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Research article

Hermite-Hadamard type inequalities for interval-valued exponential type pre-invex functions via Riemann-Liouville fractional integrals

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Abstract: In the present research, we develop Hermite-Hadamard type inequalities for interval-valued exponential type pre-invex functions in Riemann-Liouville interval-valued fractional operator settings. Moreover, we develop He's inequality for interval-valued exponential type pre-invex functions.

Keywords: pre-invex function; interval-valued function; interval-valued fractional integral operator; Hermite-Hadamard type inequality; He's inequality
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1. Introduction

The classical Hermite-Hadamard inequality is one of the most well-established inequalities in the theory of convex functions with geometrical interpretation and it has many applications. This inequality may be regarded as a refinement of the concept of convexity. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable refinements and generalizations have been studied [1, 2].

The importance of the study of set-valued analysis from a theoretical point of view as well as from their applications is well known. Many advances in set-valued analysis have been motivated by control theory and dynamical games. Optimal control theory and mathematical programming were an engine driving these domains since the dawn of the sixties. Interval analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena.

Furthermore, a few significant inequalities like Hermite-Hadamard and Ostrowski type inequalities have been established for interval valued functions in recent years. In [3, 4], Chalco-Cano et al. established Ostrowski type inequalities for interval valued functions by using Hukuhara derivatives for interval valued functions. In [5], Román-Flores et al. established Minkowski and Beckenbach's inequalities for interval valued functions. For other related results we refer to the readers [6].

In this paper, we establish Hermite-Hadamard type inequalities and He's inequality for interval-valued exponential type pre-invex functions in the Riemann-Liouville interval-valued fractional operator settings.

2. Preliminaries

We begin with recalling some basic concepts and notions in the convex analysis.

Let the space of all intervals of $\Re$ is $\Re_c$ and $\Lambda \in \Re_c$ given by

$$\Lambda_1 = [\underleftrightarrow{\Lambda}, \overleftrightarrow{\Lambda}] = \{v \in \Re \big| \underleftrightarrow{\Lambda} < v < \overleftrightarrow{\Lambda}\}, \quad \underleftrightarrow{\Lambda}, \overleftrightarrow{\Lambda} \in \Re .$$

Various binary operations are given as follows [7]:

Scalar multiplication: $\tau \in \Re$,

$$\tau.\Lambda_1 = \begin{cases} [\tau \underleftrightarrow{\Lambda}, \tau \overleftrightarrow{\Lambda}], & if \quad 0 \leq \tau, \\ 0, & if \quad \tau = 0, \\ [\tau \overleftrightarrow{\Lambda}, \tau \underleftrightarrow{\Lambda}], & if \quad `\tau \leq 0. \end{cases}$$

Difference, addition, product and reciprocal for $\Lambda_1, \Lambda_2 \in \Re_c$ are respectively given by

$$\begin{aligned} &\Lambda_1 - \Lambda_2 = [\underleftrightarrow{\Lambda_1}, \overleftrightarrow{\Lambda_1}] - [\underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_2}] = [\underleftrightarrow{\Lambda_1} - \underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1} - \overleftrightarrow{\Lambda_2}], \\ &\Lambda_1 + \Lambda_2 = [\underleftrightarrow{\Lambda_1}, \overleftrightarrow{\Lambda_1}] + [\underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_2}] = [\underleftrightarrow{\Lambda_1} + \underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1} + \overleftrightarrow{\Lambda_2}], \\ &\Lambda_1 \times \Lambda_2 = [\min\{\underleftrightarrow{\Lambda_1}\underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1}\underleftrightarrow{\Lambda_2}, \underleftrightarrow{\Lambda_1}\overleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1}\overleftrightarrow{\Lambda_2}\}, \\ &\max\{\underleftrightarrow{\Lambda_1}\underleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1}\underleftrightarrow{\Lambda_2}, \underleftrightarrow{\Lambda_1}\overleftrightarrow{\Lambda_2}, \overleftrightarrow{\Lambda_1}\overleftrightarrow{\Lambda_2}\}] = \{uv \mid u \in \Lambda_1, v \in \Lambda_2\}, \\ &\frac{1}{\Lambda} = \left\{\frac{1}{v_1} \mid 0 \neq v_1 \in \Lambda\right\} = \left[\frac{1}{\underleftrightarrow{\Lambda}}, \frac{1}{\overleftrightarrow{\Lambda}}\right], \\ &\Lambda_1.\frac{1}{\Lambda_2} = \left\{u.\frac{1}{v} \mid u \in \Lambda_1, 0 \neq v \in \Lambda_2\right\} = \left[\underleftrightarrow{\Lambda_1}.\frac{1}{\overleftrightarrow{\Lambda_2}}, \overleftrightarrow{\Lambda_1}.\frac{1}{\overleftrightarrow{\Lambda_2}}\right]. \end{aligned}$$

Let $\Re_\Lambda, \Re_\Lambda^+$ *and* $\Re_\Lambda^-$ denote the collection of all closed intervals of $\Re$, the collection of all positive intervals of $\Re$ and the collection of all negative intervals of $\Re$ respectively. In this paper, we examine a few algebraic properties of interval arithmetic.

Definition 2.1. *[7] A mapping Ω is called an interval-valued function of υ on $[a_1, b_1]$ if it assigns a nonempty interval to every $v \in [a_1, b_1]$, that is*

$$\Omega(v) = [\overleftrightarrow{\Omega}(v), \underleftrightarrow{\Omega}(v)], \tag{2.1}$$

where $\overleftrightarrow{\Omega}(\upsilon)$ and $\underleftrightarrow{\Omega}(\upsilon)$ are both real valued functions.

Consider any finite ordered subset $\complement$ be the partition of $[a_1, b_1]$, that is

$$\complement : a_1 = a_1, ..., a_n = b_1.$$

The mesh of $\complement$ is

$$mesh(\complement) = max\{a_{i+1} - a_i;\ i = 1, ..., n\}.$$

The Riemann sum of $\Omega : [a_1, b_1] \to \Re_\Lambda$ can be defined by

$$\widetilde{S}(\Omega, \complement, c) = \Sigma_{i=1}^{n} \Omega(d_i)(a_{i+1} - a_i),$$

where $mesh(\complement) < c$.

Definition 2.2. *[8] A mapping $\Omega : [a_1, b_1] \to \Re_\Lambda$ is called an interval-Riemann integrable on $[a_1, b_1]$ if $\exists\, \Lambda \in \Re_\Lambda$ such that for every $\delta > 0$ satisfying*

$$d(\widetilde{S}(\Omega, \complement, c), \Lambda) < \delta,$$

we have

$$\Lambda_1 = (IR) \int_{a_1}^{b_1} \Omega(v) dv. \tag{2.2}$$

Lemma 2.1. *[9] Let $\Omega : [a_1, b_1] \to \Re_\Lambda$ be an interval-valued function as in (2.1), then it is interval-Riemann integrable if and only if*

$$(IR) \int_{a_1}^{b_1} \Omega(v) dv = \left[(R) \int_{a_1}^{b_1} \overleftrightarrow{\Omega}(v) dv, (R) \int_{a_1}^{b_1} \underleftrightarrow{\Omega}(v) dv \right].$$

In simple words, Ω is interval-Riemann integrable if and only if $\overleftrightarrow{\Omega}(v)$ and $\underleftrightarrow{\Omega}(v)$ are both Riemann integrable functions.

Definition 2.3. *[10] Let $\Omega \in L_1[a_1, b_1]$, then the Riemann-Liouville fractional integrals of order $m > 0$ with $0 \leq a_1$ are defined by*

$$I_{a_1^+}^{m} \Omega(v) = \frac{1}{\Gamma(m)} \int_{a_1}^{v} (v - r)^{m-1} \Omega(r) dr \quad , v > a_1, \tag{2.3}$$

$$I_{b_1^-}^{m} \Omega(v) = \frac{1}{\Gamma(m)} \int_{v}^{b_1} (r - v)^{m-1} \Omega(r) dr \quad , v < b_1. \tag{2.4}$$

Definition 2.4. *[11, 12] Let $\Omega : [a_1, b_1] \to \Re_\Lambda$ be an interval-valued, interval-Riemann integrable function as in (2.1), then the interval Riemann-Liouville fractional integrals of order $m > 0$ with $0 \leq a_1$ are defined by*

$$I_{a_1^+}^{m} \Omega(v) = \frac{1}{\Gamma(m)} (IR) \int_{a_1}^{v} (v - r)^{m-1} \Omega(r) dr \quad , v > a_1, \tag{2.5}$$

$$I_{b_1^-}^{m} \Omega(v) = \frac{1}{\Gamma(m)} (IR) \int_{v}^{b_1} (r - v)^{m-1} \Omega(r) dr \quad , v < b_1. \tag{2.6}$$

Corollary 2.1. *[12] Let $\Omega : [a_1, b_1] \to \Re_\Lambda$ be an interval-valued function as in (2.1) such that $\overleftrightarrow{\Omega}(v)$ and $\underleftrightarrow{\Omega}(v)$ are Riemann integrable functions, then*

$$I^m_{a_1^+}\Omega(v) = \Big[I^m_{a_1^+}\underleftrightarrow{\Omega}(v), I^m_{a_1^+}\overleftrightarrow{\Omega}(v)\Big],$$

$$I^m_{b_1^-}\Omega(v) = \Big[I^m_{b_1^-}\underleftrightarrow{\Omega}(v), I^m_{b_1^-}\overleftrightarrow{\Omega}(v)\Big].$$

Definition 2.5. *[13] A set $\Lambda \subset \Re^n$ with respect to a vector function $\eta : \Re^n \times \Re^n \to \Re^n$ is called an invex set if*

$$b_1 + \tau\eta(a_1, b_1) \in \Lambda, \quad \forall a_1, b_1 \in \Lambda, \ \tau_1 \in [0, 1].$$

Definition 2.6. *[13] A function Ω on the invex set Λ with respect to a vector function $\eta : \Lambda \times \Lambda \to \Re^n$ is called pre-invex function if*

$$\Omega(b_1 + \tau\eta(a_1, b_1)) \leq (1 - \tau)\Omega(b_1) + \tau\Omega(a_1), \forall a_1, b_1 \in \Lambda, \ \tau_1 \in [0, 1]. \tag{2.7}$$

Lemma 2.2. *[14, 15] If Λ is open and $\eta : \Lambda \times \Lambda \to \Re$, then $\forall a_1, b_1 \in \Lambda, \ \tau, \tau_1, \tau_2 \in [0, 1]$, we have*

$$\eta(b_1, b_1 + \tau\eta(a_1, b_1)) = -\tau\eta(a_1, b_1), \tag{2.8}$$

$$\eta(a_1, b_1 + \tau\eta(a_1, b_1)) = (1 - \tau)\eta(a_1, b_1), \tag{2.9}$$

$$\eta(b_1 + \tau_2\eta(a_1, b_1), b_1 + \tau_1\eta(a_1, b_1)) = (\tau_2 - \tau_1)\eta(a_1, b_1). \tag{2.10}$$

In [16], Noor presented Hermite-Hadamard-inequality for pre-invex function, as follows:

$$\Omega\Big(\frac{2a_1 + \eta(b_1, a_1)}{2}\Big) \leq \frac{1}{\eta(b_1, a_1)} \int_{a_1}^{a_1+\eta(b_1,a_1)} \Omega(v)dv \leq \frac{\Omega(a_1) + \Omega(b_1)}{2}.$$

Definition 2.7. *[15] Let us consider an interval-valued function Ω on the set Λ, then Ω is pre-invex interval valued function with respect to η on an invex set $\Lambda \subset \Re^n$ with respect to a vector function $\eta : \Lambda \times \Lambda \to \Re^n$ if*

$$\Omega(b_1 + \tau_1\eta(a_1, b_1)) \supseteq (1 - \tau_1)\Omega(b_1) + \tau_1\Omega(a_1), \quad \forall a_1, b_1 \in \Lambda, \ \tau_1 \in [0, 1]. \tag{2.11}$$

Taking motivation from the exponential type convexity proposed in [17], we introduce the following notion:

Definition 2.8. *A function Ω on the invex set Λ is called exponential-type pre-invex function with respect to a vector function $\eta : \Lambda \times \Lambda \to \Re^n$ if*

$$\Omega(b_1 + \tau_1\eta(a_1, b_1)) \leq (e^{(1-\tau_1)} - 1)\Omega(b_1) + (e^{\tau_1} - 1)\Omega(a_1), \quad \forall a_1, b_1 \in \Lambda, \ \tau_1 \in [0, 1]. \tag{2.12}$$

It is important to note that a pre-invex function need not to be convex function. For example, the function $f(x) = -|x|$ is not a convex function but it is a pre-invex function with respect to η, where

$$\eta(v, u) = \begin{cases} u - v, & \text{if } u \leq 0, v \leq 0, v \geq 0, u \geq 0, \\ v - u, & \text{otherwise.} \end{cases}$$

Theorem 2.1. *Let $\Omega : [a_1, b_1] \to \Re$ be an exponential-type pre-invex function with respect to a vector function $\eta : \Lambda \times \Lambda \to \Re^n$. If $a_1 < b_1$ and $\Omega \in L[a_1, b_1]$, then we have*

$$\frac{1}{2(e^{\frac{1}{2}}-1)}\Omega\Big(a_1 + \frac{1}{2}\eta(b_1, a_1)\Big) \leq \frac{1}{\eta(b_1, a_1)}\int_{a_1}^{a_1+\eta(b_1,a_1)} \Omega(v)dv \leq (e-2)[\Omega(a_1) + \Omega(b_1)].$$

Proof. At first, from exponential-type-pre-invexity of Ω, we have

$$\begin{aligned}\Omega\Big(a_1 + \frac{1}{2}\eta(b_1, a_1)\Big) =&\Omega\Big(\frac{1}{2}[b_1 + \tau_1\eta(a_1, b_1)] + \frac{1}{2}[a_1 + \tau_1\eta(b_1, a_1)]\Big)\\ \leq&(e^{\frac{1}{2}}-1)\Big[\Omega(b_1 + \tau_1\eta(a_1, b_1)) + \Omega(a_1 + \tau_1\eta(b_1, a_1))\Big].\end{aligned}$$

Integrating the above inequality with respect to $\tau_1 \in [0, 1]$ yields

$$\begin{aligned}\Omega\Big(a_1 + \frac{1}{2}\eta(b_1, a_1)\Big) &\leq \Big(e^{\frac{1}{2}}-1\Big)\Big(\int_0^1 \Omega(b_1 + \tau_1\eta(a_1, b_1))d\tau_1 + \int_0^1 \Omega\left(a_1 + \tau_1\eta(b_1, a_1)\right) d\tau_1\Big)\\ &= \frac{2(e^{\frac{1}{2}}-1)}{\eta(b_1, a_1)}\int_{a_1}^{a_1+\eta(b_1,a_1)} \Omega\left(v\right) dv.\end{aligned}$$

Now, taking $v = b_1 + \tau_1\eta(a_1, b_1)$ gives

$$\begin{aligned}\frac{1}{\eta(b_1, a_1)}\int_{a_1}^{a_1+\eta(b_1,a_1)} \Omega(v)dv =& \int_0^1 \Omega(b_1 + \tau_1\eta(a_1, b_1))\, d\tau_1\\ \leq& \int_0^1 \{(e^{\tau_1}-1)\Omega(a_1) + (e^{(1-\tau_1)}-1)\Omega(b_1)\}\, d\tau_1\\ =&(e-2)[\Omega(a_1) + \Omega(b_1)].\end{aligned}$$

This completes the proof. □

By merging the concepts of pre-invexity and exponential type pre-invexity, we propose the following notion:

Definition 2.9. *Let $\Lambda \subset \Re^n$ be an invex set with respect to a vector function $\eta : \Lambda \times \Lambda \to \Re^n$. The interval valued function Ω on the set Λ is exponential-type pre-invex interval valued function with respect to η if*

$$\Omega(b_1 + \tau_1\eta(a_1, b_1)) \supseteq (e^{(1-\tau_1)}-1)\Omega(b_1) + (e^{\tau_1}-1)\Omega(a_1), \quad \forall a_1, b_1 \in \Lambda,\ \tau_1 \in [0, 1]. \tag{2.13}$$

Remark 2.1. *In Definition 2.9, by taking $h(\tau_1) = e^{\tau_1} - 1$, where $h : [0, 1] \subset [a_1, b_1] \to \Re$ and $h \neq 0$, then we get h-pre-invex interval valued function with respect to η, that is*

$$\Omega(b_1 + \tau_1\eta(a_1, b_1)) \supseteq h(1-\tau_1)\Omega(b_1) + h(\tau_1)\Omega(a_1), \quad \forall a_1, b_1 \in \Lambda,\ \tau_1 \in [0, 1]. \tag{2.14}$$

Remark 2.2. *Let $\Lambda \subset \Re^n$ be an invex set with respect to a vector function $\eta : \Re^n \times \Re^n \to \Re^n$. The interval valued function Ω on the set Λ is exponential-type-pre-invex function with respect to η if and only if $\overleftrightarrow{\Omega}$, $\underleftrightarrow{\Omega}$ are exponential-type pre-invex functions with respect to η, that is*

$$\overleftrightarrow{\Omega}(b_1 + \tau_1\eta(a_1, b_1)) \leq (e^{(1-\tau_1)}-1)\overleftrightarrow{\Omega}(b_1) + (e^{\tau_1}-1)\overleftrightarrow{\Omega}(a_1), \quad \forall a_1, b_1 \in \Lambda,\ \tau_1 \in [0, 1], \tag{2.15}$$

$$\underleftrightarrow{\Omega}(b_1 + \tau_1\eta(a_1, b_1)) \leq (e^{(1-\tau_1)}-1)\underleftrightarrow{\Omega}(b_1) + (e^{\tau_1}-1)\underleftrightarrow{\Omega}(a_1), \quad \forall a_1, b_1 \in \Lambda,\ \tau_1 \in [0, 1]. \tag{2.16}$$

Remark 2.3. *If* $\overleftrightarrow{\Omega}(v) = \underset{\leftrightarrow}{\Omega}(v)$*, then we get (2.12).*

Remark 2.4. *Since* $\tau_1 \leq e^{\tau_1} - 1$ *and* $1-\tau_1 \leq e^{1-\tau_1} - 1$ *for all* $\tau_1 \in [0,1]$*, so every nonnegative pre-invex interval valued function with respect to* η *is also exponential-type pre-invex interval valued function with respect to* η*.*

3. Interval fractional Hermite-Hadamard type inequalities

In this section, we establish fractional Hermite-Hadamard type inequality for interval-valued exponential type pre-invex. The family of Lebesgue measurable interval-valued functions is denoted by $L([v_1, v_2], \mathfrak{R}_0)$.

Theorem 3.1. *Let* $\Lambda \subset \mathfrak{R}$ *be an open invex set with respect to* $\eta : \Lambda \times \Lambda \to \mathfrak{R}$ *and* $a_1, b_1 \in \Lambda$ *with* $a_1 < a_1 + \eta(b_1, a_1)$*. If* $\Omega : [a_1, a_1 + \eta(b_1, a_1)] \to \mathfrak{R}$ *is an exponential type pre-invex interval-valued function such that* $\Omega \in L[a_1, a_1 + \eta(b_1, a_1)]$ *and* $m > 0$*, then we have (considering Lemma 2.2 holds)*

$$\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1 + \frac{1}{2}\eta(d_1, c_1)\Big) \supseteq \frac{\Gamma(m+1)}{\eta^m(d_1, c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1) + I^m_{c_1^+}\Omega(c_1 + \eta(d_1, c_1))]$$
$$\supseteq mP(\Omega(c_1 + \eta(d_1, c_1)) + \Omega(c_1)), \tag{3.1}$$

where

$$P = -\frac{1}{(m+1)(-1)^m}\Big[(\mathrm{e}m + \mathrm{e})(-1)^m\,\Gamma(m+1,1) + (-m-1)\,\Gamma(m+1,-1)$$
$$+ ((-\mathrm{e}m - \mathrm{e})(-1)^m + m + 1)\,\Gamma(m+1) + 2(-1)^m\Big]. \tag{3.2}$$

Proof. Since Ω is an exponential type pre-invex interval-valued function, so

$$\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(a_1 + \frac{1}{2}\eta(b_1, a_1)\Big) \supseteq [\Omega(a_1) + \Omega(b_1)].$$

Taking $a_1 = c_1 + (1-\tau_1)\eta(d_1, c_1)$ and $b_1 = c_1 + (\tau_1)\eta(d_1, c_1)$ gives

$$\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1 + (1-\tau_1)\eta(d_1, c_1) + \frac{1}{2}\eta(c_1 + (\tau_1)\eta(d_1, c_1), c_1 + (1-\tau_1)\eta(d_1, c_1))\Big)$$
$$\supseteq [\Omega(c_1 + (1-\tau_1)\eta(d_1, c_1)) + \Omega(c_1 + (\tau_1)\eta(d_1, c_1))],$$

implies

$$\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1 + \frac{1}{2}\eta(d_1, c_1)\Big) \supseteq [\Omega(c_1 + (1-\tau_1)\eta(d_1, c_1)) + \Omega(c_1 + (\tau_1)\eta(d_1, c_1))].$$

By multiplying by τ_1^{m-1} on both sides and integrating over $[0,1]$ with respect to τ_1, we get

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)d\tau_1\\
&\supseteq (IR)\int_0^1 \tau_1^{m-1}[\Omega(c_1+(1-\tau_1)\eta(d_1,c_1))+\Omega(c_1+(\tau_1)\eta(d_1,c_1))]d\tau_1,
\end{aligned}
$$

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)d\tau_1\\
=&\Big[(R)\int_0^1 \tau_1^{m-1}\frac{1}{(e^{\frac{1}{2}}-1)}\underset{\leftrightarrow}{\Omega}\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)d\tau_1,(R)\int_0^1 \tau_1^{m-1}\frac{1}{(e^{\frac{1}{2}}-1)}\overset{\leftrightarrow}{\Omega}\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)d\tau_1\Big],
\end{aligned}
$$

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)d\tau_1\\
=&\Big[\frac{1}{m(e^{\frac{1}{2}}-1)}\underset{\leftrightarrow}{\Omega}\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big),\frac{1}{m(e^{\frac{1}{2}}-1)}\overset{\leftrightarrow}{\Omega}\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)\Big]\\
=&\frac{1}{m(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big),
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\Omega(c_1+(\tau_1)\eta(d_1,c_1))\\
=&\Big[\frac{1}{\eta^m(d_1,c_1)}(R)\int_c^{c_1+(\tau_1)\eta(d_1,c_1)}(i-c)^{m-1}\underset{\leftrightarrow}{\Omega}(i)di,\frac{1}{\eta^m(d_1,c_1)}(R)\int_c^{c_1+(\tau_1)\eta(d_1,c_1)}(i-c)^{m-1}\overset{\leftrightarrow}{\Omega}(i)di\Big],
\end{aligned}
$$

$$
\begin{aligned}
(IR)\int_0^1 \tau_1^{m-1}\Omega(c_1+(\tau_1)\eta(d_1,c_1)) &= \frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\underset{\leftrightarrow}{\Omega}(c_1),I^m_{(c_1+\eta(d_1,c_1))^-}\overset{\leftrightarrow}{\Omega}(c_1)]\\
&= \frac{\Gamma(m)}{\eta^m(d_1,c_1)}I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).
\end{aligned}
\tag{3.4}
$$

Similarly

$$
\begin{aligned}
(IR)\int_0^1 \tau_1^{m-1}\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)) =&\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{c_1^+}\underset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1)),I^m_{c_1^+}\overset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1))]\\
=&\frac{\Gamma(m)}{\eta^m(d_1,c_1)}I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).
\end{aligned}
\tag{3.5}
$$

From (3.3)–(3.5), we get

$$
\frac{1}{m(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)\supseteq\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1))]. \tag{3.6}
$$

Now, from the interval valued exponential type pre-invexity of Ω, we have

$$\begin{aligned}\Omega(c_1+\tau_1\eta(d_1,c_1)) &= \Omega(c_1+\eta(d_1,c_1)+(1-\tau_1)\eta(c_1,c_1+\eta(d_1,c_1)))\\ &\supseteq (e^{\tau_1}-1)\Omega(c_1+\eta(d_1,c_1))+(e^{(1-\tau_1)}-1)\Omega(c_1). \end{aligned} \tag{3.7}$$

Similarly

$$\begin{aligned}\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)) &= \Omega(c_1+\eta(d_1,c_1)+(\tau_1)\eta(c_1,c_1+\eta(d_1,c_1)))\\ &\supseteq (e^{(1-\tau_1)}-1)\Omega(c_1+\eta(d_1,c_1))+(e^{\tau_1}-1)\Omega(c_1). \end{aligned} \tag{3.8}$$

Thus, by adding (3.7) and (3.8), we get

$$\Omega(c_1+\tau_1\eta(d_1,c_1))+\Omega(c_1+(1-\tau_1)\eta(d_1,c_1))\supseteq[e^{\tau_1}+e^{(1-\tau_1)}-2](\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1)).$$

By multiplying by τ_1^{m-1} on both sides and integrating over $[0,1]$ with respect to τ_1, we get

$$\begin{aligned}&(IR)\int_0^1\tau_1^{m-1}\Omega(c_1+\tau_1\eta(d_1,c_1))d\tau_1+(IR)\int_0^1\tau_1^{m-1}\Omega(c_1+(1-\tau_1)\eta(d_1,c_1))d\tau_1\\ &\supseteq (IR)\int_0^1\tau_1^{m-1}[e^{\tau_1}+e^{(1-\tau_1)}-2](\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1))d\tau_1.\end{aligned}$$

Now, from (3.2) we get

$$\begin{aligned}&(IR)\int_0^1\tau_1^{m-1}[e^{\tau_1}+e^{(1-\tau_1)}-2](\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1))d\tau_1\\ =&\Big[(R)\int_0^1\tau_1^{m-1}[e^{\tau_1}+e^{(1-\tau_1)}-2](\underset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1))+\underset{\leftrightarrow}{\Omega}(c_1))d\tau_1,\\ &(R)\int_0^1\tau_1^{m-1}[e^{\tau_1}+e^{(1-\tau_1)}-2](\overset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1))+\overset{\leftrightarrow}{\Omega}(c_1))d\tau_1\Big]\\ =&\Big[P(\underset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1))+\underset{\leftrightarrow}{\Omega}(c_1)),P(\overset{\leftrightarrow}{\Omega}(c_1+\eta(d_1,c_1))+\overset{\leftrightarrow}{\Omega}(c_1))\Big]\\ =&P(\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1)).\end{aligned} \tag{3.9}$$

Also from (3.4), (3.5) and (3.9), we get

$$\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1))]\supseteq P(\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1)). \tag{3.10}$$

Combining (3.6) and (3.10), we get

$$\begin{aligned}\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big)&\supseteq\frac{\Gamma(m+1)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1))]\\ &\supseteq mP(\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1)).\end{aligned}$$

□

Corollary 3.1. *If $\overleftrightarrow{\Omega}(v) = \underleftrightarrow{\Omega}(v)$, then (3.1) leads to the following fractional inequality for exponential type pre-invex function:*

$$\frac{1}{(e^{\frac{1}{2}}-1)}\Omega\Big(c_1+\frac{1}{2}\eta(d_1,c_1)\Big) \le \frac{\Gamma(m+1)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1))]$$
$$\le mP(\Omega(c_1+\eta(d_1,c_1))+\Omega(c_1)).$$

Theorem 3.2. *Let $\Lambda \subset \Re$ be an open invex set with respect to $\eta : \Lambda\times\Lambda \to \Re$ and $a_1, b_1 \in \Lambda$ with $a_1 < a_1+\eta(b_1,a_1)$. If $\Omega, \Omega_1 : [a_1, a_1+\eta(b_1,a_1)] \to \Re$ are exponential type pre-invex interval-valued functions such that $\Omega, \Omega_1 \in L[a_1, a_1+\eta(b_1,a_1)]$ and $m > 0$, then we have (considering Lemma 2.2 holds)*

$$\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))]$$
$$\supseteq P_1\Upsilon_1(a_1,a_1+\eta(b_1,a_1))+2P_2\Upsilon_2(a_1,a_1+\eta(b_1,a_1)), \tag{3.11}$$

where

$$P_1 = \frac{e^2\,\Gamma(m)-e^2\,\Gamma(m,2)}{2^m}+2e\,\Gamma(m,1)+\frac{2\,\Gamma(m,-1)-2\,\Gamma(m)}{(-1)^m}$$
$$+\frac{\Gamma(m)-\Gamma(m,-2)}{(-1)^m\cdot 2^m}-2e\,\Gamma(m)+\frac{2}{m}, \tag{3.12}$$

$$P_2 = e\,\Gamma(m,1)+\frac{\Gamma(m,-1)}{(-1)^m}-\frac{\Gamma(m)}{(-1)^m}-e\,\Gamma(m)+\frac{e}{m}+\frac{1}{m}, \tag{3.13}$$

$$\Upsilon_1(a_1,a_1+\eta(b_1,a_1)) = [\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1))+\Omega(a_1).\Omega_1(a_1)], \tag{3.14}$$

and

$$\Upsilon_2(a_1,a_1+\eta(b_1,a_1)) = [\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1)+\Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))]. \tag{3.15}$$

Proof. Since Ω *and* Ω_1 are exponential type pre-invex interval-valued functions, so we have

$$\Omega(a_1+\tau_1\eta(b_1,a_1)) = \Omega(a_1+\eta(b_1,a_1)+(1-\tau_1)\eta(a_1,a_1+\eta(b_1,a_1)))$$
$$\supseteq (e^{\tau_1}-1)\Omega(a_1+\eta(b_1,a_1))+(e^{(1-\tau_1)}-1)\Omega(a_1)$$

and

$$\Omega_1(a_1+\tau_1\eta(b_1,a_1)) = \Omega_1(a_1+\eta(b_1,a_1)+(1-\tau_1)\eta(a_1,a_1+\eta(b_1,a_1)))$$
$$\supseteq (e^{\tau_1}-1)\Omega_1(a_1+\eta(b_1,a_1))+(e^{(1-\tau_1)}-1)\Omega_1(a_1).$$

Since $\Omega, \Omega_1 \in \Re^+_\Lambda$, so

$$\Omega(a_1+\tau_1\eta(b_1,a_1)).\Omega_1(a_1+\tau_1\eta(b_1,a_1))$$
$$\supseteq(e^{\tau_1}-1)^2\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1))+(e^{(1-\tau_1)}-1)^2\Omega(a_1).\Omega_1(a_1)$$
$$+(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1)+\Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))]. \tag{3.16}$$

Similarly, we have

$$
\begin{aligned}
&\Omega(a_1 + (1-\tau_1)\eta(b_1,a_1)).\Omega_1(a_1 + (1-\tau_1)\eta(b_1,a_1)) \\
\supseteq&(e^{(1-\tau_1)}-1)^2\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1)) + (e^{\tau_1}-1)^2\Omega(a_1).\Omega_1(a_1) \\
&+ (e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1) + \Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))]. \qquad (3.17)
\end{aligned}
$$

Adding (3.16) and (3.17) yields

$$
\begin{aligned}
&\Omega(a_1+\tau_1\eta(b_1,a_1)).\Omega_1(a_1+\tau_1\eta(b_1,a_1)) + \Omega(a_1+(1-\tau_1)\eta(b_1,a_1)).\Omega_1(a_1+(1-\tau_1)\eta(b_1,a_1)) \\
\supseteq&[(e^{(1-\tau_1)}-1)^2 + (e^{\tau_1}-1)^2][\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1)) + \Omega(a_1).\Omega_1(a_1)] \\
&+ 2(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1) + \Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))].
\end{aligned}
$$

From (3.14) and (3.15), we have

$$
\begin{aligned}
&\Omega(a_1+\tau_1\eta(b_1,a_1)).\Omega_1(a_1+\tau_1\eta(b_1,a_1)) + \Omega(a_1+(1-\tau_1)\eta(b_1,a_1)).\Omega_1(a_1+(1-\tau_1)\eta(b_1,a_1)) \\
\supseteq&[(e^{(1-\tau_1)}-1)^2 + (e^{\tau_1}-1)^2]\Upsilon_1(a_1,a_1+\eta(b_1,a_1)) + 2(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)\Upsilon_2(a_1,a_1+\eta(b_1,a_1)).
\end{aligned}
$$

Multiplying by τ_1^{m-1} on both sides and integrating over $[0,1]$ with respect to τ_1 gives

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\Omega(a_1+\tau_1\eta(b_1,a_1)).\Omega_1(a_1+\tau_1\eta(b_1,a_1))d\tau_1 \\
&+ (IR)\int_0^1 \tau_1^{m-1}\Omega(a_1+(1-\tau_1)\eta(b_1,a_1)).\Omega_1(a_1+(1-\tau_1)\eta(b_1,a_1))d\tau_1 \\
\supseteq&(IR)\int_0^1 \tau_1^{m-1}[(e^{(1-\tau_1)}-1)^2 + (e^{\tau_1}-1)^2]\Upsilon_1(a_1,a_1+\eta(b_1,a_1))d\tau_1 \\
&+ 2(IR)\int_0^1 \tau_1^{m-1}(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)\Upsilon_2(a_1,a_1+\eta(b_1,a_1))d\tau_1.
\end{aligned}
$$

So

$$
(IR)\int_0^1 \tau_1^{m-1}\Omega(a_1+\tau_1\eta(b_1,a_1)).\Omega_1(a_1+\tau_1\eta(b_1,a_1))d\tau_1 = \frac{\Gamma(m)}{\eta^m(d_1,c_1)} I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)
$$

and

$$
\begin{aligned}
&(IR)\int_0^1 \tau_1^{m-1}\Omega(a_1+(1-\tau_1)\eta(b_1,a_1)).\Omega_1(a_1+(1-\tau_1)\eta(b_1,a_1))d\tau_1 \\
&= \frac{\Gamma(m)}{\eta^m(d_1,c_1)} I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1)).
\end{aligned}
$$

From (3.12) and (3.13), we get

$$
(IR)\int_0^1 \tau_1^{m-1}[(e^{(1-\tau_1)}-1)^2 + (e^{\tau_1}-1)^2]\Upsilon_1(a_1,a_1+\eta(b_1,a_1))d\tau_1 = P_1\Upsilon_1(a_1,a_1+\eta(b_1,a_1))
$$

and

$$
(IR)\int_0^1 \tau_1^{m-1}(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)\Upsilon_2(a_1,a_1+\eta(b_1,a_1))d\tau_1 = P_2\Upsilon_2(a_1,a_1+\eta(b_1,a_1)).
$$

Thus,

$$\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))]$$
$$\supseteq P_1\Upsilon_1(a_1,a_1+\eta(b_1,a_1))+2P_2\Upsilon_2(a_1,a_1+\eta(b_1,a_1)).$$

□

Corollary 3.2. *If $\overleftrightarrow{\Omega}(v)=\underset{\leftrightarrow}{\Omega}(v)$, then (3.11) leads to the following fractional inequality for exponential type pre-invex function:*

$$\frac{\Gamma(m)}{\eta^m(d_1,c_1)}[I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)+I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))]$$
$$\leq P_1\Upsilon_1(a_1,a_1+\eta(b_1,a_1))+2P_2\Upsilon_2(a_1,a_1+\eta(b_1,a_1)).$$

Theorem 3.3. *Let $\Lambda\subset\Re$ be an open invex set with respect to $\eta:\Lambda\times\Lambda\to\Re$ and $a_1,b_1\in\Lambda$ with $a_1<a_1+\eta(b_1,a_1)$. If $\Omega,\Omega_1:[a_1,a_1+\eta(b_1,a_1)]\to\Re$ are exponential type pre-invex interval-valued functions such that $\Omega,\Omega_1\in L[a_1,a_1+\eta(b_1,a_1)]$ and $m>0$, then from (3.12)–(3.15), we have (considering Lemma 2.2 holds)*

$$\begin{aligned}&\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))\\ \supseteq&(e^{\frac{1}{2}}-1)^2\Big[mP_1\Upsilon_2(a_1,a_1+\eta(b_1,a_1))+mP_2\Upsilon_1(a_1,a_1+\eta(b_1,a_1))\\ &+\frac{\Gamma(m+1)}{\eta^m(d_1,c_1)}[I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))+I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)]\Big].\end{aligned}\tag{3.18}$$

Proof. Since Ω is an exponential type pre-invex interval-valued function, so we have

$$\Omega\Big(a_1+\frac{1}{2}\eta(b_1,a_1)\Big)\supseteq(e^{\frac{1}{2}}-1)[\Omega(a_1)+\Omega(b_1)].$$

Taking $a_1=c_1+(1-\tau_1)\eta(d_1,c_1)$ and $b_1=c_1+(\tau_1)\eta(d_1,c_1)$ gives

$$\Omega\Big(c_1+(1-\tau_1)\eta(d_1,c_1)+\frac{1}{2}\eta(c_1+(\tau_1)\eta(d_1,c_1),c_1+(1-\tau_1)\eta(d_1,c_1))\Big)$$
$$\supseteq(e^{\frac{1}{2}}-1)[\Omega(c_1+(1-\tau_1)\eta(d_1,c_1))+\Omega(c_1+(\tau_1)\eta(d_1,c_1))],$$

implies

$$\Omega(c_1+\frac{1}{2}\eta(d_1,c_1))\supseteq(e^{\frac{1}{2}}-1)[\Omega(c_1+(1-\tau_1)\eta(d_1,c_1))+\Omega(c_1+(\tau_1)\eta(d_1,c_1))].\tag{3.19}$$

Similarly

$$\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))\supseteq(e^{\frac{1}{2}}-1)[\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))+\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))].\tag{3.20}$$

Multiplying (3.19) and (3.20) gives

$$\begin{aligned}
&\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))\\
&\supseteq(e^{\frac{1}{2}}-1)^2[\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))\\
&\quad+\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))\\
&\quad+\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))\\
&\quad+\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))].
\end{aligned}\tag{3.21}$$

Since $\Omega,\Omega_1\in\mathfrak{R}^+_\Lambda$, are exponential type pre-invex interval-valued functions for $\tau_1\in[0,1]$, so we have

$$\begin{aligned}
&\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))\\
&\supseteq(e^{(1-\tau_1)}-1)^2\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1)+(e^{\tau_1}-1)^2\Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))\\
&\quad+(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1))+\Omega(a_1).\Omega_1(a_1)].
\end{aligned}\tag{3.22}$$

Similarly

$$\begin{aligned}
&\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))\\
&\supseteq(e^{\tau_1}-1)^2\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1)+(e^{(1-\tau_1)}-1)^2\Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1))\\
&\quad+(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1))+\Omega(a_1).\Omega_1(a_1)].
\end{aligned}\tag{3.23}$$

Adding (3.22) and (3.23) yields

$$\begin{aligned}
&\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))+\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))\\
&\supseteq[(e^{\tau_1}-1)^2+(e^{(1-\tau_1)}-1)^2](\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1)+\Omega(a_1).\Omega_1(a_1+\eta(b_1,a_1)))\\
&\quad+2(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)[\Omega(a_1+\eta(b_1,a_1)).\Omega_1(a_1+\eta(b_1,a_1))+\Omega(a_1).\Omega_1(a_1)].
\end{aligned}$$

Now from (3.21), we can write

$$\begin{aligned}
&\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))\\
&\supseteq(e^{\frac{1}{2}}-1)^2\Big[[(e^{\tau_1}-1)^2+(e^{(1-\tau_1)}-1)^2]\Upsilon_2(a_1,a_1+\eta(b_1,a_1))\\
&\quad+2(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)\Upsilon_1(a_1,a_1+\eta(b_1,a_1))\\
&\quad+\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))\\
&\quad+\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))\Big].
\end{aligned}$$

Multiplying by τ_1^{m-1} on both sides and integrating over $[0,1]$ with respect to τ_1 yields

$$\begin{aligned}
&(IR)\int_0^1\tau_1^{m-1}\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))d\tau_1\\
&\supseteq(e^{\frac{1}{2}}-1)^2\Big[(IR)\int_0^1\tau_1^{m-1}[(e^{\tau_1}-1)^2+(e^{(1-\tau_1)}-1)^2]\Upsilon_2(a_1,a_1+\eta(b_1,a_1))d\tau_1
\end{aligned}$$

$$+ 2(IR)\int_0^1 \tau_1^{m-1}(e^{\tau_1}-1)(e^{(1-\tau_1)}-1)\Upsilon_1(a_1, a_1+\eta(b_1,a_1))d\tau_1$$
$$+ (IR)\int_0^1 \tau_1^{m-1}\Omega(c_1+(1-\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(1-\tau_1)\eta(d_1,c_1))d\tau_1$$
$$+ (IR)\int_0^1 \tau_1^{m-1}\Omega(c_1+(\tau_1)\eta(d_1,c_1)).\Omega_1(c_1+(\tau_1)\eta(d_1,c_1))d\tau_1\Big].$$

Thus from (3.12)–(3.15), we get

$$\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1)) \supseteq (e^{\frac{1}{2}}-1)^2\Big[mP_1\Upsilon_2(a_1,a_1+\eta(b_1,a_1))+mP_2\Upsilon_1(a_1,a_1$$
$$+\eta(b_1,a_1))+\frac{\Gamma(m+1)}{\eta^m(d_1,c_1)}[I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))+I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)]\Big].$$

□

Corollary 3.3. *If $\overleftrightarrow{\Omega}(v) = \underset{\leftrightarrow}{\Omega}(v)$, then (3.18) leads to the following fractional inequality for exponential type pre-invex function:*

$$\Omega(c_1+\frac{1}{2}\eta(d_1,c_1)).\Omega_1(c_1+\frac{1}{2}\eta(d_1,c_1))$$
$$\le(e^{\frac{1}{2}}-1)^2\Big[mP_1\Upsilon_2(a_1,a_1+\eta(b_1,a_1))+mP_2\Upsilon_1(a_1,a_1+\eta(b_1,a_1))$$
$$+\frac{\Gamma(m+1)}{\eta^m(d_1,c_1)}[I^m_{c_1^+}\Omega(c_1+\eta(d_1,c_1)).\Omega_1(c_1+\eta(d_1,c_1))+I^m_{(c_1+\eta(d_1,c_1))^-}\Omega(c_1).\Omega_1(c_1)]\Big].$$

4. Interval fractional Hermite-Hadamard type Inequality via He's fractional derivative

In this section, we establish Hermite-hadamard type inequality in the setting of the He's fractional derivatives introduced in [18].

Definition 4.1. *Let Ω be an L_1 function defined on an interval $[0, n_1]$. Then the k_1-th He's fractional derivative of $\Omega(n_1)$ is defined by*

$$I^{k_1}_{n_1}\Omega(n_1) = \frac{1}{\Gamma(i-k_1)}\frac{d^i}{dn_1^i}\int_0^{n_1}(\tau_1-n)^{i-k_1-1}\Omega(\tau_1)d\tau_1.$$

The interval He's fractional derivative based on left and right end point functions can be defined by

$$I^{k_1}_{n_1}\Omega(n_1) = \frac{1}{\Gamma(i-k_1)}\frac{d^i}{dn_1^i}\int_0^{n_1}(\tau_1-n)^{i-k_1-1}\Omega(\tau_1)d\tau_1$$
$$= \frac{1}{\Gamma(i-k_1)}\frac{d^i}{dn_1^i}\int_0^{n_1}(\tau_1-n)^{i-k_1-1}[\underset{\leftrightarrow}{\Omega}(\tau_1), \overleftrightarrow{\Omega}(\tau_1)]d\tau_1, n > n_1,$$

where

$$I^{k_1}_{n_1}\underset{\leftrightarrow}{\Omega}(n_1) = \frac{1}{\Gamma(i-k_1)}\frac{d^i}{dn_1^i}\int_0^{n_1}(\tau_1-n)^{i-k_1-1}\underset{\leftrightarrow}{\Omega}(\tau_1)d\tau_1, n > n_1 \tag{4.1}$$

and

$$I^{k_1}_{n_1}\overleftrightarrow{\Omega}(n_1) = \frac{1}{\Gamma(i-k_1)}\frac{d^i}{dn_1^i}\int_0^{n_1}(\tau_1-n)^{i-k_1-1}\overleftrightarrow{\Omega}(\tau_1)d\tau_1, n > n_1. \tag{4.2}$$

Theorem 4.1. *Let $\Omega : [n_1, n_2] \to \Re$ be an exponential type pre-invex interval-valued function defined on $[n_1, n_2] \subset \Lambda$, where Λ is an open invex set with respect to $\eta : \Lambda \times \Lambda \to \Re$ and $\Omega : [n_1, n_2] \subset \Re \to \Re_c^+$ is given by $\Omega(n) = [\underleftrightarrow{\Omega}(n), \overleftrightarrow{\Omega}(n)]$ for all $n \in [n_1, n_2]$. If $\Omega \in L_1([n_1, n_2], \Re)$, then*

$$(-1)^{i-k_1-1}\Omega\left(\frac{n_1}{2}\right) \supseteq \frac{(e^{\frac{1}{2}} - 1)n^{k_1}}{n_2^{i-k_1}}[(-1)^{i-k_1-1}I^{k_1}_{(1-n)b}\Omega((1-n)b) + I^{k_1}_{nb}\Omega(nb)]. \tag{4.3}$$

Proof. Let $\Omega : [n_1, n_2] \to \Re$ be an exponential type pre-invex interval-valued function defined on $[n_1, n_2]$, then

$$\Omega\left(n_1 + \frac{1}{2}\eta(n_2, n_1)\right) \supseteq (e^{\frac{1}{2}} - 1)[\Omega(n_2 + \tau_1\eta(n_1, n_2)) + \Omega(n_1 + \tau_1\eta(n_2, n_1))]$$

and

$$\underleftrightarrow{\Omega}\left(n_1 + \frac{1}{2}\eta(n_2, n_1)\right) \le (e^{\frac{1}{2}} - 1)[\underleftrightarrow{\Omega}(n_2 + \tau_1\eta(n_1, n_2)) + \underleftrightarrow{\Omega}(n_1 + \tau_1\eta(n_2, n_1))].$$

Taking $n_2 = 0$, $0 \le n_1$ and multiplying by $\frac{(\tau_1 - n)^{i-k_1-1}}{\Gamma(i-k_1)}$, we get

$$\frac{(\tau_1 - n)^{i-k_1-1}}{\Gamma(i-k_1)}\underleftrightarrow{\Omega}\left(\frac{n_1}{2}\right) \le (e^{\frac{1}{2}} - 1)\frac{(\tau_1 - n)^{i-k_1-1}}{\Gamma(i-k_1)}[\underleftrightarrow{\Omega}((1-\tau_1)n_1) + \underleftrightarrow{\Omega}(\tau_1 n_1)].$$

Integrating with respect to τ_1 over $[0, n_1]$ gives

$$\begin{aligned}&\underleftrightarrow{\Omega}\left(\frac{n_1}{2}\right)\frac{1}{\Gamma(i-k_1)}\int_0^{n_1}(\tau_1 - n)^{i-k_1-1}d\tau_1\\ \le&\frac{(e^{\frac{1}{2}} - 1)}{\Gamma(i-k_1)}\int_0^{n_1}(\tau_1 - n)^{i-k_1-1}\underleftrightarrow{\Omega}((1-\tau_1)n_1)d\tau_1 + \frac{(e^{\frac{1}{2}} - 1)}{\Gamma(i-k_1)}\int_0^{n_1}(\tau_1 - n)^{i-k_1-1}\underleftrightarrow{\Omega}(\tau_1 n_1)d\tau_1,\end{aligned}$$

implies

$$\begin{aligned}&\underleftrightarrow{\Omega}\left(\frac{n_1}{2}\right)\frac{(-1)^{i-k_1-1}n^{i-k_1}}{\Gamma(i-k_1)}\\ \le&\frac{(e^{\frac{1}{2}} - 1)}{\Gamma(i-k_1)}\int_0^{n_1}(\tau_1 - n)^{i-k_1-1}\underleftrightarrow{\Omega}((1-\tau_1)n_1)d\tau_1 + \frac{(e^{\frac{1}{2}} - 1)}{\Gamma(i-k_1)}\int_0^{n_1}(\tau_1 - n)^{i-k_1-1}\underleftrightarrow{\Omega}(\tau_1 n_1)d\tau_1.\end{aligned}$$

Getting i-th derivative on both sides and using (4.1), we get

$$(-1)^{i-k_1-1}\underleftrightarrow{\Omega}\left(\frac{n_1}{2}\right) \le \frac{(e^{\frac{1}{2}} - 1)n^{k_1}}{n_1^{i-k_1}}[(-1)^{i-k_1-1}I^{k_1}_{(1-n)b}\underleftrightarrow{\Omega}((1-n)b) + I^{k_1}_{nb}\underleftrightarrow{\Omega}(nb)].$$

Similarly

$$(-1)^{i-k_1-1}\overleftrightarrow{\Omega}\left(\frac{n_1}{2}\right) \le \frac{(e^{\frac{1}{2}} - 1)n^{k_1}}{n_1^{i-k_1}}[(-1)^{i-k_1-1}I^{k_1}_{(1-n)b}\overleftrightarrow{\Omega}((1-n)b) + I^{k_1}_{nb}\overleftrightarrow{\Omega}(nb)].$$

Thus, we can write

$$
\begin{aligned}
&(-1)^{i-k_1-1}\left[\underleftrightarrow{\Omega}\left(\frac{n_1}{2}\right), \overleftrightarrow{\Omega}\left(\frac{n_1}{2}\right)\right] \\
&\supseteq \frac{(e^{\frac{1}{2}}-1)n^{k_1}}{n_1^{i-k_1}}\left[(-1)^{i-k_1-1} I^{k_1}_{(1-n)b}[\underleftrightarrow{\Omega}((1-n)b), \overleftrightarrow{\Omega}((1-n)b)] + I^{k_1}_{nb}[\underleftrightarrow{\Omega}(nb), \overleftrightarrow{\Omega}(nb)]\right].
\end{aligned}
$$

So,

$$
(-1)^{i-k_1-1}\Omega\left(\frac{n_1}{2}\right) \supseteq \frac{(e^{\frac{1}{2}}-1)n^{k_1}}{n_1^{i-k_1}}[(-1)^{i-k_1-1} I^{k_1}_{(1-n)b}\Omega((1-n)b) + I^{k_1}_{nb}\Omega(nb)].
$$

□

Corollary 4.1. *If $\overleftrightarrow{\Omega}(v) = \underleftrightarrow{\Omega}(v)$, then (4.3) leads to the following fractional inequality for exponential type pre-invex function:*

$$
(-1)^{i-k_1-1}\Omega\left(\frac{n_1}{2}\right) \leq \frac{(e^{\frac{1}{2}}-1)n^{k_1}}{n_1^{i-k_1}}[(-1)^{i-k_1-1} I^{k_1}_{(1-n)b}\Omega((1-n)b) + I^{k_1}_{nb}\Omega(nb)].
$$

5. Conclusions

In this paper we studied the interval-valued exponential type pre-invex functions. We established He's and Hermite-Hadamard type inequalities for interval-valued exponential type pre-invex functions in the setting of Riemann-Liouville interval-valued fractional operator.

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Conflict of interest

The author declares no conflict of interest.

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