Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2989–3005.
DOI: 10.3934/math.2022165
Received: 27 September 2021
Accepted: 15 November 2021
Published: 23 November 2021

Research article

Certain class of bi-univalent functions defined by quantum calculus operator associated with Faber polynomial

Sheza. M. El-Deeb[1,2,*], Gangadharan Murugusundaramoorthy[3], Kaliyappan Vijaya[3] and Alhanouf Alburaikan[4]

[1] Department of Mathematics, College of Science and Arts in Al-Badaya, Qassim University, Buraidah, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt

[3] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, TN, India

[4] Department of Mathematics, College of Science and Arts in Al-Badaya, Qassim University, Buraidah, Saudi Arabia

* **Correspondence:** Email: shezaeldeeb@yahoo.com.

Abstract: In this paper, we introduce a new class of bi-univalent functions defined in the open unit disc and connected with a q-convolution. We find estimates for the general Taylor-Maclaurin coefficients of the functions in this class by using Faber polynomial expansions, and we obtain an estimation for Fekete-Szegö problem for this class.

Keywords: Faber polynomial; bi-univalent functions; convolution; q-derivative operator
Mathematics Subject Classification: 30C45, 30C80.

1. Introduction and preliminary definitions

In his survey-cum-expository review article, Srivastava [1] presented and motivated about brief expository overview of the classical q-analysis versus the so-called (p,q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok-Srivastava, Srivastava-Wright and Srivastava-Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p,q)-analysis has important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional q-calculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see Srivastava and Karlsson [2, pp. 350–351], Srivastava [3, 4]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical q-calculus and the fractional q-calculus in geometric function theory of complex analysis have the potential to encourage and motivate significant further researches on many of these and other related subjects. Jackson [5, 6] was the first that gave some application of q-calculus and introduced the q-analogue of derivative and integral operator (see also [7, 8]), we apply the concept of q-convolution in order to introduce and study the general Taylor-Maclaurin coefficient estimates for functions belonging to a new class of normalized analytic in the open unit disk, which we have defined here.

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) := z + \sum_{m=2}^{\infty} a_m z^m, \qquad z \in \Delta := \{z \in \mathbb{C} : |z| < 1\} \tag{1.1}$$

and let $\mathcal{S} \subset \mathcal{A}$ consisting on functions that are univalent in Δ. If the function $h \in \mathcal{A}$ is given by

$$h(z) := z + \sum_{m=2}^{\infty} b_m z^m, \ (z \in \Delta). \tag{1.2}$$

The Hadamard product (or convolution) of f and h, given by (1.1) and (1.2), respectively, is defined by

$$(f * h)(z) := z + \sum_{m=2}^{\infty} a_m b_m z^m, \ z \in \Delta. \tag{1.3}$$

If f and F are analytic functions in Δ, we say that f is subordinate to F, written as $f(z) \prec F(z)$, if there exists a Schwarz function s, which is analytic in Δ, with $s(0) = 0$, and $|s(z)| < 1$ for all $z \in \Delta$, such that $f(z) = F(s(z))$, $z \in \Delta$. Furthermore, if the function F is univalent in Δ, then we have the following equivalence ([9, 10])

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\Delta) \subset F(\Delta).$$

The Koebe one-quarter theorem (see [11]) prove that the image of Δ under every univalent function $f \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} that satisfies

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$\begin{aligned} g(w) &= f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \\ &= w + \sum_{m=2}^{\infty} A_m w^m \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ represent the class of bi-univalent functions in Δ given by (1.1). The class of analytic bi-univalent functions was first familiarised by Lewin [12], where it was shown that $|a_2| < 1.51$. Brannan and Clunie [13] enhanced Lewin's result to $|a_2| < \sqrt{2}$ and later Netanyahu [14] proved that $|a_2| < \frac{4}{3}$.

Note that the functions

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \qquad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \qquad f_3^{-1}(w) = \frac{e^w-1}{e^w}$$

are elements of Σ (see [15, 16]). For a brief history and exciting examples in the class Σ (see [17]). Brannan and Taha [18] (see also [16]) presented certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α $(0 \le \alpha < 1)$, respectively (see [17, 19, 20]). Ensuing Brannan and Taha [18], a function $f \in \mathcal{A}$ is said to be in the class $S^*_\Sigma(\alpha)$ of bi-starlike functions of order α $(0 < \alpha \le 1)$, if each of the following conditions are satisfied:

$$f \in \Sigma, \quad \text{with} \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta),$$

and

$$\left|\arg\frac{wg'(w)}{g(w)}\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta),$$

where the function g is the analytic extension of f^{-1} to Δ, given by

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots \qquad (w \in \Delta). \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be in the class $K_\Sigma(\alpha)$ of bi-convex functions of order α $(0 < \alpha \le 1)$, if each of the following conditions are satisfied:

$$f \in \Sigma, \quad \text{with} \quad \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta),$$

and

$$\left|\arg\left(1 + \frac{wg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta).$$

The classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α $(0 < \alpha \le 1)$, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ ([16, 18]).

1.1. Faber polynomial expansion of functions $f \in \mathcal{A}$

The Faber polynomials introduced by Faber [21] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis (see, for details, [22]). In 2013, Hamidi and Jahangiri [23–25] took a new approach to show that the initial coefficients of classes of bi- starlike functions e as well as provide an estimate for the general coefficients of such functions subject to a given gap series condition.Recently,their idea of application of Faber polynomials triggered a number of related publications by several authors (see, for example, [26–28] and also references cited threin) investigated some interesting and useful properties for analytic functions. Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ has the form (1.1), the coefficients of its inverse map may be expressed as

$$g(w) = f^{-1}(w) = w + \sum_{m=2}^{\infty} \frac{1}{m} K_{m-1}^{-m}(a_2, a_3, ...)w^m, \tag{1.5}$$

where

$$\begin{aligned} \mathcal{K}_{m-1}^{-m}(a_2, a_3, ...) &= \tfrac{(-m)!}{(-2m+1)!\ (m-1)!} a_2^{m-1} + \tfrac{(-m)!}{(2(-m+1))!\ (m-3)!} a_2^{m-3} a_3 \\ &\quad + \tfrac{(-m)!}{(-2m+3)!\ (m-4)!} a_2^{m-4} a_4 + \tfrac{(-m)!}{(2(-m+2))!\ (m-5)!} a_2^{m-5} \left[a_5 + (-m+2)\, a_3^2\right] \\ &\quad + \tfrac{(-m)!}{(-2m+5)!\ (m-6)!} a_2^{m-6} \left[a_6 + (-2m+5)\, a_3 a_4\right] + \sum_{i \geq 7} a_2^{m-i} U_i, \end{aligned} \tag{1.6}$$

such that U_i with $7 \leq i \leq m$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_m$, In particular, the first three terms of $\mathcal{K}_{m-1}^{-m}$ are

$$\begin{aligned} \mathcal{K}_1^{-2} &= -2a_2, \\ \mathcal{K}_2^{-3} &= 3\left(2a_2^2 - a_3\right), \\ \mathcal{K}_3^{-4} &= -4\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned}$$

In general, an expansion of $\mathcal{K}_m^{-n}$ $(n \in \mathbb{N})$ is (see [29–33])

$$\mathcal{K}_m^{-n} = na_m + \frac{n(n-1)}{2}\mathcal{D}_m^2 + \frac{n!}{3!\,(n-3)!}\mathcal{D}_m^3 + ... + \frac{n!}{m!\,(n-m)!}\mathcal{D}_m^m,$$

where $\mathcal{D}_m^n = \mathcal{D}_m^n(a_2, a_3, ...)$ and

$$\mathcal{D}_m^p(a_1, a_2, ...a_m) = \sum_{m=1}^{\infty} \frac{p!}{i_1!...i_m!} a_1^{i_1} ... a_m^{i_m},$$

while $a_1 = 1$ and the sum is taken over all non-negative integers $i_1...i_m$ satisfying

$$\begin{aligned} i_1 + i_2 + ... + i_m &= p \\ i_1 + 2i_2 + ... + mi_m &= m. \end{aligned}$$

Evidently

$$\mathcal{D}_m^m(a_1, a_2, ...a_m) = a_1^m.$$

1.2. Quantum calculus operator

Srivastava [1] made use of several operators of q-calculus and fractional q-calculus and recollecting the definition and representations. The q-shifted factorial is defined for $\kappa, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\kappa; q)_m = \begin{cases} 1 & , \quad m = 0 \\ (1-\kappa)(1-\kappa q)\ldots\left(1-\kappa q^{k-1}\right) & , \quad m \in \mathbb{N} \end{cases}.$$

By using the q-Gamma function $\Gamma_q(z)$, we get

$$(q^\kappa; q)_m = \frac{(1-q)^m \; \Gamma_q(\kappa + m)}{\Gamma_q(\kappa)} \qquad (m \in \mathbb{N}_0),$$

where (see [34])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty} \qquad (|q| < 1).$$

Also, we note that

$$(\kappa; q)_\infty = \prod_{m=0}^{\infty} (1 - \kappa q^m) \qquad (|q| < 1),$$

and, the q-Gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \; \Gamma_q(z),$$

where $[m]_q$ symbolizes the basic q-number defined as follows

$$[m]_q := \begin{cases} \frac{1-q^m}{1-q} & , \quad m \in \mathbb{C} \\ 1 + \sum\limits_{j=1}^{m-1} q^j & , \quad m \in \mathbb{N} \end{cases}. \tag{1.7}$$

Using the definition formula (1.7) we have the next two products:
(i) For any non-negative integer m, the q-shifted factorial is given by

$$[m]_q! := \begin{cases} 1, & \text{if} \quad m = 0, \\ \prod\limits_{n=1}^{m} [n]_q, & \text{if} \quad m \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the q-generalized Pochhammer symbolis defined by

$$[r]_{q,m} := \begin{cases} 1, & \text{if} \quad m = 0, \\ \prod\limits_{n=r}^{r+m-1} [n]_q, & \text{if} \quad m \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) Gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-}\left\{\frac{(q^\kappa;q)_m}{(1-q)^m}\right\}=(\kappa)_m\,,$$

where $(\kappa)_m$ is the familiar Pochhammer symbol defined by

$$(\kappa)_m=\begin{cases}1, & \text{if}\quad m=0,\\ \kappa(\kappa+1)\dots(\kappa+m-1), & \text{if}\quad m\in\mathbb{N}.\end{cases}$$

For $0<q<1$, the *q-derivative operator (or, equivalently, the q-* difference operator) El-Deeb et al. [35] defined D_q for $f*h$ given by (1.3) is defined by (see [5, 6])

$$\begin{aligned}D_q\,(f*h)\,(z):&=D_q\left(z+\sum_{m=2}^{\infty}a_mb_mz^m\right)\\&=\frac{(f*h)\,(z)-(f*h)\,(qz)}{z(1-q)}\\&=1+\sum_{m=2}^{\infty}[m]_qa_mb_mz^{m-1}\qquad(z\in\Delta)\,,\end{aligned}$$

where, as in the definition (1.7)

$$[m]_q:=\begin{cases}\frac{1-q^m}{1-q}=1+\sum\limits_{j=1}^{m-1}q^j & (m\in\mathbb{N}),\\ 0 & (m=0).\end{cases}\tag{1.8}$$

For $\kappa>-1$ and $0<q<1$, El-Deeb et al. [35] (see also) defined the linear operator $\mathcal{H}_h^{\kappa,q}:\mathcal{A}\to\mathcal{A}$ by

$$\mathcal{H}_h^{\kappa,q}f(z)*\mathcal{M}_{q,\kappa+1}(z)=z\,D_q\,(f*h)\,(z)\qquad(z\in\Delta)\,,$$

where the function $\mathcal{M}_{q,\kappa+1}$ is given by

$$\mathcal{M}_{q,\kappa+1}(z):=z+\sum_{m=2}^{\infty}\frac{[\kappa+1]_{q,m-1}}{[m-1]_q!}z^m\qquad(z\in\Delta)\,.$$

A simple computation shows that

$$\mathcal{H}_h^{\kappa,q}f(z):=z+\sum_{m=2}^{\infty}\frac{[m]_q!}{[\kappa+1]_{q,m-1}}\,a_mb_m\,z^m\qquad(\kappa>-1,\ 0<q<1,\ z\in\Delta).\tag{1.9}$$

From the definition relation (1.9), we can easily verify that the next relations hold for all $f\in\mathcal{A}$:

$$\text{(i)}\quad[\kappa+1]_q\mathcal{H}_h^{\kappa,q}f(z)=[\kappa]_q\mathcal{H}_h^{\kappa+1,q}f(z)+q^\kappa\,z\,D_q\left(\mathcal{H}_h^{\kappa+1,q}f(z)\right)\qquad(z\in\Delta)\,;$$

$$\text{(ii)}\quad\mathcal{I}^\kappa_hf(z):=\lim_{q\to1^-}\mathcal{H}_h^{\kappa,q}f(z)=z+\sum_{m=2}^{\infty}\frac{m!}{(\kappa+1)_{m-1}}\,a_mb_mz^m\qquad(z\in\Delta)\,.\tag{1.10}$$

Remark 1. Taking precise cases for the coefficients b_m we attain the next special cases for the operator $\mathcal{H}_h^{\kappa,q}$:

(i) For $b_m = 1$, we obtain the operator $\mathcal{I}_q^\kappa$ defined by Srivastava [32] and Arif et al. [36] as follows

$$\mathcal{I}_q^\kappa f(z) := z + \sum_{m=2}^{\infty} \frac{[m]_q!}{[\kappa+1]_{q,m-1}} a_m z^m \qquad (\kappa > -1,\ 0 < q < 1,\ z \in \Delta); \tag{1.11}$$

(ii) For $b_m = \dfrac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\Gamma(m+\upsilon)}$, $\upsilon > 0$, we obtain the operator $\mathcal{N}_{\upsilon,q}^\kappa$ defined by El-Deeb and Bulboacă [37] and El-Deeb [38] as follows

$$\begin{aligned}\mathcal{N}_{\upsilon,q}^\kappa f(z) &:= z + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\Gamma(m+\upsilon)} \cdot \frac{[m]_q!}{[\kappa+1]_{q,m-1}} a_m z^m \\ &= z + \sum_{m=2}^{\infty} \frac{[m]_q!}{[\kappa+1]_{q,m-1}} \psi_m a_m z^m \qquad (\upsilon > 0,\ \kappa > -1,\ 0 < q < 1,\ z \in \Delta),\end{aligned} \tag{1.12}$$

where

$$\psi_m := \frac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\Gamma(m+\upsilon)}; \tag{1.13}$$

(iii) For $b_m = \left(\dfrac{n+1}{n+m}\right)^\alpha$, $\alpha > 0$, $n \geq 0$, we obtain the operator $\mathcal{M}_{n,q}^{\kappa,\alpha}$ defined by El-Deeb and Bulboacă [39] and Srivastava and El-Deeb [40] as follows

$$\mathcal{M}_{n,q}^{\kappa,\alpha} f(z) := z + \sum_{m=2}^{\infty} \left(\frac{n+1}{n+m}\right)^\alpha \cdot \frac{[m]_q!}{[\kappa+1]_{q,m-1}} a_m z^m \qquad (z \in \Delta); \tag{1.14}$$

(iv) For $b_m = \dfrac{\rho^{m-1}}{(m-1)!} e^{-\rho}$, $\rho > 0$, we obtain the q-analogue of Poisson operator defined by El-Deeb et al. [35] (see [41]) as follows

$$\mathcal{I}_q^{\kappa,\rho} f(z) := z + \sum_{m=2}^{\infty} \frac{\rho^{m-1}}{(m-1)!} e^{-\rho} \cdot \frac{[m]_q!}{[\kappa+1]_{q,m-1}} a_m z^m \qquad (z \in \Delta). \tag{1.15}$$

(v) For $b_m = \left[\dfrac{1+\ell+\mu(m-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\mu \geq 0$, we obtain the q-analogue of Prajapat operator defined by El-Deeb et al. [35] (see also [42]) as follows

$$\mathcal{J}_{q,\ell,\mu}^{\kappa,n} f(z) := z + \sum_{m=2}^{\infty} \left[\frac{1+\ell+\mu(m-1)}{1+\ell}\right]^n \cdot \frac{[m,q]!}{[\kappa+1,q]_{m-1}} a_m z^m \qquad (z \in \Delta); \tag{1.16}$$

(vi) For $b_m = \begin{pmatrix} n+m-2 \\ m-1 \end{pmatrix} \theta^{m-1} (1-\theta)^n$, $n \in \mathbb{N}$, $0 \leq \theta \leq 1$, we obtain the q-analogue of the Pascal distribution operator defined by Srivastava and El-Deeb [28] (see also [35, 43, 44]) as follows

$$\ominus_{q,\theta}^{\kappa,n} f(z) := z + \sum_{m=2}^{\infty} \begin{pmatrix} n+m-2 \\ m-1 \end{pmatrix} \theta^{m-1} (1-\theta)^n \cdot \frac{[m,q]!}{[\kappa+1,q]_{m-1}} a_m z^m \qquad (z \in \Delta). \tag{1.17}$$

The purpose of the paper is to present a new subclass of functions $\mathcal{L}_{\Sigma}^{q,\kappa}(\eta; h; \Phi)$ of the class Σ, that generalize the previous defined classes. This subclass is defined with the aid of a general $\mathcal{H}_{h}^{\kappa,q}$ linear operator defined by convolution products composed with the aid of q-derivative operator. This new class extend and generalize many preceding operators as it was presented in Remark 1, and the main goal of the paper is find estimates on the coefficients $|a_2|$, $|a_3|$, and for the Fekete-Szegö functional for functions in these new subclasses. These classes will be introduced by using the subordination and the results are obtained by employing the techniques used earlier by Srivastava et al. [16]. This last work represents one of the most important study of the bi-univalent functions, and inspired many investigations in this area including the present paper, while many other recent papers deals with problems initiated in this work, like [33, 44–48], and many others. Inspired by the work of Silverman and Silvia [49] (also see [50]) and recent study by Srivastava et al [51], in this article, we define the following new subclass of bi-univalent functions $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, h)$ as follows:

Definition 1. Let $\varpi \in (-\pi, \pi]$ and let the function $f \in \Sigma$ be of the form (1.1) and h is given by (1.2), the function f is said to be in the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, h)$ if the following conditions are satisfied:

$$\Re\left(\left(\mathcal{H}_{h}^{\kappa,q} f(z)\right)' + \frac{(1+e^{i\varpi})}{2} z\left(\mathcal{H}_{h}^{\kappa,q} f(z)\right)''\right) > \vartheta, \tag{1.18}$$

and

$$\Re\left(\left(\mathcal{H}_{h}^{\kappa,q} g(w)\right)' + \frac{(1+e^{i\varpi})}{2} w\left(\mathcal{H}_{h}^{\kappa,q} g(w)\right)''\right) > \vartheta \tag{1.19}$$

with $\kappa > -1$, $0 < q < 1$, $0 \le \vartheta < 1$ and $z, w \in \Delta$, where the function g is the analytic extension of f^{-1} to Δ, and is given by (1.4).

Definition 2. Let $\varpi = 0$ and let the function $f \in \Sigma$ be of the form (1.1) and h is given by (1.2), the function f is said to be in the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\vartheta, h)$ if the following conditions are satisfied:

$$\Re\left(\left(\mathcal{H}_{h}^{\kappa,q} f(z)\right)' + z\left(\mathcal{H}_{h}^{\kappa,q} f(z)\right)''\right) > \vartheta, \tag{1.20}$$

and

$$\Re\left(\left(\mathcal{H}_{h}^{\kappa,q} g(w)\right)' + w\left(\mathcal{H}_{h}^{\kappa,q} g(w)\right)''\right) > \vartheta \tag{1.21}$$

with $\kappa > -1$, $0 < q < 1$, $0 \le \vartheta < 1$ and $z, w \in \Delta$, where the function g is the analytic extension of f^{-1} to Δ, and is given by (1.4).

Definition 3. Let $\varpi = \pi$ and let the function $f \in \Sigma$ be of the form (1.1) and h is given by (1.2), the function f is said to be in the class $\mathcal{HM}_{\Sigma}^{q,\kappa}(\vartheta, h)$ if the following conditions are satisfied:

$$\Re\left(\left(\mathcal{H}_{h}^{\kappa,q} f(z)\right)'\right) > \vartheta \qquad \textit{and} \qquad \Re\left(\left(\mathcal{H}_{h}^{\kappa,q} g(w)\right)'\right) > \vartheta \tag{1.22}$$

with $\kappa > -1$, $0 < q < 1$, $0 \le \vartheta < 1$ and $z, w \in \Delta$, where the function g is the analytic extension of f^{-1} to Δ, and is given by (1.4).

Remark 2. (i) Putting $q \to 1^-$ we obtain that $\lim\limits_{q\to 1^-} \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h) =: \mathcal{G}_{\Sigma}^{\kappa}(\varpi, \vartheta; h)$, where $\mathcal{G}_{\Sigma}^{\kappa}(\varpi, \vartheta; h)$ represents the functions $f \in \Sigma$ that satisfy (1.18) and (1.19) for $\mathcal{H}_h^{\kappa,q}$ replaced with $\mathcal{I}_h^{\kappa}$ (1.10).

(ii) Fixing $b_m = \dfrac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\Gamma(m+\upsilon)}$, $\upsilon > 0$, we obtain the class $\mathcal{B}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, \upsilon)$, that represents the functions $f \in \Sigma$ that satisfy (1.18) and (1.19) for $\mathcal{H}_h^{\kappa,q}$ replaced with $\mathcal{N}_{\upsilon,q}^{\kappa}$ (1.12).

(iii) Taking $b_m = \left(\dfrac{n+1}{n+m}\right)^{\alpha}$, $\alpha > 0$, $n \geq 0$, we obtain the class $\mathcal{L}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, n, \alpha)$, that represents the functions $f \in \Sigma$ that satisfy (1.18) and (1.19) for $\mathcal{H}_h^{\kappa,q}$ replaced with $\mathcal{M}_{n,q}^{\kappa,\alpha}$ (1.14).

(iv) Fixing $b_m = \dfrac{\rho^{m-1}}{(m-1)!}e^{-\rho}$, $\rho > 0$, we obtain the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, \rho)$, that represents the functions $f \in \Sigma$ that satisfy (1.18) and (1.19) for $\mathcal{H}_h^{\kappa,q}$ replaced with $\mathcal{I}_q^{\kappa,\rho}$ (1.15).

(v) Choosing $b_m = \left[\dfrac{1+\ell+\mu(m-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\mu \geq 0$, we obtain the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta, n, \ell, \mu)$, that represents the functions $f \in \Sigma$ that satisfy (1.18) and (1.19) for $\mathcal{H}_h^{\kappa,q}$ replaced with $\mathcal{J}_{q,\ell,\mu}^{\kappa,n}$ (1.16).

2. Coefficient bounds for $f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$

Throughout this paper, we assume that

$$\varpi \in (-\pi; \pi], \quad \kappa > -1, \quad 0 \leq \vartheta < 1, \quad 0 < q < 1.$$

Recall the following Lemma which will be needed to prove our results.

Lemma 1. *(Caratheodory Lemma [11]) If $\phi \in \mathcal{P}$ and $\phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ then $|c_n| \leq 2$ for each n, this inequality is sharp for all n where $\mathcal{P}$ is the family of all functions ϕ analytic and having positive real part in Δ with $\phi(0) = 1$.*

We firstly introduce a bound for the general coefficients of functions belong to the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$.

Theorem 2. *Let the function f given by* (1.1) *belongs to the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$. If $a_k = 0$ for $2 \leq k \leq m-1$, then*

$$|a_m| \leq \frac{4(1-\vartheta)[\kappa+1, q]_{m-1}}{m\,|2+(1+e^{i\varpi})(m-1)|\;[m,q]!\,b_m}.$$

Proof. If $f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$, from (1.18), (1.19), we have

$$\begin{aligned} &\left(\left(\mathcal{H}_h^{\kappa,q} f(z)\right)' + \frac{(1+e^{i\varpi})}{2} z \left(\mathcal{H}_h^{\kappa,q} f(z)\right)''\right) \\ = \;& 1 + \sum_{m=2}^{\infty} \frac{m}{2}\left[2+\left(1+e^{i\varpi}\right)(m-1)\right]\frac{[m,q]!}{[\kappa+1,q]_{m-1}}\, b_m a_m z^{m-1} \qquad (z \in \Delta), \end{aligned} \tag{2.1}$$

and

$$\left(\left(\mathcal{H}_h^{\kappa,q} g(w)\right)' + \frac{(1+e^{i\varpi})}{2} z \left(\mathcal{H}_h^{\kappa,q} g(w)\right)''\right)$$

$$= \quad 1 + \sum_{m=2}^{\infty} \frac{m}{2}\left[2 + \left(1+e^{i\varpi}\right)(m-1)\right] \tfrac{[m,q]!}{[\kappa+1,q]_{m-1}} b_m \, A_m w^{m-1}$$

$$= 1 + \sum_{m=2}^{\infty} \frac{m}{2}\left[2 + \left(1+e^{i\varpi}\right)(m-1)\right] \tfrac{[m,q]!}{[\kappa+1,q]_{m-1}} b_m \, \tfrac{1}{m}\mathcal{K}_{m-1}^{-m}(a_2, ..., a_m) w^{m-1} \qquad (w \in \Delta). \tag{2.2}$$

Since

$$f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h) \qquad \text{and} \qquad g = f^{-1} \in \mathcal{M}_{\Sigma}^{q,\kappa}(\gamma, \eta, \vartheta; h),$$

we know that there are two positive real part functions:

$$U(z) = 1 + \sum_{m=1}^{\infty} c_m z^m,$$

and

$$V(w) = 1 + \sum_{m=1}^{\infty} d_m w^m,$$

where

$$\Re(U(z)) > 0 \qquad \text{and} \qquad \Re(V(w)) > 0 \qquad (z, w \in \Delta),$$

so that

$$\begin{aligned} \left(\mathcal{H}_h^{\kappa,q} f(z)\right)' + \frac{(1+e^{i\theta})}{2} z \left(\mathcal{H}_h^{\kappa,q} f(z)\right)'' &= \vartheta + (1-\vartheta)\, U(z) \\ &= 1 + (1-\vartheta) \sum_{m=1}^{\infty} c_m z^m, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \left(\mathcal{H}_h^{\kappa,q} g(w)\right)' + \frac{(1+e^{i\theta})}{2} z \left(\mathcal{H}_h^{\kappa,q} g(w)\right)'' &= \vartheta + (1-\vartheta)\, V(w) \\ &= 1 + (1-\vartheta) \sum_{m=1}^{\infty} d_m w^m. \end{aligned} \tag{2.4}$$

Using (2.1) and comparing the corresponding coefficients in (2.3), we obtain

$$\frac{m}{2}\left[2 + \left(1+e^{i\varpi}\right)(m-1)\right] \tfrac{[m,q]!}{[\kappa+1,q]_{m-1}} \, b_m a_m = (1-\vartheta)\, c_{m-1}, \tag{2.5}$$

and similarly, by using (2.2) in the equality (2.4), we have

$$\frac{m}{2}\left[2 + \left(1+e^{i\varpi}\right)(m-1)\right] \tfrac{[m,q]!}{[\kappa+1,q]_{m-1}} \, b_m \frac{1}{m}\mathcal{K}_{m-1}^{-m}(a_2, a_3, ...a_m) = (1-\vartheta)\, d_{m-1}, \tag{2.6}$$

under the assumption $a_k = 0$ for $0 \leq k \leq m-1$, we obtain $A_m = -a_m$ and so

$$\frac{m}{2}\left[2 + \left(1+e^{i\varpi}\right)(m-1)\right] \frac{[m,q]!}{[\kappa+1,q]_{m-1}} \, b_m a_m = (1-\vartheta)\, c_{m-1}, \tag{2.7}$$

and

$$-\frac{m}{2}\left[2+\left(1+e^{i\varpi}\right)(m-1)\right]\frac{[m,q]!}{[\kappa+1,q]_{m-1}}\,b_m a_m=(1-\vartheta)\,d_{m-1}, \tag{2.8}$$

Taking the absolute values of (2.7) and (2.8), we conclude that

$$\begin{aligned} |a_m| &= \left|\frac{2\,(1-\vartheta)\,[\kappa+1,q]_{m-1}c_{m-1}}{m\,[2+(1+e^{i\varpi})\,(m-1)]\ \ [m,q]!\,b_m}\right| \\ &= \left|\frac{-2\,(1-\vartheta)\,[\kappa+1,q]_{m-1}d_{m-1}}{m\,[2+(1+e^{i\varpi})\,(m-1)]\ \ [m,q]!\,b_m}\right|. \end{aligned}$$

Applying the Caratheodory Lemma 1, we obtain

$$|a_m|\leq\frac{4\,(1-\vartheta)\,[\kappa+1,q]_{m-1}}{m\,|2+(1+e^{i\varpi})\,(m-1)|\ \ [m,q]!\,b_m},$$

which completes the proof of Theorem.

□

Theorem 3. *Let the function f given by* (1.1) *belongs to the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi,\vartheta;h)$, then*

$$|a_2|\leq\begin{cases}\frac{2(1-\vartheta)[\kappa+1,q]}{\left|3+e^{i\varpi}\right|[2,q]!b_2} & ,\quad 0\leq\vartheta<1-\frac{\left|3+e^{i\varpi}\right|^2\,([2,q]!)^2\,[\kappa+2,q]b_2^2}{3\left|2+e^{i\varpi}\right|\,[3,q]!\,[\kappa+1,q]b_3} \\ \sqrt{\frac{2(1-\vartheta)[\kappa+1,q]_2}{3\left|2+e^{i\varpi}\right|\,[3,q]!\,b_3}} & ,\quad 1-\frac{\left|3+e^{i\varpi}\right|^2\,([2,q]!)^2\,[\kappa+2,q]b_2^2}{3\left|2+e^{i\varpi}\right|\,[3,q]!\,[\kappa+1,q]b_3}\leq\vartheta<1\end{cases}, \tag{2.9}$$

$$|a_3|\leq\frac{2\,(1-\vartheta)\,[\kappa+1,q]_2}{3\,|2+e^{i\varpi}|\,[3,q]!b_3}, \tag{2.10}$$

and

$$\left|a_3-2a_2^2\right|\leq\frac{2\,(1-\vartheta)\,[\kappa+1,q]_2}{3\,|2+e^{i\varpi}|\ \ [3,q]!\,b_3}. \tag{2.11}$$

Proof. Fixing $m=2$ and $m=3$ in (2.5), (2.6), we have

$$\left(3+e^{i\varpi}\right)\ \frac{[2,q]!}{[\kappa+1,q]}\,b_2a_2=(1-\vartheta)\,c_1, \tag{2.12}$$

$$3\left(2+e^{i\varpi}\right)\ \frac{[3,q]!}{[\kappa+1,q]_2}\,b_3a_3=(1-\vartheta)\,c_2, \tag{2.13}$$

$$-\left(3+e^{i\varpi}\right)\ \frac{[2,q]!}{[\kappa+1,q]}\,b_2a_2=(1-\vartheta)\,d_1, \tag{2.14}$$

and

$$-\,3\left(2+e^{i\varpi}\right)\ \frac{[3,q]!}{[\kappa+1,q]_2}\,b_3\left(2a_2^2-a_3\right)=(1-\vartheta)\,d_2. \tag{2.15}$$

From (2.12) and (2.14), by using the Caratheodory Lemma1, we obtain

$$
\begin{aligned}
|a_2| &= \frac{(1-\vartheta)[\kappa+1,q]\,|c_1|}{|3+e^{i\varpi}|\,[2,q]!b_2} = \frac{(1-\vartheta)[\kappa+1,q]\,|d_1|}{|3+e^{i\varpi}|\,[2,q]!b_2} \\
&\leq \frac{2(1-\vartheta)[\kappa+1,q]}{|3+e^{i\varpi}|\,[2,q]!b_2}. \qquad (2.16)
\end{aligned}
$$

Also, from (2.13) and (2.15), we have

$$6\left(2+e^{i\varpi}\right)\frac{[3,q]!}{[\kappa+1,q]_2}\,b_3a_2^2 = (1-\vartheta)(c_2+d_2),$$

$$a_2^2 = \frac{(1-\vartheta)[\kappa+1,q]_2}{6(2+e^{i\varpi})[3,q]!\,b_3}(c_2+d_2), \qquad (2.17)$$

and by using the Caratheodory Lemma 1, we obtain

$$|a_2| \leq \sqrt{\frac{2(1-\vartheta)[\kappa+1,q]_2}{3|2+e^{i\varpi}|\ [3,q]!\,b_3}}. \qquad (2.18)$$

From (2.16) and (2.18), we obtain the desired estimate on the coefficient as asserted in (2.9).

To find the bound on the coefficient $|a_3|$, we subtract (2.15) from (2.13). we get

$$6\left(2+e^{i\varpi}\right)\frac{[3,q]!}{[\kappa+1,q]_2}\,b_3\left(a_3-a_2^2\right) = (1-\vartheta)(c_2-d_2),$$

or

$$a_3 = a_2^2 + \frac{(1-\vartheta)(c_2-d_2)[\kappa+1,q]_2}{6(2+e^{i\varpi})[3,q]!b_3}, \qquad (2.19)$$

substituting the value of a_2^2 from (2.12) into (2.19), we obtain

$$a_3 = \frac{(1-\vartheta)^2[\kappa+1,q]^2c_1^2}{(3+e^{i\varpi})^2([2,q]!)^2\,b_2^2} + \frac{(1-\vartheta)(c_2-d_2)[\kappa+1,q]_2}{6(2+e^{i\varpi})[3,q]!b_3}.$$

Using the Caratheodory Lemma 1, we find that

$$|a_3| \leq \frac{4(1-\vartheta)^2[\kappa+1,q]^2}{|3+e^{i\varpi}|^2([2,q]!)^2\,b_2^2} + \frac{2(1-\vartheta)[\kappa+1,q]_2}{3|2+e^{i\varpi}|[3,q]!b_3}, \qquad (2.20)$$

and from (2.13), we have

$$a_3 = \frac{(1-\vartheta)[\kappa+1,q]_2\,c_2}{3(2+e^{i\varpi})[3,q]!b_3}.$$

Appling the Caratheodory Lemma 1, we obtain

$$|a_3| \leq \frac{2(1-\vartheta)[\kappa+1,q]_2}{3|2+e^{i\varpi}|[3,q]!b_3}. \qquad (2.21)$$

Combining (2.20) and (2.21), we have the desired estimate on the coefficient $|a_3|$ as asserted in (2.10).

Finally, from (2.15), we deduce that

$$\left|a_3-2a_2^2\right| \leq \frac{(1-\vartheta)[\kappa+1,q]_2\,|d_2|}{3|2+e^{i\varpi}|\ [3,q]!\,b_3} = \frac{2(1-\vartheta)[\kappa+1,q]_2}{3|2+e^{i\varpi}|\ [3,q]!\,b_3}.$$

Thus the proof of Theorem 3 was completed.

□

3. Fekete-Szegö inequality for $f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$

Fekete and Szegö [52] introduced the generalized functional $|a_3 - \aleph a_2^2|$, where $\aleph$ is some real number. Due to Zaprawa [53], (also see [54]) in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$.

Theorem 4. *Let the function f given by* (1.1) *belongs to the class $\mathcal{M}_{\Sigma}^{q,\kappa}(\varpi, \vartheta; h)$ and $\aleph \in \mathbb{R}$. Then we have*

$$|a_3 - \aleph a_2^2| \leq \left(\frac{(1-\vartheta)\,[\kappa+1, q]_2}{3\,|2+e^{i\varpi}|\,[3, q]!\,b_3}\right)\{|2-\aleph| + |\aleph|\}.$$

Proof. From (2.17) and (2.19)we obtain

$$\begin{aligned} a_3 - \aleph a_2^2 &= \frac{(1-\aleph)\,(1-\vartheta)\,[\kappa+1, q]_2}{6\,(2+e^{i\varpi})\,[3, q]!\,b_3}\,(c_2+d_2) \\ &+ \frac{(1-\vartheta)\,[\kappa+1, q]_2}{6\,(2+e^{i\varpi})\,[3, q]!b_3}\,(c_2-d_2)\,, \\ &= \left(\frac{(1-\vartheta)\,[\kappa+1, q]_2}{6\,(2+e^{i\varpi})\,[3, q]!\,b_3}\right)\{[(1-\aleph)+1]c_2 + [(1-\aleph)-1]d_2\}. \end{aligned}$$

So we have

$$a_3 - \aleph a_2^2 = \left(\frac{(1-\vartheta)\,[\kappa+1, q]_2}{6\,(2+e^{i\varpi})\,[3, q]!\,b_3}\right)\{(2-\aleph)c_2 + (-\aleph)d_2\}. \tag{3.1}$$

Then, by taking modulus of (3.1), we conclude that

$$|a_3 - \aleph a_2^2| \leq \left(\frac{(1-\vartheta)\,[\kappa+1, q]_2}{3\,|2+e^{i\varpi}|\,[3, q]!\,b_3}\right)\{|2-\aleph| + |\aleph|\}$$

□

Taking $\aleph = 1$, we have the following result.

$$|a_3 - a_2^2| \leq \frac{2\,(1-\vartheta)\,[\kappa+1, q]_2}{3\,|2+e^{i\varpi}|\,[3, q]!\,b_3}.$$

4. Conclusions and observations

In the current paper, we mainly get upper bounds of the initial Taylors coefficients of bi-univalent functions related with $q-$ calculus operator. By fixing b_m as demonstrated in Remark 1, one can effortlessly deduce results correspondents to Theorems 2 and 3 associated with various operators listed in Remark 1. Further allowing $q \to 1^-$ as itemized in Remark 2 we can outspread the results for new subclasses stated in Remark 2. Moreover by fixing $\varpi = 0$ and $\varpi = \pi$ in Theorems 2 and 3, we can easily state the results for $f \in \mathcal{M}_{\Sigma}^{q,\kappa}(\vartheta; h)$ and $f \in \mathcal{HM}_{\Sigma}^{q,\kappa}(\vartheta; h)$. Further by suitably fixing the parameters in Theorem 4, we can deduce Fekete-Szegö functional for these function classes.

By using the subordination technique, we can extend the study by defining a new class

$$\left[\left(\mathcal{H}_h^{\kappa,q} f(z)\right)' + \left(\frac{1+e^{i\varpi}}{2}\right) z \left(\mathcal{H}_h^{\kappa,q} f(z)\right)''\right] \prec \Psi(z)$$

where $\Psi(z)$ the function Ψ is an analytic univalent function such that $\Re\ (\Psi) > 0$ in Δ with $\Psi(0) = 1$, $\Psi'(0) > 0$ and Ψ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis and is given by $\Psi(z) = z + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0)$. Also, motivating further researches on the subject-matter of this,we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications (see, for example, [1, 2, 4]). and developments in the area of mathematical analysis. In conclusion,we choose to reiterate an important observation, which was presented in the recently-published review-cum-expository review article by Srivastava ([1], p. 340), who pointed out the fact that the results for the above-mentioned or new $q-$ analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called $(p; q)-$analogues(with $0 < |q| < p \leq 1$)by applying some obvious parametric and argument variations with the additional parameter p being redundant.

Acknowledgments

The researcher(s) would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.The authors are grateful to the reviewers for their valuable remarks, comments, and advices that help us to improve the quality of the paper.

Conflict of interest

The authors declare that they have no competing interests.

References

1. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. doi: 10.1007/s40995-019-00815-0.
2. H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian hypergeometric series*, New York: Wiley, 1985.
3. H. M. Srivastava, Certain q-polynomial expansions for functions of several variables. II, *IMA J. Appl. Math.*, **33** (1984), 205–209. doi: 10.1093/imamat/33.2.205.
4. H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: H. M. Srivastava, S. Owa, *Univalent functions, fractional calculus, and their applications*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989, 329–354.
5. F. H. Jackson, On q-functions and a certain difference operator, *Trans. Royal Soc. Edinburgh*, **46** (1909), 253–281, doi: 10.1017/S0080456800002751.
6. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193–203.
7. M. H. Abu Risha, M. H. Annaby, Z. S. Mansour, M. E. H. Ismail, Linear q-difference equations, *Z. Anal. Anwend.*, **26** (2007), 481–494.
8. F. M. Sakar, M. Naeem, S. Khan, S. Hussain, Hankel determinant for class of analytic functions involving Q-derivative operator, *J. Adv. Math. Stud.*, **14** (2021), 265–278.

9. T. Bulboacă, *Differential subordinations and superordinations*, Recent Results, House of Scientific, Cluj-Napoca, 2005.
10. S. S. Miller, P. T. Mocanu, *Differential subordinations: Theory and applications*, CRC Press, 2000.
11. P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, New York: Springer, 1983.
12. M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63–68. doi: 10.2307/2035225.
13. D. A. Brannan, J. Clunie, *Aspects of contemporary complex analysis*, Academic Press, 1980.
14. E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Ration. Mech. Anal.*, **32** (1969), 100–112. doi: 10.1007/BF00247676.
15. H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat*, **29** (2015), 1839–1843.
16. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), 1188–1192. doi: 10.1016/j.aml.2010.05.009.
17. D. A. Brannan, J. Clunie, W. E. Kirwan, Coefficient estimates for a class of starlike functions, *Can. J. Math.*, **22** (1970), 476–485.
18. D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, *Math. Anal. Appl.*, 1988, 53–60. doi: 10.1016/B978-0-08-031636-9.50012-7.
19. M. Çaglar, H. Orhan, N. Yagmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat*, **27** (2013), 1165–1171.
20. H. M. Srivastava, S. Bulut, M. Caglar, N. Yagmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat*, **27** (2013), 831–842.
21. G. Faber, Ů̊ber polynomische Entwickelungen, *Math. Ann.*, **57** (1903), 389–408. doi: 10.1007/BF01444293.
22. P. G. Todorov, On the Faber polynomials of the univalent functions of class Σ, *J. Math. Anal. Appl.*, **162** (1991), 268–276. doi: 10.1016/0022-247X(91)90193-4.
23. S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, *C. R. Math.*, **352** (2014), 17–20. doi: 10.1016/j.crma.2013.11.005.
24. S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by subordinations, *Bull. Iran. Math. Soc.*, **41** (2015), 1103–1119.
25. S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient of bi-subordinate functions, *C. R. Math.*, **354** (2016), 365–370. doi: 10.1016/j.crma.2016.01.013.
26. H. M. Srivastava, G. Murugusundaramoorthy, S. M. El-Deeb, Faber polynomial coefficient estimates of bi-close-convex functions connected with the Borel distribution of the Mittag-Leffler type, *J. Nonlinear Var. Anal.*, **5** (2021), 103–118. doi: 10.23952/jnva.5.2021.1.07.
27. M. Naeem, S. Khan, F. M. Sakar, Faber polynomial coefficients estimates of bi-univalent functions, *Internat. J. Maps Math.*, **3** (2020), 57–67.

28. H. M. Srivastava, S. M. El-Deeb, The Faber polynomial expansion method and the Taylor-Maclaurin coefficient estimates of bi-close-to-convex functions connected with the q-convolution, *AIMS Math.*, **5** (2020), 7087–7106. doi: 10.3934/math.2020454.

29. H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, *Bull. Iran. Math. Soc.*, **44** (2018), 149–157. doi: 10.1007/s41980-018-0011-3.

30. H. M. Srivastava, A. Motamednezhad, E. A. Adegan, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, *Mathematics*, **8** (2020), 1–12. doi: 10.3390/math8020172.

31. H. M. Srivastava, F. M. Sakar, H. O. Güney, Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination, *Filomat*, **32** (2018), 1313–1322. doi: 10.2298/FIL1804313S.

32. H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babeş-Bolyai Math.*, **63** (2018), 419–436. doi: 10.24193/subbmath.2018.4.01.

33. S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions, *Filomat*, **30** (2016), 1567–1575.

34. G. Gasper, M. Rahman, *Basic hypergeometric series*, Encyclopedia of mathematics and its applications, Cambridge: Cambridge University Press, 1990.

35. S. M. El-Deeb, T. Bulboacă, B. M. El-Matary, Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative, *Mathematics*, **8** (2020), 1–14, doi: 10.3390/math8030418.

36. M. Arif, M. Ul Haq, J. L. Liu, A subfamily of univalent functions associated with q-analogue of Noor integral operator, *J. Funct. Spaces*, **2018** (2018), 1–5. doi: 10.1155/2018/3818915.

37. S. M. El-Deeb, T. Bulboacă, Fekete-Szegő inequalities for certain class of analytic functions connected with q-anlogue of Bessel function, *J. Egypt. Math. Soc.*, **27** (2019), 1–11. doi: 10.1186/s42787-019-0049-2.

38. S. M. El-Deeb, Maclaurin Coefficient estimates for new subclasses of bi-univalent functions connected with a q-analogue of Bessel function, *Abstr. Appl. Anal.*, **2020** (2020), 1–7. doi: 10.1155/2020/8368951.

39. S. M. El-Deeb, T. Bulboacă, Differential sandwich-type results for symmetric functions connected with a q-analog integral operator, *Mathematics*, **7** (2019), 1–17. doi: 10.3390/math7121185.

40. H. M. Srivastava, S. M. El-Deeb, A certain class of analytic functions of complex order with a q-analogue of integral operators, *Miskolc Math. Notes*, **21** (2020), 417–433. doi: 10.18514/MMN.2020.3102.

41. S. Porwal, An application of a Poisson distribution series on certain analytic functions, *J. Complex Anal.*, **2014** (2014), 1–3. doi: 10.1155/2014/984135.

42. J. K. Prajapat, Subordination and superordination preserving properties for generalized multiplier transformation operator, *Math. Comput. Model.*, **55** (2012), 1456–1465. doi: 10.1016/j.mcm.2011.10.024.
43. S. M. El-Deeb, T. Bulboaca, Differential sandwich-type results for symmetric functions associated with Pascal distribution series, *J. Contemp. Math. Anal.*, **56** (2021), 214–224. doi: 10.3103/S1068362321040105.
44. H. Aldweby, M. Darus, On a subclass of bi-univalent functions associated with the q-derivative operator, *J. Math. Comput. Sci.*, **19** (2019), 58–64. doi: 10.22436/jmcs.019.01.08.
45. M. Çaglar, E. Deniz, Initial coefficients for a subclass of bi-univalent functions defined by Sălăgean differential operator, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **66** (2017), 85–91.
46. S. Elhaddad, M. Darus, Coefficient estimates for a subclass of bi-univalent functions defined by q-derivative operator, *Mathematics*, **8** (2020), 306. doi: 10.3390/math8030306.
47. W. Zhi-Gang, S. Bulut, A note on the coefficient estimates of bi-close-to-convex functions, *C. R. Math.*, **355** (2017), 876–880.
48. P. N. Kamble, M. G. Shrigan, Coefficient estimates for a subclass of bi-univalent functions defined by Sălăgean type q-calculus operator, *Kyungpook Math. J.*, **58** (2018), 677–688, doi: 10.5666/KMJ.2018.58.4.677.
49. H. Silverman, E. M. Silvia, Characterizations for subclasses of univalent functions, *Math. Jpn.*, **50** (1999), 103–109.
50. H. Silverman, A class of bounded starlike functions, *Int. J. Math. Math. Sci.*, **17** (1994), 249–252. doi: 10.1155/S0161171294000360.
51. H. M. Srivastava, D. Raducanu, P. A. Zaprawa, Certain subclass of analytic functions defined by means of differential subordination, *Filomat*, **30** (2016), 3743–3757.
52. M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Functionen, *J. London Math. Soc.*, **8** (1933), 85–89. doi: 10.1112/jlms/s1-8.2.85.
53. P. Zaprawa, On the Fekete-Szegö problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **21** (2014), 169–178. doi: 10.36045/bbms/1394544302.
54. M. Hesam, Coefficient and Fekete-Szegö problem estimates for certain subclass of analytic and bi-univalent functions, *Filomat*, **34** (2020), 4637–4647. doi: 10.2298/FIL2014637M.

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)