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Research article

Stability in a Ross epidemic model with road diffusion

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Abstract: Reaction-diffusion equations have been used to describe the dynamical behavior of epidemic models, where the spreading of infectious disease has the same speed in every direction. A natural question is how to describe the dynamical system when the spreading of infectious disease is directed diffusion. We introduce the road diffusion into a Ross epidemic model which describes the spread of infected Mosquitoes and humans. With the comparison principle the system is proved to have a unique global solution. By the approach of upper and lower solutions, we show that the disease-free equilibrium is asymptotically stable if the basic reproduction number is lower than 1 while the endemic equilibrium asymptotically stable if the basic reproduction number is greater than 1.

Keywords: epidemic model; basic reproduction number; asymptotic stability; road diffusion
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1. Introduction

Mosquito was firstly discovered a transmission vector of Malaria by Ronald Ross in 1898. Ross was also the first to use differential equation to study epidemiology. In order to build an intra-host epidemic model, the mosquito population is divided into two compartments: healthy mosquitoes (susceptible mosquitoes) and infected mosquitoes, while the human population is divided into two compartments: susceptible humans and infected humans. He constructed a two-component system composed with the infected humans u and the infectious mosquitoes v as the following form:

$$\begin{cases} \frac{du}{dt} = mab_1v(1-u) - \gamma_1 u, \\ \frac{dv}{dt} = ab_2(1-v)u - \gamma_2 v. \end{cases} \tag{1.1}$$

He proposed out the threshold (the predecessor of basic reproduction number) to determine the asymptotic stability of the disease-free equilibrium and endemic equilibrium.

It is assumed that the densities of mosquitoes and humans are homogeneous in the above model. The heterogeneous spatial distribution has formed a reasonable basis for studying insect dispersal (Lewis [1]). For example, in 1986 Aedes albopictus was found for the first time in northern counties in Florida (Peacock et al. [2]). Then it kept spreading southward, slowly but steadily, and had spread all over the 67 counties (O'Meara et al. [3]) in six years' time. By 2008, Aedes albopictus had spread over to 36 states and was still continuing its expansion (Hawley et al. [4], Enserink [5], and Hahnet et al. [6]). In 2013, Rochlin et al. [7] predicted that, especially in urban areas, in the next 20 years, Aedes albopictus population, will be over three times more. A recent survey on the current distribution of Aedes albopictus (Parker et al. [8]) shows that Aedes albopictus have been tracked in 56 of all 67 Florida counties. The mosquito of Aedes albopictus had also been reported to spread along roads. Bennett et al. [9] found that the infestation rate of Aedes albopictus was high in garages trading used tires along the highways, and this road provided a channel for fast dispersal across Panama.

Considering the effect of highways on the spreading of mosquito, Berestycki, Roquejoffre and Rossi [10] introduced the road diffusion into a reaction-diffusion system. They proposed the model where a two-dimensional environment contains a "line" inside which fast diffusion occurs, while reproduction and the usual diffusion occur only outside. Once "plane" is a field and "line" is a road, this system is a combination of the density of the field with that of the road, between which exists a population exchange satisfied Fickian conservation law. Berestycki et al. have studied the qualitative properties of road-field system (see [11, 12]). Since the fact the motion of mosquitoes and humans obey the Gaussian distribution, we take the effect of the dispersal into account by using a Laplacian diffusion. Moreover, we study the spread of mosquitoes along the highways with the help of the road diffusion. Thus we improve Ross epidemic model as the form:

$$\begin{cases} \partial_t u - d_1 \Delta u = mab_1 v(1-u) - \gamma_1 u, & (x,y) \in \mathbb{R} \times \mathbb{R}^+, \\ \partial_t v - d_2 \Delta v = ab_2(1-v)u - \gamma_2 v, & (x,y) \in \mathbb{R} \times \mathbb{R}^+, \\ \partial_t w - D\partial_{xx} w = \nu v|_{y=0} - \mu w, & x \in \mathbb{R}, \\ -d_2 \partial_y v|_{y=0} = \mu w - \nu v|_{y=0}, & x \in \mathbb{R}, \\ \partial_y u|_{y=0} = 0, & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

where $u(x, y, t)$ represents the density of the infected humans, $v(x, y, t)$ represents the density of the infected mosquitoes, and $w(x, t)$ represents the density of the infected humans living on the road. Meanwhile, $\mathbb{R} \times \mathbb{R}^+$ and $\mathbb{R}$ represent field and road respectively. The first equation accounts for the infected humans dynamic on the field, the second and third for the infected mosquitoes dynamic in the field and road respectively, and the fourth for the exchanges between the field and the road. d_1, d_2 and D denote the diffusion rates. ν represents the ratio moving from field to road, μ for the ratio moving from road to field. m is the rate of the bitten mosquitoes over the whole mosquito community. a is the number of the humans bitted by a mosquito per unit of time. b_1 is the probability of a susceptible hunman becoming infectious after a bite of an infected mosquito. b_2 is the probability of a susceptible mosquito becoming infectious after it bites a infectd human. γ_1 and γ_2 are the infection cycles of the mosquitos and humans respectively. Mosquito and human infectious individual remains infectious on a mean time $\frac{1}{\gamma_1}$ and $\frac{1}{\gamma_2}$ respectively. Epidemic models without road diffusion have been applied to study the spreading of infectious disease through the investigation of asymptotic behavior, seeing for example Allen et al. [13, 14], Yang et al. [15], Lin and Zhu [16], Li et al. [17], Lei et al. [18], and the references therein. Besides the classical Laplacian diffusion in (1.2), the new generalized fractional derivative has been presented in [19–21] to study the memory effect of epidemic model dynamics. Recently, some important real examples of epidemic model dynamics have been proposed in [22–24].

This paper is mainly aimed to study the existence and stability of the solutions to system (1.2). Section 2 proves the global existence and uniqueness of solutions to the road diffusion problem (1.2). Section 3 investigates the asymptotic stabilities of the disease-free equilibrium and endemic equilibrium based on the theory of the basic reproduction number. Section 4 gives out some discussions and conclusions.

2. Existence and uniqueness

We use the approach of upper and lower solutions to study the existence and uniqueness of solutions.

Definition 2.1. Suppose that $\tilde{\mathbf{u}} := (\tilde{u}, \tilde{v}, \tilde{w})$ and $\hat{\mathbf{u}} := (\hat{u}, \hat{v}, \hat{w})$ is continuous for $t \in [0, \infty)$. $\hat{\mathbf{u}}$ is called a *lower solution* of system (1.2) if it satisfies (1.2) with the = signs replaced by $\leq$ signs. Similarly, $\tilde{\mathbf{u}}$ is called a *upper solution satisfies* (1.2) *with the = signs replaced by* $\geq$ *signs.*

In order to show the uniqueness, we need the following comparison principle.

Theorem 2.1. ***(Comparison principle)*** *If nonnegative* $(\tilde{u}, \tilde{v}, \tilde{w})$ *and* $(\hat{u}, \hat{v}, \hat{w})$ *are respectively a upper solution and lower solution of system (1.2) satisfying* $\hat{u} \leq \tilde{u} \leq 1, \hat{v} \leq \tilde{v} \leq 1, \hat{w} \leq \tilde{w} \leq 1$ *at* $t = 0$. *Then* $\hat{u} \leq \tilde{u}, \hat{v} \leq \tilde{v}, \hat{w} \leq \tilde{w}$ *for all* $t > 0$.

Proof. For $K > 0$, we define functions $(\underline{u}, \underline{v}, \underline{w}) = (\hat{u}, \hat{v}, \hat{w})e^{-Kt}, (\overline{u}, \overline{v}, \overline{w}) = (\tilde{u}, \tilde{v}, \tilde{w})e^{-Kt}$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function satisfying:

$$\chi(x) = 0 \text{ in } x \in [0, 2x_0],\ \chi'' \leq \min\{\frac{1}{2}, \frac{1}{2d_1}, \frac{1}{2d_2}, \frac{1}{D}\} \text{ in } \mathbb{R}, \tag{2.1}$$

where χ'' is the second derivative of χ, $x_0 = \max\{x_1, x_2\}$, here x_1 and x_2 are determined in the following (2.8) and (2.12).

For $\varepsilon > 0$, set

$$\begin{aligned} \breve{u}(x, y, t) &:= \overline{u}(x, y, t) + \frac{\mu}{\nu}\varepsilon(\chi(|x|) + \chi(y) + t + 1), \\ \breve{v}(x, y, t) &:= \overline{v}(x, y, t) + \frac{\mu}{\nu}\varepsilon(\chi(|x|) + \chi(y) + t + 1), \\ \breve{w}(x, t) &:= \overline{w}(x, t) + \varepsilon(\chi(|x|) + t + 1). \end{aligned} \tag{2.2}$$

It is easy to verify that $(\breve{u}, \breve{v}, \breve{w})$ is strictly above $(\underline{u}, \underline{v}, \underline{w})$ at $t = 0$. In order to show $\hat{u} \leq \tilde{u}, \hat{v} \leq \tilde{v}, \hat{w} \leq \tilde{w}$ for all $t > 0$, we need to show that $\underline{u} < \breve{u}, \underline{v} < \breve{v}, \underline{w} < \breve{w}$ for all $t > 0$ owing to the arbitrariness of ε. Assume by contradiction that this property is not true for all $t > 0$. Then,

$$T = \sup\{\tau \geq 0, \underline{u} \leq \breve{u} \in \mathbb{R} \times \mathbb{R}^+ \times [0, \tau], \underline{v} \leq \breve{v} \in \mathbb{R} \times \mathbb{R}^+ \times [0, \tau], \underline{w} \leq \breve{w} \in \mathbb{R} \times [0, \tau]\} \in (0, +\infty).$$

According to the continuity of the functions, $\breve{u}$, $\breve{v}$ and $\breve{w}$ tend to $+\infty$ as the space variable goes to infinity, uniformly in time, implies that $T > 0$ and $\breve{u} - \underline{u} = 0$ or $\breve{v} - \underline{v} = 0$ or $\breve{w} - \underline{w} = 0$ at time T. By choosing

$$K = 2mab_1 + 2ab_2, \tag{2.3}$$

we argue differently depending on $\check{u} - \underline{u} = 0$ or $\check{v} - \underline{v} = 0$ or $\check{w} - \underline{w} = 0$ at time T.

Case 1. In the case $\min_{\mathbb{R}}(\check{w} - \underline{w})(\cdot, T) = 0$. In view of the definition of $\overline{w}$, we have

$$\partial_t \overline{w} - D\partial_{xx}\overline{w} + (K+\mu)\overline{w} \geq \nu \overline{v}|_{y=0}.$$

Applying (2.2) to the above inequality, we have

$$\begin{aligned}(\partial_t - D\partial_{xx} + (K+\mu))\,\check{w} &= (\partial_t - D\partial_{xx} + (K+\mu))\,\overline{w} + \varepsilon\left(1 - D\chi''(|x|) + (K+\mu)(\chi(|x|) + t + 1)\right)\\ &\geq \nu\overline{v}|_{y=0} + \varepsilon\left(1 - D\chi''(|x|) + (K+\mu)(\chi(|x|) + t + 1)\right)\\ &\geq \nu\overline{v}|_{y=0} + \varepsilon(K+\mu)(\chi(|x|) + t + 1),\end{aligned}$$

where the last inequality comes from (2.1).

By (2.2), it follows from the above inequality that

$$\begin{aligned}(\partial_t - D\partial_{xx} + (K+\mu))\,\check{w} \;&\geq\; \nu\check{v}|_{y=0} + \varepsilon(K+\mu)(\chi(|x|) + t + 1) - \mu\varepsilon(\chi(|x|) + t + 1)\\ &\geq\; \nu\check{v}|_{y=0}.\end{aligned} \tag{2.4}$$

On the other hand, we have

$$(\partial_t - D\partial_{xx} + (K+\mu))\,\underline{w} \leq \nu\underline{v}|_{y=0}. \tag{2.5}$$

Combining (2.4) and (2.5), for $x \in \mathbb{R}$ and $t \in (0, T]$, we have

$$(\partial_t - D\partial_{xx} + (K+\mu))\,(\check{w} - \underline{w}) \geq \nu\check{v}|_{y=0} - \nu\underline{v}|_{y=0} \geq 0.$$

Using the parabolic strong maximum principle, we can yield that $\min_{\mathbb{R}}(\check{w} - \underline{w}) = 0$ in $\mathbb{R} \times [0, T]$, which is contrary to $\check{w}(x,0) - \underline{w}(x,0) > 0$. Thus $\min_{\mathbb{R}}(\check{w} - \underline{w})(\cdot, T) = 0$ is not true.

Case 2. In the case $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{u} - \overline{u})(\cdot, T) = 0$. By the definition of $\underline{u}$ and $\overline{u}$, we have

$$\begin{aligned}&(\partial_t - d_1\Delta + K)\,\underline{u} \leq e^{-Kt}(mab_1\hat{v}(1-\hat{u}) - \gamma_1\hat{u}),\\ &(\partial_t - d_1\Delta + K)\,\overline{u} \geq e^{-Kt}(mab_1\tilde{v}(1-\tilde{u}) - \gamma_1\tilde{u}).\end{aligned} \tag{2.6}$$

By (2.2), it follows from (2.6) that

$$\begin{aligned}(\partial_t - d_1\Delta + K)\,\check{u} &= (\partial_t - d_1\Delta + K)\,\overline{u} + \frac{\mu}{\nu}\varepsilon\left(1 - d_1\chi''(|x|) - d_1\chi''(y) + K(\chi(|x|) + \chi(y) + t + 1)\right)\\ &\geq (\partial_t - d_1\Delta + K)\,\overline{u} + \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1)\\ &\geq e^{-Kt}(mab_1\tilde{v}(1-\tilde{u}) - \gamma_1\tilde{u}) + \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1),\end{aligned}$$

where the second inequality is induced from (2.1).

Thus we have

$$\begin{aligned}(\partial_t - d_1\Delta + K)\,(\check{u} - \underline{u}) &\geq \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1)\\ &\qquad + e^{-Kt}[mab_1\tilde{v}(1-\tilde{u}) - \gamma_1\tilde{u} - (mab_1\hat{v}(1-\hat{u}) - \gamma_1\hat{u})],\\ &= \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1)\\ &\qquad + e^{-Kt}(-\gamma_1 - mab_1\tilde{v})(\tilde{u} - \hat{u}) + e^{-Kt}(mab_1 - mab_1\hat{u})(\tilde{v} - \hat{v}),\\ &= \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1)\\ &\qquad + (-\gamma_1 - mab_1\tilde{v})(\overline{u} - \underline{u}) + (mab_1 - mab_1\hat{u})(\overline{v} - \underline{v}),\\ &\geq (-\gamma_1 - mab_1\tilde{v})(\check{u} - \underline{u}) + (mab_1 - mab_1\hat{u})(\check{v} - \underline{v}),\end{aligned}$$

where the last inequality is due to the definition of K from (2.3).

Owing to $\hat{u} \le 1$, we have

$$(\partial_t - d_1\Delta + K + \gamma_1 + mab_1\tilde{v})(\check{u} - \underline{u}) \ge (mab_1 - mab_1\hat{u})(\check{v} - \underline{v}) \ge 0, \tag{2.7}$$

for $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ and $t \in (0, T]$. By applying the parabolic strong maximum principle, $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{u} - \underline{u})(\cdot, T) = 0$ must be attained on the boundary of $\mathbb{R} \times \mathbb{R}^+$. Thus we have

$$(\check{u} - \underline{u})(x_1, 0, T) = 0, \text{ for some } x_1 \in \mathbb{R}. \tag{2.8}$$

Moreover, we have

$$\begin{aligned}
(\check{u} - \underline{u})(x_1, 0, T) &\ge (\overline{u} - \underline{u})(x_1, 0, T) = e^{-Kt}(\tilde{u} - \hat{u})(x_1, 0, T) \\
&\ge \frac{e^{-Kt}}{\nu}[\partial_y(\tilde{u}(x_1, 0, T) - \hat{u}(x_1, 0, T)) + \mu(\tilde{w}(x_1, 0, T) - \hat{w}(x_1, 0, T)] \\
&\ge \frac{e^{-Kt}}{\nu}\partial_y(\tilde{u}(x_1, 0, T) - \hat{u}(x_1, 0, T)) \\
&= \frac{1}{\nu}\partial_y(\overline{u}(x_1, 0, T) - \underline{u}(x_1, 0, T)) \\
&= \frac{1}{\nu}\partial_y(\check{u}(x_1, 0, T) - \underline{u}(x_1, 0, T)) > 0,
\end{aligned} \tag{2.9}$$

where the last inequality is by using the Hopf lemma.

This contradiction of (2.8) and (2.9) implies that $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{u} - \underline{u})(\cdot, T) = 0$ is not true.

Case 3. In the case $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{v} - \underline{v})(\cdot, T) = 0$. Directly calculations show that

$$\begin{aligned}
(\partial_t - d_2\Delta + K)\,\underline{v} &\le e^{-Kt}(ab_2(1 - \hat{v})\hat{u} - \gamma_2\hat{v}), \\
(\partial_t - d_2\Delta + K)\,\overline{v} &\ge e^{-Kt}(ab_2(1 - \tilde{v})\tilde{u} - \gamma_2\tilde{v}).
\end{aligned} \tag{2.10}$$

By (2.2), it follows from (2.10) that

$$\begin{aligned}
(\partial_t - d_2\Delta + K)\,\check{v} &= (\partial_t - d_2\Delta + K)\,\overline{v} + \frac{\mu}{\nu}\varepsilon\,(1 - d_2\chi''(|x|) - d_2\chi''(y) + K(\chi(|x|) + \chi(y) + t + 1)) \\
&\ge (\partial_t - d_2\Delta + K)\,\overline{v} + \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1) \\
&\ge e^{-Kt}(ab_2(1 - \tilde{v})\tilde{u} - \gamma_2\tilde{v}) + \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1),
\end{aligned}$$

where the second inequality is induced from (2.1). Inserting (2.10) into the above inequality, we have

$$\begin{aligned}
(\partial_t - d_2\Delta + K)(\check{v} - \underline{v}) &\ge \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1) \\
&+ e^{-Kt}[ab_2(1 - \tilde{v})\tilde{u} - \gamma_2\tilde{v} - (ab_2(1 - \hat{v})\hat{u} - \gamma_2\hat{v})] \\
&= \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1) \\
&+ e^{-Kt}[(-\gamma_2 - ab_2\tilde{u})(\tilde{v} - \hat{v}) + (ab_2 - ab_2\hat{v})(\tilde{u} - \hat{u})] \\
&= \frac{\mu}{\nu}\varepsilon K(\chi(|x|) + \chi(y) + t + 1)
\end{aligned}$$

$$\begin{aligned} &+ \quad [(-\gamma_2 - ab_2\tilde{u})(\overline{v} - \underline{v}) + (ab_2 - ab_2\hat{v})(\overline{u} - \underline{u})] \\ &\geq \quad (-\gamma_2 - ab_2\tilde{u})(\check{v} - \underline{v}) + (ab_2 - ab_2\hat{v})(\check{u} - \underline{u}), \end{aligned}$$

where the last inequality is because of the definition of K from (2.3).

Owing to $\hat{v} \leq 1$, we have

$$(\partial_t - d_2\Delta + K + \gamma_2 + ab_2\tilde{u})\,(\check{v} - \underline{v}) \geq (ab_2 - ab_2\hat{v})(\check{u} - \underline{u}) \geq 0, \tag{2.11}$$

for $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ and $t \in (0, T]$. By applying the parabolic strong maximum principle, $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{v} - \underline{v})(\cdot, T) = 0$ must be attained on the boundary of $\mathbb{R} \times \mathbb{R}^+$. Thus we have

$$(\check{v} - \underline{v})(x_2, 0, T) = 0, \text{ for some } x_2 \in \mathbb{R}. \tag{2.12}$$

Using the Hopf lemma yields that

$$\partial_y(\check{v} - \underline{v})|_{y=0}(x_2, 0, T) > 0, \text{ for some } x_2 \in \mathbb{R}. \tag{2.13}$$

In view of (2.2) and (2.12), we have

$$\begin{aligned} (\tilde{v} - \hat{v})(x_2, 0, T) &= e^{KT}(\overline{v} - \underline{v})(x_2, 0, T), \\ &= e^{KT}(\check{v} - \underline{v})(x_2, 0, T) - \frac{\mu e^{KT}}{\nu}\varepsilon(\chi(|x_2|) + T + 1). \end{aligned} \tag{2.14}$$

In view of (2.2), we have

$$\begin{aligned} \partial_y|_{y=0}(\tilde{v} - \hat{v})(x_2, 0, T) &= e^{KT}\partial_y|_{y=0}(\check{v} - \underline{v})(x_2, 0, T) - \frac{\mu e^{KT}\varepsilon}{\nu}\partial_y|_{y=0}(\chi(|x_2|) + \chi(y) + T + 1), \\ &= e^{KT}\partial_y|_{y=0}(\check{v} - \underline{v})(x_2, 0, T) > 0, \end{aligned} \tag{2.15}$$

where the last inequality is because of the definition of χ in (2.1). The conclusion (2.15) contradicts to $\partial_y\tilde{v}|_{y=0} \leq 0 \leq \partial_y\hat{v}|_{y=0}$. Thus $\min_{\mathbb{R}\times\mathbb{R}^+}(\check{v} - \underline{v})(\cdot, T) = 0$ is not true.

In the above three cases, we all reached a contradiction. This completes the proof. □

As for a given pair of coupled upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, we denote

$$\Lambda := \{\mathbf{u} \in C(E^*) \times C(E^*) \times C(E^{**}) : \ \hat{u} \leq u \leq \tilde{u}, \ \hat{v} \leq v \leq \tilde{v}, \ \hat{w} \leq w \leq \tilde{w}\}. \tag{2.16}$$

where

$$E^* := \{(t, x, y) : \ t \in (0, \infty), \ (x, y) \in \mathbb{R} \times \mathbb{R}^+\}, \ E^{**} := \{(t, x) : \ t \in (0, \infty), \ x \in \mathbb{R}\}. \tag{2.17}$$

We denote the functions of system (1.2) by

$$f_1(u, v) := mab_1v(1 - u) - \gamma_1 u, \ f_2(u, v) := ab_2(1 - v)u - \gamma_2 v, \ g(v, w) := \nu v|_{y=0} - \mu w. \tag{2.18}$$

We denote the linear operators of system (1.2) by

$$\mathcal{L}_1(u) := \partial_t u - d_1\Delta u,\ \mathcal{L}_2(v) := \partial_t v - d_2\Delta v,$$
$$\mathcal{L}_3(w) := \partial_t w - D\partial_{xx}w,\ \mathcal{L}_4(v) := -d_2\partial_y v|_{y=0},\ \mathcal{L}_5(u) := -\partial_y u|_{y=0}. \tag{2.19}$$

By using $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ and $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ as the initial iterations we can construct sequences $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ and $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ satisfying the same initial functions from the iteration process of scalar equations

$$\begin{cases} \mathcal{L}_1(\overline{u}^{(m)}) + K_1\overline{u}^{(m)} = f_1(\overline{u}^{(m-1)}, \overline{v}^{(m-1)}) + K_1\overline{u}^{(m-1)}, & \text{in } E^*, \\ \mathcal{L}_2(\overline{v}^{(m)}) + K_2\overline{v}^{(m)} = f_2(\overline{u}^{(m-1)}, \overline{v}^{(m-1)}) + K_2\overline{v}^{(m-1)}, & \text{in } E^*, \\ \mathcal{L}_3(\overline{w}^{(m)}) + \mu\overline{w}^{(m)} = g(\overline{v}^{(m-1)}, \overline{w}^{(m-1)}) + \mu\overline{w}^{(m-1)}, & \text{in } E^{**}, \\ (\mathcal{L}_4 + \nu)(\overline{v}^{(m)}) = -g(\overline{v}^{(m-1)}, \overline{w}^{(m-1)}) + \nu\overline{v}^{(m-1)}, & \text{in } E^{**}, \\ \mathcal{L}_5(\overline{u}^{(m)}) = 0, & \text{in } E^{**}. \end{cases} \tag{2.20}$$

Here we choose

$$K_1 = \gamma_1 + mab_1,\ K_2 = \gamma_2 + ab_2. \tag{2.21}$$

$\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ satisfies the above equation with the superscripts replaced by subscripts. Indeed, (2.20) is reduced to a linear parabolic equation with half-space homogeneous Neumamm condition with respect to u and v, and a Cauchy problem of linear parabolic equation with respect to w. Thus $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ are well-defined. By using the monotone dynamical system method (Smith [25]), we have the following lemma.

Lemma 2.1. *The sequences $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ governed by (2.20) possess the monotonicity property*

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}^{(\mathbf{m})} \le \underline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m})} \le \tilde{\mathbf{u}}\ \textit{for}\ m = 1, 2, \cdots \tag{2.22}$$

Moreover, for each $m = 1, 2, \cdots$, $\overline{\mathbf{u}}^{(\mathbf{m})}$ and $\underline{\mathbf{u}}^{(\mathbf{m})}$ are coupled upper and lower solutions of (1.2).

Proof. Let $\underline{s}^{(1)} = \underline{u}^{(1)} - \underline{u}^{(0)}$. We necessarily apply the comparison principle of standard parabolic equation to $\underline{s}^{(1)}$. By (2.20), $\underline{s}^{(1)}$ satisfies

$$\begin{cases} \mathcal{L}_1(\underline{s}^{(1)}) + K_1\underline{s}^{(1)} = f_1(\underline{u}^{(0)}, \underline{v}^{(0)}) + K_1\underline{u}^{(0)} - (\mathcal{L}_1(\underline{u}^{(0)}) + K_1\underline{u}^{(0)}) & \\ \qquad = -\mathcal{L}_1(\hat{u}) + f_1(\hat{u}, \hat{v}) \ge 0, & \text{in } E^*, \\ \mathcal{L}_5(\underline{s}^{(1)}) = 0, & \text{in } E^{**}, \\ \underline{s}^{(1)}(0, x, y) = 0, & \text{in } \mathbb{R} \times \mathbb{R}^+. \end{cases}$$

Using comparison principle of standard parabolic equation yields $\underline{s}^{(1)} \ge 0$. It follows that $\underline{u}^{(0)} \le \underline{u}^{(1)}$. On the other hand, we set $\underline{q}^{(1)} = \underline{v}^{(1)} - \underline{v}^{(0)}$. By (2.20), $\underline{q}^{(1)}$ satisfies

$$\begin{cases} \mathcal{L}_2(\underline{q}^{(1)}) + K_2\underline{q}^{(1)} = f_2(\underline{u}^{(0)}, \underline{v}^{(0)}) + K_2\underline{u}^{(0)} - (\mathcal{L}_2(\underline{v}^{(0)}) + K_2\underline{v}^{(0)}) & \\ \qquad = -\mathcal{L}_2(\hat{u}) + f_2(\hat{u}, \hat{v}) \ge 0, & \text{in } E^*, \\ (\mathcal{L}_4 + \nu)(\underline{q}^{(1)}) = \mu\underline{w}^{(0)} - \nu\underline{v}^{(0)}|_{y=0} + \nu\underline{v}^{(0)}_{y=0} - \mathcal{L}_4\hat{v} - \nu\hat{v}|_{y=0} \ge 0, & \text{in } E^{**}, \\ \underline{q}^{(1)}(0, x, y) = 0, & \text{in } \mathbb{R} \times \mathbb{R}^+. \end{cases}$$

the parabolic comparison yields $\underline{v}^{(0)} \le \underline{v}^{(1)}$. Meanwhile, using the similar process yields $\underline{w}^{(0)} \le \underline{w}^{(1)}$. Thus, we have $\underline{\mathbf{u}}^{(\mathbf{0})} \le \underline{\mathbf{u}}^{(\mathbf{1})}$. Likewise, we have $\overline{\mathbf{u}}^{(\mathbf{0})} \ge \overline{\mathbf{u}}^{(\mathbf{1})}$.

Letting $z^{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$, it follows from (2.21) that

$$\begin{cases} \mathcal{L}_1(z^{(1)}) + K_1 z^{(1)} = f_1(\overline{u}^{(0)}, \overline{v}^{(0)}) + K_1\overline{u}^{(0)} - (f_1(\underline{u}^{(0)}, \underline{v}^{(0)}) + K_1\underline{u}^{(0)}) \ge 0, & \text{in } E^*, \\ (\mathcal{L}_4 + \nu)(z^{(1)}) = \mu(\overline{w}^{(1)} - \underline{w}^{(1)}) - \nu(\overline{u}^{(1)} - \underline{u}^{(1)})|_{y=0} + \nu(\overline{u}^{(1)} - \underline{u}^{(1)})|_{y=0} \ge 0, & \text{in } E^{**}, \\ z^{(1)}(0, x, y) = 0, & \text{in } \mathbb{R} \times \mathbb{R}^+. \end{cases}$$

It follows again from comparison principle that $\overline{u}^{(1)} \ge \underline{u}^{(1)}$, and thus we have $\underline{v}^{(1)} \le \overline{v}^{(1)}$. The above conclusions show that

$$\underline{\mathbf{u}}^{(0)} \le \underline{\mathbf{u}}^{(1)} \le \overline{\mathbf{u}}^{(1)} \le \overline{\mathbf{u}}^{(0)}. \tag{2.23}$$

Now we show that $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ are upper and lower solutions of (1.2).

Next we use an induction method. By choosing $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ as the coupled upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, after a similar argument as above, we have

$$\underline{\mathbf{u}}^{(1)} \le \underline{\mathbf{u}}^{(2)} \le \overline{\mathbf{u}}^{(2)} \le \overline{\mathbf{u}}^{(1)}, \tag{2.24}$$

so $\overline{\mathbf{u}}^{(2)}$ and $\underline{\mathbf{u}}^{(2)}$ are coupled upper and lower solutions of (1.2). The conclusion of the lemma follows from the induction principle. □

In view of Lemma 2.1, the pointwise limits

$$\lim_{m\to\infty} \overline{\mathbf{u}}^{(\mathbf{m})} = \overline{\mathbf{u}}, \quad \lim_{m\to\infty} \underline{\mathbf{u}}^{(\mathbf{m})} = \underline{\mathbf{u}} \tag{2.25}$$

exist. In the following theorem we show that $(\overline{u}, \overline{v}, \overline{w})$ and $(\underline{u}, \underline{v}, \underline{w})$ are respectively the maximal and minimal solutions of system (1.2).

Theorem 2.2. *Assume that* $\tilde{\mathbf{u}}$ *and* $\hat{\mathbf{u}}$ *be upper and lower solutions of system* (1.2). *Let* $(\overline{u}, \overline{v}, \overline{w})$ *and* $(\underline{u}, \underline{v}, \underline{w})$ *be given by* (2.25). *Then* $(\overline{u}, \overline{v}, \overline{w})$ *and* $(\underline{u}, \underline{v}, \underline{w})$ *are the solutions of system* (1.2). *Moreover,*

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}^{(\mathbf{m})} \le \underline{\mathbf{u}}^{(\mathbf{m+1})} \le \underline{\mathbf{u}} = \overline{\mathbf{u}} \le \overline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m})} \le \tilde{\mathbf{u}}. \tag{2.26}$$

Proof. (i) To show that the limit $(\overline{u}, \overline{v}, \overline{w})$ in (2.25) is the solution of (1.2), we use an integral representation for the solution of the Cauchy problem and the linear parabolic Eq (2.20) under Neumann boundary condition. Let $\mathcal{G}(t, x, y; \tau, \xi_1, \xi_2)$ be the Green function given by

$$\begin{aligned} \mathcal{G}_1(t, x, y; \tau, \xi_1, \xi_2) &:= \frac{e^{-K_1 t}}{4\pi d_1(t-\tau)} \left(e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4d_1(t-\tau)}} + e^{-\frac{(x-\xi_1)^2+(-y-\xi_2)^2}{4d_1(t-\tau)}} \right), \\ \mathcal{G}_2(t, x, y; \tau, \xi_1, \xi_2) &:= \frac{e^{-K_2 t}}{4\pi d_2(t-\tau)} \left(e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4d_2(t-\tau)}} + e^{-\frac{(x-\xi_1)^2+(-y-\xi_2)^2}{4d_2(t-\tau)}} \right), \\ \mathcal{G}_3(t, x; \tau, \xi) &:= \frac{e^{-\mu t}}{2\sqrt{\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \end{aligned}$$

We denote

$$
\begin{aligned}
I_1(t,x,y) &:= \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_1(t,x,y;0,\xi_1,\xi_2)u_0(\xi_1,\xi_2)d\xi_1 d\xi_2 \\
I_2(t,x,y) &:= \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_2(t,x,y;0,\xi_1,\xi_2)v_0(\xi_1,\xi_2)d\xi_1 d\xi_2 \\
I_3(t,x) &:= \int_{\mathbb{R}} \mathcal{G}_3(t,x;0,\xi)w_0(\xi)d\xi.
\end{aligned}
$$

By the integral representation of linear parabolic boundary value problems [26], the solution of (2.20) is given by

$$
\begin{aligned}
\overline{u}^{(m)}(t,x,y) =& I_1(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_1(t,x,y;\tau,\xi_1,\xi_2)\left(f_1(\overline{u}^{(m-1)},\overline{v}^{(m-1)}) + K_1\overline{u}^{(m-1)}\right)d\xi_1 d\xi_2 \\
\overline{v}^{(m)}(t,x,y) =& I_2(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_2(t,x,y;\tau,\xi_1,\xi_2)\left(f_2(\overline{u}^{(m-1)},\overline{v}^{(m-1)}) + K_2\overline{v}^{(m-1)}\right)d\xi_1 d\xi_2 \\
&- \frac{1}{d_2}\int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_2(t,x,0;\tau,\xi_1,0)\left(-g(\overline{v}^{(m-1)},\overline{w}^{(m-1)}) + \nu(\overline{v}^{(m-1)} - \overline{v}^{(m)})\right)d\xi_1 \\
\overline{w}^{(m)}(t,x) =& I_3(t,x) + \int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_3(t,x;\tau,\xi)\left(g(\overline{v}^{(m-1)},\overline{w}^{(m-1)}) + \mu\overline{w}^{(m-1)}\right)d\xi
\end{aligned}
$$

The integrands in the above equation are integrable because the definition of Green function $\mathcal{G}$ satisfies the admissible growth in the time variable. Using the dominated convergence theorem yields that the limits $(\overline{u},\overline{v},\overline{w})$ satisfy the relation

$$
\begin{aligned}
\overline{u}(t,x,y) =& I_1(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_1(t,x,y;0,\xi_1,\xi_2)\left(f_1(\overline{u},\overline{v}) + K_1\overline{u}\right)d\xi_1 d\xi_2 \\
\overline{v}(t,x,y) =& I_2(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}\times\mathbb{R}^+} \mathcal{G}_2(t,x,y;0,\xi_1,\xi_2)\left(f_2(\overline{u},\overline{v}) + K_2\overline{v}\right)d\xi_1 d\xi_2 \\
&- \frac{1}{d_2}\int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_2(t,x,0;\tau,\xi_1,0)\left(-g(\overline{v},\overline{w})\right)d\xi_1 \\
\overline{w}(t,x) =& I_3(t,x) + \int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_3(t,x;0,\xi)\left(g(\overline{v},\overline{w}) + \mu\overline{w}\right)d\xi
\end{aligned}
\tag{2.27}
$$

Then $(\overline{u},\overline{v},\overline{w})$ is a solution of (1.2). A similar argument shows that $(\underline{u},\underline{v},\underline{w})$ is also a solution of (1.2).

(ii) We denote by

$$
E_T^* := \{(t,x,y):\ t\in(0,T],\ (x,y)\in\mathbb{R}\times\mathbb{R}^+\},\ E_T^{**} := \{(t,x):\ t\in(0,T],\ x\in\mathbb{R}\}.
$$

By plugging (2.18) into (2.27), we obtain

$$
\begin{aligned}
\overline{w}(t,x) =& \ I_3(t,x) + \textstyle\int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_3(t,x;\tau,\xi)\nu\overline{v}d\xi, \\
\underline{w}(t,x) =& \ I_3(t,x) + \textstyle\int_0^t d\tau \int_{\mathbb{R}} \mathcal{G}_3(t,x;\tau,\xi)\nu\underline{v}d\xi.
\end{aligned}
$$

In view of $\mathcal{G}_3$, we have

$$\begin{aligned}(\overline{w}-\underline{w}) &= \nu\int_0^t d\tau\int_{\mathbb{R}}\mathcal{G}_3(t,x;\tau,\xi)(\overline{v}-\underline{v})d\xi\\ &\le \nu\|\overline{v}-\underline{v}\|_{L^\infty(E_t)}\int_0^t e^{-\mu t}d\tau\int_{\mathbb{R}}\frac{1}{2\sqrt{\pi D(t-\tau)}}e^{-\frac{(x-\xi)^2}{4D(t-\tau)}}d\xi\\ &= \frac{\nu}{\mu}(1-e^{-\mu t})\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}. \qquad (2.28)\end{aligned}$$

We also obtain

$$\begin{aligned}\overline{u}(t,x,y) &= I_1(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}\times\mathbb{R}^+}\mathcal{G}_1(t,x,y;\tau,\xi_1,\xi_2)(f_1(\overline{u},\overline{v})+K_1\overline{u})d\xi_1 d\xi_2\\ \underline{u}(t,x,y) &= I_1(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}\times\mathbb{R}^+}\mathcal{G}_1(t,x,y;\tau,\xi_1,\xi_2)(f_1(\underline{u},\underline{v})+K_1\underline{u})d\xi_1 d\xi_2\end{aligned}$$

Since the definition of K_1 in (2.16), we have $|f_1(\overline{u},\underline{v})+K_1\overline{u}-f_1(\underline{u},\overline{v})-K_1\underline{u}|\le 2K_1(|\overline{u}-\underline{u}|+|\overline{v}-\underline{v}|)$. Recalling $\mathcal{G}_2$, we have

$$\begin{aligned}(\overline{u}-\underline{u}) &\le 2K_1\int_0^t d\tau\int_{\mathbb{R}\times\mathbb{R}^+}\mathcal{G}_1(t,x,y;\tau,\xi_1,\xi_2)(|\overline{u}-\underline{u}|+|\overline{v}-\underline{v}|)d\xi_1 d\xi_2\\ &\le 2K_1(\|\overline{u}-\underline{u}\|_{L^\infty(E_t)}+\|\overline{v}-\underline{v}\|_{L^\infty(E_t)})\int_0^t e^{-K_1 t}d\tau\\ &\quad\int_{\mathbb{R}\times\mathbb{R}^+}\frac{1}{4\pi d_1(t-\tau)}\left(e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4d_1(t-\tau)}}+e^{-\frac{(x-\xi_1)^2+(-y-\xi_2)^2}{4d_1(t-\tau)}}\right)d\xi_1 d\xi_2\\ &= 2(1-e^{-K_1 t})(\|\overline{u}-\underline{u}\|_{L^\infty(E_t^*)}+\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}). \qquad (2.29)\end{aligned}$$

By (2.27), we have

$$\begin{aligned}\overline{v}(t,x,y) =& I_2(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}\times\mathbb{R}^+}\mathcal{G}_2(t,x,y;\tau,\xi_1,\xi_2)(f_2(\overline{u},\overline{v})+K_2\overline{v})d\xi_1 d\xi_2\\ &-\frac{1}{d_2}\int_0^t d\tau\int_{\mathbb{R}}\mathcal{G}_2(t,x,0;\tau,\xi_1,0)\,(-g(\overline{v},\overline{w}))\,d\xi_1\\ \underline{v}(t,x,y) =& I_2(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}\times\mathbb{R}^+}\mathcal{G}_2(t,x,y;\tau,\xi_1,\xi_2)(f_2(\underline{u},\underline{v})+K_2\underline{v})d\xi_1 d\xi_2\\ &-\frac{1}{d_2}\int_0^t d\tau\int_{\mathbb{R}}\mathcal{G}_2(t,x,0;\tau,\xi_1,0)\left(-g(\underline{v},\underline{w})\right)d\xi_1\end{aligned}$$

Since the definition of K_2 in (2.21), we have $|f_2(\overline{u},\overline{v})+K_2\overline{v}-f_2(\underline{u},\underline{v})-K_2\underline{v}|\le 2K_2(|\overline{u}-\underline{u}|+|\overline{v}-\underline{v}|)$. We also have $|g(\overline{u},\overline{w})-g(\underline{u},\underline{w})|\le(\mu+\nu)(|\overline{u}-\underline{u}|+|\overline{w}-\underline{w}|)$. Recalling $\mathcal{G}_2$, we have

$$\begin{aligned}(\overline{v}-\underline{v}) &\le 2(1-e^{-K_2 t})(\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}+\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)})\\ &+\frac{1}{d_2}(\mu+\nu)(\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}+\|\overline{w}-\underline{w}\|_{L^\infty(E_t^{**})})\int_0^t d\tau\int_{\mathbb{R}}\mathcal{G}_2(t,x,0;\tau,\xi_1,0)d\xi_1\\ &\le 2(1-e^{-K_2 t})(\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}+\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)})\\ &+\frac{1}{d_2}(\mu+\nu)(\|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)}+\|\overline{w}-\underline{w}\|_{L^\infty(E_t^{**})})\frac{1-e^{-K_2 t}}{K_2} \qquad (2.30)\end{aligned}$$

Combining (2.28), (2.29) and (2.30), by choosing $K := \max\{\frac{\nu}{\mu}, \frac{\mu+\nu}{d_2K_2} + 2\}$ and $\gamma = \max\{K_1, K_2, \mu\}$, we obtain

$$\begin{aligned}(\|\overline{u}-\underline{u}\|_{L^\infty(E_t^*)} + \|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)} + \|\overline{w}-\underline{w}\|_{L^\infty(E_t^{**})}) \leq K(1-e^{-\gamma t})\\ (\|\overline{u}-\underline{u}\|_{L^\infty(E_t^*)} + \|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)} + \|\overline{w}-\underline{w}\|_{L^\infty(E_t^{**})}),\ \text{for } t\in(0,\infty).\end{aligned} \tag{2.31}$$

Thus there exist a constant $T := ln\frac{K}{K-1}$ such that $\|\overline{u}-\underline{u}\|_{L^\infty(E_t^*)} + \|\overline{v}-\underline{v}\|_{L^\infty(E_t^*)} + \|\overline{w}-\underline{w}\|_{L^\infty(E_t^{**})} = 0$, that is $\overline{u} \equiv \underline{u}$, $\overline{v} \equiv \underline{v}$, and $\overline{w} \equiv \underline{w}$ for $t \in (0, T]$. Owing to the fact the above γ and K do not depend on the initial value, T can be extended to the time ∞. □

Notice that the above theorem is also valid when the upper and the lower solutions are constant vectors. Moreover we can induce the global asymptotic convergence of system (1.2). Suppose that there are constant vectors $\tilde{\mathbf{c}} := (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ and $\hat{\mathbf{c}} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)$ such that $f_i(\tilde{c}_1, \tilde{c}_2) \leq 0$ and $f_i(\hat{c}_1, \hat{c}_2) \geq 0$ for $i = 1, 2$. Here $\tilde{c}_3 = \frac{\nu}{\mu}\tilde{c}_2$, $\hat{c}_3 = \frac{\nu}{\mu}\hat{c}_2$, and $\hat{c}_i \leq \tilde{c}_i$ for $i = 1, 2, 3$. In view of Definition 2.1, $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$ are upper and lower solutions of system (1.2). Thus using $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$ as initial iteration, after the procedure (2.20), we can obtain the maximal solution $\overline{\mathbf{c}} := (\overline{c}_1, \overline{c}_2, \overline{c}_3)$ and minimal solution $\underline{\mathbf{c}} := (\underline{c}_1, \underline{c}_2, \underline{c}_3)$. Since the solution of system (2.20) is unique, the sequences $\{\overline{c}_i^{(m)}\}$ and $\{\underline{c}_i^{(m)}\}$ are constants. The limits of $\{\overline{c}_i^{(m)}\}$ and $\{\underline{c}_i^{(m)}\}$ satisfy

$$\begin{cases} f_1(\underline{c}_1, \underline{c}_2) = 0 = f_1(\overline{c}_1, \overline{c}_2),\\ f_2(\underline{c}_1, \underline{c}_2) = 0 = f_2(\overline{c}_1, \overline{c}_2),\\ \overline{c}_3 = \frac{\nu}{\mu}\overline{c}_2,\ \underline{c}_3 = \frac{\nu}{\mu}\underline{c}_2. \end{cases} \tag{2.32}$$

We give the convergence result in the next theorem.

Theorem 2.3. *Suppose that the initial functions of system* (1.2) *satisfy* $\hat{c}_1 \leq u_0(x,y) \leq \tilde{c}_1$, $\hat{c}_2 \leq v_0(x,y) \leq \tilde{c}_2$, *and* $\hat{c}_3 \leq w_0(x) \leq \tilde{c}_3$. *If* $\overline{c}_i = \underline{c}_i := c_i$ *for* $i = 1, 2, 3$, *then for* $(x, y) \in \mathbb{R}\times\mathbb{R}^+$,

$$\lim_{t\to\infty} u(t,x,y) = c_1,\ \lim_{t\to\infty} v(t,x,y) = c_2,\ \lim_{t\to\infty} w(t,x) = c_3.$$

Proof. We denote $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ by the solution of the following system

$$\begin{cases} \partial_t u - d_1\Delta u = f_1(u,v), & t>0, (x,y)\in\mathbb{R}\times\mathbb{R}^+,\\ \partial_t v - d_2\Delta v = f_2(u,v), & t>0, (x,y)\in\mathbb{R}\times\mathbb{R}^+,\\ \partial_t w - D\partial_{xx} w = \nu u|_{y=0} - \mu w, & t>0, x\in\mathbb{R},\\ \partial_y u|_{y=0} = 0, & t>0, x\in\mathbb{R},\\ -d_2\partial_y v|_{y=0} = \mu w - \nu v|_{y=0}, & t>0, x\in\mathbb{R},\\ u(0,x,y) = \tilde{c}_1,\ v(0,x,y) = \tilde{c}_2, & (x,y)\in\mathbb{R}\times\mathbb{R}^+,\\ w(0,x) = \tilde{c}_3, & x\in\mathbb{R}. \end{cases} \tag{2.33}$$

It is easy to see that $\overline{\mathbf{c}}$ is a lower solution of (2.33). Then we have $\overline{c}_i \leq \tilde{u}_i$ for $i = 1, 2, 3$. By using Theorem 2.1, $\tilde{u}_i$ is time-nonincreasing. The limit of $\tilde{u}_i$ exists as $t \to \infty$. We denote by $\lim_{t\to\infty}\tilde{u}_i(t,\cdot) =$

$\overline{u}_i(\cdot)$. Thanks to the Schauder estimates, $\overline{u}_i$ is the stationary solution of system (1.2). That is to say, $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ satisfies

$$\begin{cases} -d_1\Delta u = f_1(u,v), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ -d_2\Delta v = f_2(u,v), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ -D\partial_{xx} w = \nu u|_{y=0} - \mu w, & x \in \mathbb{R}, \\ \partial_y u|_{y=0} = 0, & x \in \mathbb{R}, \\ -d_2\partial_y v|_{y=0} = \mu w - \nu v|_{y=0}, & x \in \mathbb{R}. \end{cases} \tag{2.34}$$

Meanwhile, $\overline{\mathbf{c}}$ is also a stationary solution of system (1.2). Moreover, in view of the iterative process of $\overline{\mathbf{c}}$, $(\overline{c}_1^{(1)}, \overline{c}_2^{(1)}, \overline{c}_3^{(1)})$ satisfies

$$\begin{cases} -d_1\Delta \overline{c}_1^{(1)} = f_1(\tilde{c}_1, \tilde{c}_2) \ge f_1(\overline{u}_1, \overline{u}_2), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ -d_2\Delta \overline{c}_2^{(1)} = f_2(\tilde{c}_1, \tilde{c}_2) \ge f_2(\overline{u}_1, \overline{u}_2), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ (-D\partial_{xx} + \mu)\overline{c}_3^{(1)} = \nu\tilde{c}_2|_{y=0} \ge \nu\overline{u}_2|_{y=0}, & x \in \mathbb{R}, \\ \partial_y c_1^{(1)}|_{y=0} = 0 \ge \partial_y\overline{u}_1|_{y=0}, & x \in \mathbb{R}, \\ (-d_2\partial_y + \nu)c_2^{(1)}|_{y=0} = \mu\tilde{c}_3 \ge \mu\overline{u}_3, & x \in \mathbb{R}, \end{cases} \tag{2.35}$$

where the righthand inequality sign is due to $\overline{u}_i \le \tilde{c}_i$ for $i = 1,2,3$. By using comparison principle (Theorem 2.1), we induce that $\overline{c}_i^{(1)} \ge \overline{u}_i$ for $i = 1,2,3$. By using an induction method, $(\overline{c}_1^{(2)}, \overline{c}_2^{(2)}, \overline{c}_3^{(2)})$ satisfies

$$\begin{cases} -d_1\Delta \overline{c}_1^{(2)} = f_1(\overline{c}_1^{(1)}, \overline{c}_2^{(1)}) \ge f_1(\overline{u}_1, \overline{u}_2), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ -d_2\Delta \overline{c}_2^{(2)} = f_2(\overline{c}_1^{(1)}, \overline{c}_2^{(1)}) \ge f_2(\overline{u}_1, \overline{u}_2), & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ (-D\partial_{xx} + \mu)\overline{c}_3^{(2)} = \nu\overline{c}_2^{(1)}|_{y=0} \ge \nu\overline{u}_2|_{y=0}, & x \in \mathbb{R}, \\ \partial_y c_1^{(2)}|_{y=0} = 0 \ge \partial_y\overline{u}_1|_{y=0}, & x \in \mathbb{R}, \\ (-d_2\partial_y + \nu)c_2^{(2)}|_{y=0} = \mu\overline{c}_3^{(1)} \ge \mu\overline{u}_3, & x \in \mathbb{R}. \end{cases} \tag{2.36}$$

By using comparison principle (Theorem 2.1), we induce that $\overline{c}_i^{(2)} \ge \overline{u}_i$ for $i = 1,2,3$. After an induction argument, we have $\overline{c}_i^{(m)} \ge \overline{u}_i$ for $i = 1,2,3$. Here $m = 1,2,\cdots$. By letting $m \to \infty$, we have

$$\overline{c}_i \ge \overline{u}_i = \lim_{t\to\infty} \tilde{u}_i(t,\cdot), \text{ for } i = 1,2,3. \tag{2.37}$$

On the other hand, We denote $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ by the solution of the following system

$$\begin{cases} \partial_t u - d_1\Delta u = f_1(u,v), & t>0, (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ \partial_t v - d_2\Delta v = f_2(u,v), & t>0, (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ \partial_t w - D\partial_{xx} w = \nu u|_{y=0} - \mu w, & t>0, x \in \mathbb{R}, \\ \partial_y u|_{y=0} = 0, & t>0, x \in \mathbb{R}, \\ -d_2\partial_y v|_{y=0} = \mu w - \nu v|_{y=0}, & t>0, x \in \mathbb{R}, \\ u(0,x,y) = \hat{c}_1, \ v(0,x,y) = \hat{c}_2, & (x,y) \in \mathbb{R}\times\mathbb{R}^+, \\ w(0,x) = \hat{c}_3, & x \in \mathbb{R}. \end{cases} \tag{2.38}$$

In a similar way, we can induce

$$\underline{c}_i \leq \underline{u}_i = \lim_{t\to\infty} \hat{u}_i(t,\cdot), \text{ for } i = 1,2,3. \tag{2.39}$$

Since $\overline{c}_i = \underline{c}_i := c_i$ for $i = 1,2,3$, it follows from (2.38) and (2.39) that

$$\lim_{t\to\infty} \tilde{u}_i(t,\cdot) = \lim_{t\to\infty} \hat{u}_i(t,\cdot) = c_i, \text{ for } i = 1,2,3. \tag{2.40}$$

For any initial functions satisfying $\hat{c}_1 \leq u_0(x,y) \leq \tilde{c}_1$, $\hat{c}_2 \leq v_0(x,y) \leq \tilde{c}_2$, and $\hat{c}_3 \leq w_0(x) \leq \tilde{c}_3$, the solution (u,v,w) is a lower solution of system (2.33) and a upper solution of system (2.38). By using comparison principle (Theorem 2.1), we have $(\hat{u}_1,\hat{u}_2,\hat{u}_3) \leq (u,v,w) \leq (\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$. Thus we induce that

$$\lim_{t\to\infty}(u,v,w) = (c_1,c_2,c_3).$$

□

Theorem 2.4. *If the initial functions of system* (1.2) *satisfy* $0 \leq u_0(x,y) \leq 1$, $0 \leq v_0(x,y) \leq 1$, *and* $0 \leq w_0(x) \leq \frac{\nu}{\mu}$. *then system* (1.2) *possesses a unique solution* (u,v,w) *for* $t \in (0,\infty)$. *Moreover, for* $(x,y) \in \mathbb{R}\times\mathbb{R}^+$,

$$0 \leq u(t,x,y) \leq 1, \; 0 \leq v(t,x,y) \leq 1, \; 0 \leq w(t,x) \leq \frac{\nu}{\mu}. \tag{2.41}$$

Proof. In order to utilize Theorem 2.2, we need to construct upper and lower solutions of system (1.2). We set

$$M_1 = 1, \; M_2 = 1, \; M_3 = \frac{\nu}{\mu}.$$

It is easy to verify that (M_1,M_2,M_3) and $(0,0,0)$ are upper and lower solutions of system (1.2). Using Theorem 2.2, system (1.2) has a unique solution. Moreover we can obtain (2.41). □

3. Stabilities of the equilibria

Based on the basic reproduction $R_0 := \frac{ma^2 b_1 b_2}{\gamma_1\gamma_2}$, we examine the asymptotic stability of the disease-free equilibrium $(0,0,0)$ and endemic equilibrium.

Theorem 3.1. *Suppose that the initial functions of system* (1.2) *satisfy* $0 \leq u_0(x,y) \leq 1$, $0 \leq v_0(x,y) \leq 1$ *and* $0 \leq w_0(x) \leq \frac{\nu}{\mu}$, *if* $R_0 < 1$, *then the solution to system* (1.2) *satisfies*

$$\lim_{t\to\infty}(u,v,w) = (0,0,0). \tag{3.1}$$

Proof. We use the method of upper and lower solutions. we set

$$M_1 = 1, \; M_2 = 1, \; M_3 = \frac{\nu}{\mu}.$$

It is easy to verify that (M_1, M_2, M_3) and $(0,0,0)$ are upper and lower solutions of system (1.2). By using Theorem 2.3, we can construct the maximal solution $\bar{\mathbf{c}}$ and minimal solution $\underline{\mathbf{c}}$, which satisfy

$$\begin{cases} mab_1\bar{c}_2(1-\bar{c}_1) - \gamma_1\bar{c}_1 = 0 = mab_1\underline{c}_2(1-\underline{c}_1) - \gamma_1\underline{c}_1, \\ ab_2(1-\bar{c}_2)\bar{c}_1 - \gamma_2\bar{c}_2 = 0 = ab_2(1-\underline{c}_2)\underline{c}_1 - \gamma_2\underline{c}_2, \\ \bar{c}_3 = \frac{\nu}{\mu}\bar{c}_2,\ \underline{c}_3 = \frac{\nu}{\mu}\underline{c}_2. \end{cases}$$

By a directly computation, we have

$$\bar{c}_1\bar{c}_2[\frac{ma^2b_1b_2}{\gamma_1\gamma_2}(1-\bar{c}_1)(1-\bar{c}_2) - 1] = 0.$$

Since $R_0 < 1$, we induce that $\bar{c}_1\bar{c}_2 = 0$. Combing with $mab_1\bar{c}_2(1-\bar{c}_1) - \gamma_1\bar{c}_1 = 0$, we have

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = 0. \tag{3.2}$$

On the other hand ,a similar argument yields that

$$\underline{c}_1 = \underline{c}_2 = \underline{c}_3 = 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\underline{c}_i = \bar{c}_i = 0 \text{ for } i = 1,2,3. \tag{3.4}$$

By applying Theorem 2.3, we have

$$\lim_{t\to\infty}(u, v, w) = (\underline{c}_1, \underline{c}_2, \underline{c}_3) = (0,0,0).$$

□

Theorem 3.2. *Suppose that δ is an arbitrary small positive constant such that $\delta \le u_0(x,y) \le 1$, $\delta \le v_0(x,y) \le 1$ and $\delta \le w_0(x,y) \le \frac{\nu}{\mu}$. If $R_0 > 1$, then the solution to system* (1.2) *satisfies*

$$\lim_{t\to\infty}(u, v, w) = (u^*, v^*, w^*), \tag{3.5}$$

where (u^*, v^*) is a unique positive solution of

$$\begin{cases} f_1(u^*, v^*) = 0, \\ f_2(u^*, v^*) = 0. \end{cases} \tag{3.6}$$

and $w^* = \frac{\nu}{\mu}v^*$.

Proof. We first consider the nullclines of system (1.2), which is written by

$$\begin{cases} v = \frac{\gamma_1 u}{mab_1(1-u)}, \\ u = \frac{\gamma_2 v}{ab_2(1-v)}. \end{cases} \tag{3.7}$$

By taking the derivatives of (3.6), we have

$$\begin{cases} \frac{dv}{du} = \frac{\gamma_1}{mab_1(1-u)^2} > 0, \\ \frac{du}{dv} = \frac{\gamma_2}{ab_2(1-v)^2} > 0. \end{cases} \tag{3.8}$$

Hence the two nullclines are monotone increasing. In the domain $(u, v) \in [0, 1] \times [0, \infty)$, they have the same start point $(0, 0)$, but the different end points $(1, \infty)$ and $(1, \frac{ab_2}{ab_2+\gamma_2})$. The nullclines has a unique positive cross points (u^*, v^*).

Next we use the method of upper and lower solutions. we set

$$M_1 = 1,\ M_2 = 1,\ M_3 = \frac{\nu}{\mu}.$$

It is easy to verify that (M_1, M_2, M_3) and (δ, δ, δ) are upper and lower solutions of system (1.2). By using Theorem 2.3, we can construct the maximal solution $\bar{\mathbf{c}}$ and minimal solution $\underline{\mathbf{c}}$, which satisfy $\bar{c} \geq \underline{c}_i > 0$ for $i = 1, 2, 3$. Moreover, $(\bar{c}_1, \bar{c}_2)$ and $(\underline{c}_1, \underline{c}_2)$ are the positive solutions of the following nullclines

$$\begin{cases} v = \frac{\gamma_1 u}{mab_1(1-u)}, \\ u = \frac{\gamma_2 v}{ab_2(1-v)}. \end{cases}$$

Since the nullclines has a unique positive cross points (u^*, v^*), we obtain that $\bar{c}_1 = \underline{c}_1 = u^*$, $\bar{c}_2 = \underline{c}_2 = v^*$. Then $\bar{c}_3 = \underline{c}_3 = w^*$.

By applying Theorem 2.3, we have

$$\lim_{t\to\infty}(u, v, w) = (u^*, v^*, w^*).$$

□

4. Discussion

Since mosquito is a prominent vector of Malaria, it is of crucial importance to study the qualitative spreading behavior of mosquitoes in implementing vector control strategies and preventing mosquito-borne diseases. A large proportion of the current studies on epidemic transmission dynamics, using ordinary differential systems, focuses on the temporal development and control of infectious diseases, thus uncertainty still exists concerning how population mobility affects on epidemic outbreaks. The spatial factor should be taken into consideration in the modeling processes in order to study the geographic spread of infectious diseases. Laplacian diffusion systems with spatially homogeneous parameters (Murray [27], Ruan [28]) and spatially heterogeneous parameters (Allen et al. [14], Lei et al. [18], Li et al. [17], Song et al. [1]) have been proposed to study the spatio-temporal dynamics epidemic models. These studies exhibit an equal probability of mosquitoes' movement to any direction. Yet our road diffusion model is different since mosquitoes move at a faster speed along the highway. We studied the long-time dynamical behaviors of Ross epidemic model. To the best of our knowledge, the road diffusion has never been applied to describe the two compartments spread of infectious diseases in any epidemic model in the literature.

Our results indicate that mosquitoes has an impact on long-time dynamics of infectious diseases in road diffusion models. According to Theorem 3.1, when the basic reproduction number $R_0 < 1$, system (1.2) admits a globally asymptotically stable disease-free equilibrium, and an asymptotic stable endemic equilibrium when the basic reproduction number $R_0 > 1$ in view of Theorem 3.2. So we have generalized the threshold dynamics of the classical Ross model to that in a road diffusion model. Although many factors related to a real disease have been simplified in our model, we capture the dynamics of mosquito-borne diseases, which helps us to control and prevent the spread of diseases.

5. Conclusions

Since Ronald Ross discovers the transmission of Malaria by mosquitoes, the differential equations have been used to study the spread of infectious disease (see [29–31]). A natural question is how to describe the dynamical system when the infectious disease spreads along the directed diffusion. We introduce the road-field diffusion into a Ross epidemic model which describes the dynamical behaviors of infected Mosquitoes and humans. Our result reveals that the disease-free equilibrium is asymptotically stable if the basic reproduction number is lower than 1 while the endemic equilibrium asymptotically stable if the basic reproduction number is greater than 1.

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Conflict of interest

The authors declare that they have no conflict of interest.

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