m-1)\rho X\right)\zeta. \tag{2.8}$$

Choosing $X = \zeta$ in above equation, while using $S(\zeta) = -(m-1)\rho\zeta$, we have

$$\left(D_\zeta S\right)(\zeta) = 0. \tag{2.9}$$

Differentiating $S(\zeta) = -(m-1)\rho\zeta$ in the direction of ζ and using Eq (1.1), we arrive at

$$D_\zeta S(\zeta) = -(m-1)\zeta(\rho)\zeta - (m-1)\rho\left(h\zeta + \gamma(\zeta)\zeta\right).$$

Moreover, using Eq (1.1) and $S(\zeta) = -(m-1)\rho\zeta$, we have

$$S\left(D_\zeta\zeta\right) = hS(\zeta) + \gamma(\zeta)S(\zeta) = -(m-1)\rho\left(h + \gamma(\zeta)\right)\zeta.$$

Combining last two equations, we arrive at

$$\left(D_\zeta S\right)(\zeta) = -(m-1)\zeta(\rho)\zeta, \tag{2.10}$$

which in view (2.9) and $\zeta \neq 0$ on a connected N implies

$$\zeta(\rho) = 0. \tag{2.11}$$

We denote by A_h be the Hessian operator of the function h. Using Eqs (1.1), (2.6) and (2.7), we have

$$A_h X = X(h)\mathbf{w} + hBX + X(\rho)\zeta + \rho\left(hX + \gamma(X)\zeta\right),$$

that is,

$$A_h X = hBX + \rho hX + \left[X(h)\mathbf{w} + \left(X(\rho) + \rho\gamma(X)\right)\zeta\right].$$

Using symmetry of A_h, we get

$$X(h)\gamma(Y) + \left(X(\rho) + \rho\gamma(X)\right)\beta(Y) = Y(h)\gamma(X) + \left(Y(\rho) + \rho\gamma(Y)\right)\beta(X),$$

which in view of (2.7) in the form $X(h) = h\gamma(X) + \rho\beta(X)$, $X \in \mathfrak{X}(N)$ implies

$$X(\rho)\beta(Y) = Y(\rho)\beta(X), \quad X, Y \in \mathfrak{X}(N).$$

Now, the above equation with $Y = \zeta$ while keeping in view Eq (2.11), gives

$$\|\zeta\|^2 X(\rho) = 0, \quad X \in \mathfrak{X}(N).$$

As, $\zeta \neq 0$ on connected N, we conclude ρ is a constant c.

Next, we take the inner product in Eq (2.8) with ζ and use symmetry of the operator S and Eq (2.9), to arrive at

$$\gamma(S(X) + (m-1)cX)\|\zeta\|^2 = 0,$$

which on connected N with $\zeta \neq 0$ implies

$$\gamma(S(X) + (m-1)cX) = 0, \quad X \in \mathfrak{X}(N).$$

This proves

$$S(\mathbf{w}) = -(m-1)c\mathbf{w}. \tag{2.12}$$

□

3. Characterizing spheres

Given a non-trivial TFVF ζ on a connected Riemannian manifold (N, g) with torsed function h and torsed form γ, there is a dual vector field $\mathbf{w}$ to γ. We have observed that if ζ annihilates the skew-symmetric operator φ associated to torsed form γ, then $\varphi = 0$ and there is a function ρ defined on N that satisfies $\nabla h = h\mathbf{w} + \rho\zeta$ and $S(\zeta) = -(m-1)\rho\zeta$. Furthermore, we have seen that if in addition the Ricci operator S of (N, g) is invariant under the TFVF ζ, then the function $\rho = c$ a constant and that $S(\mathbf{w}) = -(m-1)c\mathbf{w}$. These constraints on TFVF ζ are having an effect on the vector field $\mathbf{w}$. We also have an operator B associated to $\mathbf{w}$ satisfying Eq (2.6). We denote by $f = trB$ and this is the third function on N associated to a non-trivial TFVF ζ. As we are interested in seeking further conditions so that (N, g) is isometric to an m-sphere $\mathbf{S}^m(a)$, naturally, we need to ask for the Ricci curvature $T(\mathbf{w}, \mathbf{w}) > 0$. We prove the following characterization of the spheres using a non-trivial TFVF ζ on a compact and connected Riemannian manifold (N, g).

Theorem 3.1. Let ζ be a non-trivial TFVF on an m-dimensional compact and connected Riemannian manifold (N, g), with torsed function h, torsed form γ and Ricci curvature $T(\mathbf{w}, \mathbf{w}) > 0$. Then (N, g) is isometric to $\mathbf{S}^m(a)$ if and only if, ζ annihilates the skew-symmetric operator φ associated to γ, the Ricci operator S is invariant under ζ and the Ricci curvature $T(\mathbf{w}, \mathbf{w})$ satisfies

$$\int_M T(\mathbf{w}, \mathbf{w}) \geq \frac{m-1}{m} \int_M (div\mathbf{w})^2.$$

Proof. First notice that with condition $T(\mathbf{w}, \mathbf{w}) > 0$, in view of Eq (2.12), the constant $c < 0$ and we put $c = -a$ for a positive constant a. Note that Eq (2.6) implies

$$div\mathbf{w} = f, \quad f = trB. \tag{3.1}$$

Choose a local frame $\{u_1, ..., u_m\}$ on N and use Eq (2.6), to compute

$$divB\mathbf{w} = \sum_i g\left(D_{u_i}B\mathbf{w}, u_i\right) = \sum_i g\left(\left(D_{u_i}B\right)(\mathbf{w}) + B^2u_i, u_i\right).$$

Using symmetry of the operator B, we get

$$divB\mathbf{w} = \|B\|^2 + g\left(\mathbf{w}, \sum_i \left(D_{u_i}B\right)(u_i)\right). \tag{3.2}$$

Now, using Eq (2.6), we have

$$R(X, Y)\mathbf{w} = \left(D_XB\right)(Y) - \left(D_YB\right)(X), \quad X, Y \in \mathfrak{X}(N), \tag{3.3}$$

which implies

$$T(Y, \mathbf{w}) = g\left(Y, \sum_i \left(D_{u_i}B\right)(u_i)\right) - Y(f).$$

Thus,

$$T(\mathbf{w}, \mathbf{w}) = g\left(\mathbf{w}, \sum_i \left(D_{u_i}B\right)(u_i)\right) - \mathbf{w}(f).$$

Using this equation in (3.2), we arrive at

$$divB\mathbf{w} = \|B\|^2 + T(\mathbf{w}, \mathbf{w}) + \mathbf{w}(f). \tag{3.4}$$

Observe that $div\,(f\mathbf{w}) = \mathbf{w}(f) + f div\mathbf{w}$ and using (3.1), we have $div\,(f\mathbf{w}) = \mathbf{w}(f) + (div\mathbf{w})^2$. Thus, Eq (3.4) becomes

$$divB\mathbf{w} = \|B\|^2 + T(\mathbf{w}, \mathbf{w}) + div\,(f\mathbf{w}) - (div\mathbf{w})^2,$$

which on integration yields

$$\int_M \|B\|^2 = \int_M \left((div\mathbf{w})^2 - T(\mathbf{w}, \mathbf{w})\right).$$

Using above equation in view of Eq (3.1), we have

$$\int_M \left(\|B\|^2 - \frac{1}{m}f^2\right) = \int_M \left(\frac{m-1}{m}(div\mathbf{w})^2 - T(\mathbf{w}, \mathbf{w})\right). \tag{3.5}$$

Now, in view of the condition in the statement the right hand integral is non-positive and we have

$$\int_M \left(\|B\|^2 - \frac{1}{m}f^2\right) \le 0. \tag{3.6}$$

The Schwartz's inequality $\|B\|^2 \ge \frac{1}{m}f^2$ and inequality (3.6) implies

$$\left(\|B\|^2 - \frac{1}{m}f^2\right) = 0.$$

Thus, we have the equality $\|B\|^2 = \frac{1}{m} f^2$, and it holds if and only if

$$B = \frac{f}{m} I. \tag{3.7}$$

Next, we see that Eq (3.7) implies

$$\left(D_X B\right)(Y) = \frac{1}{m} X(f) Y, \quad X, Y \in \mathfrak{X}(N)$$

and combining it with Eq (3.3), we get

$$R(X, Y)\mathbf{w} = \frac{1}{m}\left(X(f)Y - Y(f)X\right), \quad X, Y \in \mathfrak{X}(N).$$

This equation implies

$$T\left(Y, \mathbf{w}\right) = -\frac{m-1}{m} Y(f),$$

that is,

$$S\left(\mathbf{w}\right) = -\frac{m-1}{m} \nabla f.$$

Using Lemma 2.2 and $c = -a$, we get

$$\nabla f = -ma\mathbf{w}, \tag{3.8}$$

where a is a positive constant. Note that if f is a constant, then Eq (3.8) will imply $\mathbf{w} = 0$, that is, the torsed form $\gamma = 0$ and it contradicts the fact that ζ is a non-trivial TFVF. Thus f is a not a constant. Differentiating (3.8) with respect to $X \in \mathfrak{X}(N)$ while using Eqs (2.6) and (3.7), we get

$$D_X \nabla f = -afX, \quad X \in \mathfrak{X}(N).$$

This proves that (N, g) is isometric to $\mathbf{S}^m(a)$ (cf. [22, 23]).

Conversely, we have already seen in the introduction that the sphere $\mathbf{S}^m(a)$ admits a non-trivial TFVF ζ with torsed function $h = -\sqrt{a}\sigma e^{-\sigma}$ and torsed form γ given by

$$\gamma(X) = -\sqrt{a} g\left(\mathbf{v}, X\right).$$

The vector field $\mathbf{w} = -\sqrt{a}\mathbf{v}$. Then using Eq (1.3), we get that $d\gamma = 0$ and that the skew-symmetric operator φ associated to γ has to be $\varphi = 0$. Thus, ζ annihilates φ. Furthermore, the Ricci operator S for the sphere $\mathbf{S}^m(a)$ is given by $S = (m-1)aI$ and therefore is invariant under ζ. The Ricci curvature $T(\mathbf{w}, \mathbf{w}) > 0$ and is given by

$$T(\mathbf{w}, \mathbf{w}) = (m-1)a \|\mathbf{w}\|^2 = (m-1)a^2 \|\mathbf{v}\|^2.$$

Using Eq (1.3), we have $\nabla\sigma = \sqrt{a}\mathbf{v}$, which in view of above equation implies

$$\int_{\mathbf{S}^m(a)} T(\mathbf{w}, \mathbf{w}) = (m-1)a \int_{\mathbf{S}^m(a)} \|\nabla\sigma\|^2. \tag{3.9}$$

Note that on using Eq (1.3), we have $div\mathbf{v} = -\sqrt{a}m\sigma$ and $\Delta\sigma = -am\sigma$. This last equation implies

$$\int_{\mathbf{S}^m(a)} \|\nabla\sigma\|^2 = am \int_{\mathbf{S}^m(a)} \sigma^2. \tag{3.10}$$

Also, we have $div\mathbf{w} = -am\sigma$, that is,

$$\frac{m-1}{m}\int_{\mathbf{S}^m(a)}(div\mathbf{w})^2 = m(m-1)a^2\int_{\mathbf{S}^m(a)}\sigma^2.$$

Using Eq (3.10), we have

$$\frac{m-1}{m}\int_{\mathbf{S}^m(a)}(div\mathbf{w})^2 = (m-1)a\int_{\mathbf{S}^m(a)}\|\nabla\sigma\|^2. \tag{3.11}$$

Combining Eqs (3.9) and (3.11), we conclude that requirements in the statement are fulfilled. □

4. Conclusions

We have noticed in Theorem 3.1 that a TFVF ζ on a compact Riemannian manifold (M, g) with torsed function h and torsed form γ that annihilates the operator φ associated to γ and the Ricci operator invariant under ζ can be used to find a characterization of a sphere. Naturally, it will be of interest to know whether we could use other conditions such as the operator φ is invariant under the TFVF ζ instead of $\varphi(\zeta) = 0$, keeping other conditions same to reach the same conclusion of Theorem 3.1. Furthermore, one would be interested to find characterizations of Euclidean spaces and Hyperbolic spaces using a TFVF on a complete Riemannian manifold.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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