Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2568–2584.
DOI: 10.3934/math.2022144
Received: 22 September 2021
Accepted: 08 November 2021
Published: 17 December 2021

Research article

A study of the Fekete-Szegö functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination condition and associated with the Gegenbauer polynomials

H. M. Srivastava[1,2,3,4,*], **Muhammet Kamalı**[5] **and Anarkül Urdaletova**[6]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

[4] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

[5] Department of Mathematics, Faculty of Science, Kyrgyz-Turkish Manas University, Chyngz Aitmatov Avenue, 720038 Biskek, Chuy Province, Kyrgyz Republic

[6] Department of Mathematics, Faculty of Science, Kyrgyz-Turkish Manas University, Chyngz Aitmatov Avenue, 720038 Biskek, Chuy Province, Kyrgyz Republic

* **Correspondence:** Email: harimsri@math.uviv.ca; Tel: +12504725313, +12504776960; Fax: +12507218962.

Abstract: In this paper, we introduce and study a new subclass of normalized analytic functions, denoted by

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha,\delta,\mu,H(z,C_n^{(\lambda)}(t))\Big),$$

satisfying the following subordination condition and associated with the Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(t)$ of order λ and degree n in t:

$$\alpha\left(\frac{zG'(z)}{G(z)}\right)^{\delta}+(1-\alpha)\left(\frac{zG'(z)}{G(z)}\right)^{\mu}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\mu}\prec H(z,C_n^{(\lambda)}(t)),$$

where

$$H(z,C_n^{(\lambda)}(t))=\sum_{n=0}^{\infty}C_n^{(\lambda)}(t)\,z^n=\left(1-2tz+z^2\right)^{-\lambda},$$

$$G(z)=\gamma\beta z^2 f''(z)+(\gamma-\beta)\,zf'(z)+(1-\gamma+\beta)\,f(z),$$

$0 \leqq \alpha \leqq 1$, $1 \leqq \delta \leqq 2$, $0 \leqq \mu \leqq 1$, $0 \leqq \beta \leqq \gamma \leqq 1$, $\lambda \geqq 0$ and $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$. For functions in this function class, we first derive the estimates for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and then examine the Fekete-Szegö functional. Finally, the results obtained are applied to subclasses of normalized analytic functions satisfying the subordination condition and associated with the Legendre and Chebyshev polynomials. The basic or quantum (or q-) calculus and its so-called trivially inconsequential (p, q)-variations have also been considered as one of the concluding remarks.

Keywords: analytic functions; univalent functions; principle of subordination; Gegenbauer (or ultraspherical) polynomials; coefficient estimates; Fekete-Szegö functional; Legendre and Chebyshev polynomials; Horadam and related polynomials; basic or quantum (or q-) calculus and its so-called trivially inconsequential (p, q)-variation
Mathematics Subject Classification: Primary 30C45; Secondary 11M35, 30C50, 33C45

1. Introduction, definitions and motivation

Let $\mathcal{A}$ denote the family of all analytic functions, which are defined on the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by the following condition:

$$f(0) = f'(0) - 1 = 0.$$

Such functions $f \in \mathcal{A}$ have the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Furthermore, by $\mathcal{S}$ we denote the class of all functions $f \in \mathcal{A}$ that are also univalent in $\mathbb{U}$.

With a view to recalling the principle of subordination between analytic functions, let the functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We then say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

If the function g is univalent function in $\mathbb{U}$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The concept of arithmetic means of functions and other entities is frequently used in mathematics, especially in geometric function theory of complex analysis. Making use of the concept of arithmetic means, Mocanu [12] introduced the class of α-convex functions ($0 \leqq \alpha \leqq 1$) as follows:

$$M_\alpha = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0 \quad (z \in \mathbb{U}) \right\}, \tag{1.2}$$

which, in some case, corresponds to the class of starlike functions and, in another case, to the class of convex functions. In general, the class of α-convex functions determines the arithmetic bridge between starlikeness and convexity.

By using the geometric means, Lewandowski et al. [9] defined the class of μ-starlike functions ($0 \leqq \mu \leqq 1$) consisting of functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{\mu} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu} \right] > 0 \quad (z \in \mathbb{U}). \tag{1.3}$$

We note that the class of μ-starlike functions constitutes the geometric bridge between starlikeness and convexity.

We now recall that a function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+z} \qquad (z \in \mathbb{U}) \tag{1.4}$$

On the other hand, a function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a convex domain if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1-z}{1+z} \qquad (z \in \mathbb{U}). \tag{1.5}$$

It is well known that, if a function $f \in \mathcal{A}$ satisfies (1.4), then f is univalent and starlike in $\mathbb{U}$.

Let $\beta \in [0, 1)$. A function $f \in \mathcal{A}$ is said to be starlike of order β and convex of order β, if

$$\frac{zf'(z)}{f(z)} \prec \frac{1-(1-2\beta)z}{1+z} \qquad (z \in \mathbb{U}) \tag{1.6}$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1-(1-2\beta)z}{1+z} \qquad (z \in \mathbb{U}), \tag{1.7}$$

respectively.

In the year 1933, Fekete and Szegö [6] obtained a sharp bound of the functional $a_3 - \nu a_2^2$, with real ν ($0 \leqq \nu \leqq 1$) for a univalent function f. Since then, the problem of finding the sharp bounds for the Fekete-Szegö functional of any compact family of functions or for $f \in \mathcal{A}$ with any complex ν is known as the classical Fekete-Szegö problem (see, for details, [14, 21]). More recently, in the year 1994. Szynal [25] introduced and investigated the class $\mathcal{T}(\lambda)$ $(\lambda \geqq 0)$ as a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z, t)\, d\sigma(t), \tag{1.8}$$

where

$$k(z,t) = \frac{z}{(1-2tz+z^2)^{\lambda}} \qquad (z \in \mathbb{U};\ -1 \leqq t \leqq 1) \tag{1.9}$$

and σ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$. The function $k(z,t)$ has the following Taylor-Maclaurin series expansion:

$$k(z,t) = z + C_1^{(\lambda)}(t)\, z^2 + C_2^{(\lambda)}(t)\, z^3 + C_3^{(\lambda)}(t)\, z^4 + \cdots, \tag{1.10}$$

where $C_n^{(\lambda)}(t)$ denotes the Gegenbauer (or ultraspherical) polynomials of order λ and degree n in t, which are generated by (see, for details, [18])

$$H(z, C_n^{(\lambda)}(t)) = \sum_{n=0}^{\infty} C_n^{(\lambda)}(t)\, z^n = \left(1-2tz+z^2\right)^{-\lambda}. \tag{1.11}$$

If a function $f \in \mathcal{T}(\lambda)$ is given by (1.8), then the coefficients of this function can be written as follows:

$$a_n = \int_{-1}^{1} C_{n-1}^{(\lambda)}(t)\ d\sigma(t). \tag{1.12}$$

We note that $\mathcal{T}(1) =: \mathcal{T}$ is the well-known class of typically real functions.

The Gengenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(t)$ as well as their relatively more familiar special or limit cases such as the Legendre (or spherical) polynomials $P_n(t)$, the Chebyshev polynomials $T_n(t)$ of the first kind, and the Chebyshev polynomials $U_n(t)$ of the second kind, are orthogonal over the interval $[-1,1]$. In fact, we have

$$P_n(t) = C_n^{(\frac{1}{2})}(t), \quad T_n(t) = \frac{1}{2}\, n \lim_{\lambda\to\infty} \left\{\frac{C_n^{(\lambda)}(t)}{\lambda}\right\} \quad \text{and} \quad U_n(t) = C_n^{(1)}(t). \tag{1.13}$$

The subject of Geometric Function Theory of Complex Analysis has been a fast-growing area of research in recent years. Noteworthy developments and studies involving various old (or traditional) as well as newly-introduced subclasses of the class of normalized analytic or meromorphic functions, together with the multivalent analogues in each case, can be found in the remarkably vast literature on this subject. A good source for some recent researches and developments in Geometric Function Theory of Complex Analysis is the 888-page edited volume by Milovanović and Rassias [11].

Our present investigation is motivated by the above-mentioned developments as well as by many recent works on the Fekete-Szegö functional and other coefficient estimate problems by (for example) Dziok et al. [5], Ałtinkaya and Yalçin [2], Srivastava et al. [20], Szatmari and Ałtinkaya [24], and Çağlar et al. [4] (see also [1, 3, 8, 10, 13, 14, 17, 19, 21]). Here, in this paper, we introduce and study a new subclass of normalized analytic functions $\mathcal{A}$ in $\mathbb{U}$, which we denote by

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big).$$

We say that a function $f \in \mathcal{A}$ of the form (1.1) is in the following class:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big),$$

if it satisfies the following subordination condition associated the Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(t)$ of order λ and degree n in t:

$$\alpha\left(\frac{zG'(z)}{G(z)}\right)^{\delta}+(1-\alpha)\left(\frac{zG'(z)}{G(z)}\right)^{\mu}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\mu}\prec H(z,C_n^{(\lambda)}(t)), \tag{1.14}$$

where $H(z,C_n^{(\lambda)}(t))$ is given by the generating relation (1.11),

$$G(z)=\gamma\beta z^2 f''(z)+(\gamma-\beta)zf'(z)+(1-\gamma+\beta)f(z), \tag{1.15}$$

and

$$0\leqq\alpha\leqq 1,\quad 1\leqq\delta\leqq 2,\quad 0\leqq\mu\leqq 1,\quad 0\leqq\beta\leqq\gamma\leqq 1,\quad \lambda\geqq 0\quad\text{and}\quad t\in\left(\frac{1}{\sqrt{2}},1\right].$$

For functions in this subclass, we first derive the estimates for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and then examine the corresponding Fekete-Szegö inequality. Finally, the results obtained are applied to subclasses of normalized analytic functions satisfying the subordination condition and associated with the Legendre and Chebyshev polynomials. In the concluding section, we have indicated the possibility of using the basic or quantum (or q-) calculus and we have also exposed the so-called trivial and inconsequential (p,q)-variations by forcing-in an obviously redundant (or superfluous) parameter p in the familiar q-calculus.

2. Initial coefficient bounds for the function class $\mathcal{F}_{(\beta,\gamma)}\Big(\alpha,\delta,\mu,H(z,C_n^{(\lambda)}(t))\Big)$

Our first result (Theorem 1 below) provides bounds for the initial Taylor-Maclaurin coefficients a_2 and a_3 in (1.1).

Theorem 1. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha,\delta,\mu,H(z,C_n^{(\lambda)}(t))\Big).$$

Then

$$|a_2|\leqq\frac{2\lambda t}{\{\alpha\delta+(1-\alpha)(2-\mu)\}(2\gamma\beta+\gamma-\beta+1)} \tag{2.1}$$

and

$$|a_3|\leqq\frac{\Big(\lambda(\lambda+1)[\alpha\delta+(1-\alpha)(2-\mu)]^2-\Big[\alpha\left(\delta^2-3\delta\right)+(1-\alpha)\left(\mu^2+5\mu-8\right)\Big]\lambda^2\Big)2t^2}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2[\alpha\delta+(1-\alpha)(3-2\mu)]\cdot[2(3\gamma\beta+\gamma-\beta)+1]}-\frac{\lambda}{2[\alpha\delta+(1-\alpha)(3-2\mu)]\cdot[2(3\gamma\beta+\gamma-\beta)+1]}, \tag{2.2}$$

provided that

$$0\leqq\alpha\leqq 1,\quad 1\leqq\delta\leqq 2,\quad 0\leqq\mu\leqq 1,\quad 0\leqq\beta\leqq\gamma\leqq 1,\quad \lambda\geqq 0\quad\textit{and}\quad t\in\left(\frac{1}{\sqrt{2}},1\right].$$

Proof. Under the hypotheses of Theorem 1, we find from (1.1) and (1.15) that

$$\begin{aligned} G(z) &= \gamma\beta z^2 f''(z) + (\gamma-\beta) z f'(z) + (1-\gamma+\beta) f(z) \\ &= [\gamma-\beta+(1-\gamma+\beta)]\, z + \sum_{n=2}^{\infty} [\gamma\beta n(n-1) + (\gamma-\beta) n + (1-\gamma+\beta)]\, a_n z^n \\ &= z + \sum_{n=2}^{\infty} [(n-1)(\gamma\beta n + \gamma - \beta) + 1]\, a_n z^n \\ &= z + [(2\gamma\beta+\gamma-\beta)+1]\, a_2 z^2 + [2(3\gamma\beta+\gamma-\beta)+1]\, a_3 z^3 + \cdots . \end{aligned} \tag{2.3}$$

Now, upon setting $\nabla := 2\gamma\beta + \gamma - \beta$ in (2.3), we can write

$$G(z) = z + (\nabla+1)\, a_2 z^2 + [2(\nabla+\gamma\beta)+1]\, a_3 z^3 + \cdots ,$$

which readily yields

$$\begin{aligned} \frac{zG'(z)}{G(z)} &= \frac{z + 2(\nabla+1)\, a_2 z^2 + 3[2(\nabla+\gamma\beta)+1]\, a_3 z^3 + \cdots}{z + (\nabla+1)\, a_2 z^2 + [2(\nabla+\gamma\beta)+1]\, a_3 z^3 + \cdots} \\ &= 1 + (\nabla+1)\, a_2 z + \left([4(\nabla+\gamma\beta)+2]\, a_3 - (\nabla+1)^2 a_2^2\right) z^2 + \cdots , \end{aligned}$$

$$\left(\frac{zG'(z)}{G(z)}\right)^{\delta} = 1 + \delta(\nabla+1)\, a_2 z + \frac{\left(\delta^2 - 3\delta\right)(\nabla+1)^2 a_2^2 + 4\delta[2(\nabla+\gamma\beta)+1]\, a_3}{2} z^2 + \cdots ,$$

$$1 + \frac{zG''(z)}{G'(z)} = 1 + 2(\nabla+1)\, a_2 z + \left(6[2(\nabla+\gamma\beta)+1]\, a_3 - 4(\nabla+1)^2 a_2^2\right) z^2 + \cdots ,$$

$$\begin{aligned} \left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\mu} &= 1 + 2(1-\mu)(\nabla+1)\, a_2 z + \left[2\mu(\mu-1)(\nabla+1)^2 a_2^2 + (1-\mu)\right] \\ &\qquad \cdot \left(6[2(\nabla+\gamma\beta)+1]\, a_3 - 4(\nabla+1)^2 a_2^2\right) z^2 + \cdots , \end{aligned}$$

$$\begin{aligned} \left(\frac{zG'(z)}{G(z)}\right)^{\mu} \left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\mu} &= 1 + (2-\mu)(\nabla+1)\, a_2 z \\ &\quad + \left[\frac{\mu^2+5\mu-8}{2}(\nabla+1)^2 a_2^2 + 2(3-2\mu)[2(\nabla+\gamma\beta)+1]\, a_3\right] z^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} &(1-\alpha)\left(\frac{zG'(z)}{G(z)}\right)^{\mu} \left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\mu} \\ &= (1-\alpha) + (1-\alpha)(2-\mu)(\nabla+1)\, a_2 z \\ &\quad + \frac{1-\alpha}{2}\left[\left(\mu^2+5\mu-8\right)(\nabla+1)^2 a_2^2 + 4(3-2\mu)[2(\nabla+\gamma\beta)+1]\, a_3\right] z^2 + \cdots . \end{aligned}$$

If we make use of the above expressions and apply (1.14), we see that

$$\alpha\left(\frac{zG'(z)}{G(z)}\right)^{\delta}+(1-\alpha)\left(\frac{zG'(z)}{G(z)}\right)^{\mu}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\mu}$$
$$=1+C_1^{(\lambda)}(t)\,p(z)+C_2^{(\lambda)}(t)\,(p(z))^2+C_3^{(\lambda)}(t)\,(p(z))^3+\cdots \tag{2.4}$$

for some analytic function $p(z)$ given by

$$p(z)=p_1z+p_2z^2+p_3z^3+\cdots \qquad (z\in\mathbb{U}),$$

such that

$$p(0)=0 \qquad \text{and} \qquad |p(z)|<1 \qquad (z\in\mathbb{U}).$$

Then, for all $j\in\mathbb{N}$, we have

$$\left|p_j\right|\leqq 1. \tag{2.5}$$

Also, for all $\xi\in\mathbb{R}$, we obtain

$$\left|p_2-\xi p_1^2\right|\leqq \max\{1,|\xi|\}. \tag{2.6}$$

It follows from (2.4) that

$$\alpha\left(\frac{zG'(z)}{G(z)}\right)^{\delta}+(1-\alpha)\left(\frac{zG'(z)}{G(z)}\right)^{\mu}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\mu}$$
$$=1+C_1^{(\lambda)}(t)\,p_1z+\left[C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2\right]z^2+\cdots, \tag{2.7}$$

which leads us to the following consequences:

$$\{\alpha\delta+(1-\alpha)(2-\mu)\}(\nabla+1)\,a_2=C_1^{(\lambda)}(t)\,p_1 \tag{2.8}$$

and

$$\left[\frac{\alpha}{2}\left(\delta^2-3\delta\right)+\frac{1-\alpha}{2}\left(\mu^2+5\mu-8\right)\right](\nabla+1)^2a_2^2$$
$$+\left[2\alpha\delta+2(1-\alpha)(3-2\mu)\right]\left[2(\nabla+\gamma\beta)+1\right]a_3$$
$$=C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2. \tag{2.9}$$

Now, from (1.11), (2.5) and (2.8), we can write

$$\left[\alpha\delta+(1-\alpha)(2-\mu)\right](\nabla+1)\,a_2=C_1^{(\lambda)}(t)\,p_1$$
$$\Longrightarrow \{\alpha\delta+(1-\alpha)(2-\mu)\}(\nabla+1)\,a_2=2\lambda t p_1.$$

We thus obtain the first coefficient bound (2.1) asserted by Theorem 1:

$$|a_2|\leqq\frac{2\lambda t}{\{\alpha\delta+(1-\alpha)(2-\mu)\}(\nabla+1)}. \tag{2.10}$$

Similarly, from (1.11), (2.5) and (2.9), we can show that

$$
\begin{aligned}
&[2\alpha\delta + 2(1-\alpha)(3-2\mu)][2(3\gamma\beta+\gamma-\beta)+1]\,a_3 \\
&= 2\lambda t p_2 + \left(\left[\frac{\lambda(\lambda+1)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \left[\alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right]\lambda^2}{[\alpha\delta+(1-\alpha)(2-\mu)]^2}\right]2t^2 - \lambda\right)p_1^2 \\
&= 2\lambda t\left(p_2 - \frac{1}{2t}\left[1 - \left\{\frac{(\lambda+1)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \left[\alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right]\lambda}{[\alpha\delta+(1-\alpha)(2-\mu)]^2}\right\}2t^2\right]p_1^2\right),
\end{aligned}
$$

which, in conjunction with (2.6), yields

$$
\begin{aligned}
|a_3| \leqq{} & \frac{2\lambda t}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\gamma\beta+\gamma-\beta)+1]} \\
& \cdot \max\left\{1, \frac{1}{2t}\left|\frac{(\lambda+1)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \left[\alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right]\lambda}{[\alpha\delta+(1-\alpha)(2-\mu)]^2}\,2t^2 - 1\right|\right\}.
\end{aligned}
$$

Finally, by making use of the parametric constraints given with Theorem 1, we find eventually that

$$
\begin{aligned}
|a_3| \leqq{} & \frac{\left(\lambda(\lambda+1)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \left[\alpha\left(\delta^2-3\delta\right)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right]\lambda^2\right)2t^2}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\gamma\beta+\gamma-\beta)+1]} \\
& - \frac{\lambda}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\gamma\beta+\gamma-\beta)+1]},
\end{aligned}
$$

which is precisely the coefficient bound (2.2) of Theorem 1. This completes our proof of Theorem 1. □

The following corollaries and consequences of Theorem 1 are worthy of note.

I. If we set $\alpha = \delta = 1$ or $\alpha = \mu - 1 = 0$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $f(z)$ given by* (1.1) *be in the following class*:

$$
\mathcal{F}_{(\beta,\gamma)}\left(1, 1, \mu, H(z, C_n^{(\lambda)}(t))\right) \equiv \mathcal{F}_{(\beta,\gamma)}\left(0, \delta, 1, H(z, C_n^{(\lambda)}(t))\right).
$$

Then

$$
|a_2| \leqq \frac{2\lambda t}{2\gamma\beta+\gamma-\beta+1}
$$

and

$$
|a_3| \leqq \frac{\lambda}{2[2(3\gamma\beta+\gamma-\beta)+1]}\left\{(3\lambda+1)2t^2 - 1\right\},
$$

provided that

$$
0 \leqq \beta \leqq \gamma \leqq 1, \quad \lambda \geqq 0 \quad \text{and} \quad t \in \left(\frac{1}{\sqrt{2}}, 1\right].
$$

II. Taking $\beta = \gamma = 0$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(0,0)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big).$$

Then

$$|a_2| \leqq \frac{2\lambda t}{\alpha\delta + (1-\alpha)(2-\mu)}$$

and

$$|a_3| \leqq \frac{\lambda}{2\left[\alpha\delta + (1-\alpha)(3-2\mu)\right]} \cdot \left(\frac{(\lambda+1)\left[\alpha\delta + (1-\alpha)(2-\mu)\right]^2 - \left[\alpha\left(\delta^2 - 3\delta\right) + (1-\alpha)\left(\mu^2 + 5\mu - 8\right)\right]\lambda}{\left[\alpha\delta + (1-\alpha)(2-\mu)\right]^2} 2t^2 - 1 \right),$$

provided that

$$0 \leqq \alpha \leqq 1, \quad 1 \leqq \delta \leqq 2, \quad 0 \leqq \mu \leqq 1, \quad \lambda \geqq 0 \quad \textit{and} \quad t \in \left(\frac{1}{\sqrt{2}}, 1\right].$$

III. If we put $\delta - 1 = \mu = 0$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, 1, 0, H(z, C_n^{(\lambda)}(t))\Big).$$

Then

$$|a_2| \leqq \frac{2\lambda t}{(2-\alpha)(2\gamma\beta + \gamma - \beta + 1)}$$

and

$$|a_3| \leqq \frac{\lambda}{(6-4\alpha)\left[2(3\gamma\beta + \gamma - \beta) + 1\right]} \left(\frac{(\lambda+1)(2-\alpha)^2 - (6\alpha - 8)\lambda}{(2-\alpha)^2} 2t^2 - 1 \right),$$

provided that

$$0 \leqq \alpha \leqq 1, \quad 0 \leqq \beta \leqq \gamma \leqq 1, \quad \lambda \geqq 0 \quad \textit{and} \quad t \in \left(\frac{1}{\sqrt{2}}, 1\right].$$

IV. Taking $\delta - 1 = \mu = 0$ and $\beta = \gamma = 0$ in Theorem 1, we obtain the following corollary.

Corollary 4. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(0,0)}\Big(\alpha, 1, 0, H(z, C_n^{(\lambda)}(t))\Big).$$

Then

$$|a_2| \leqq \frac{2\lambda t}{(2-\alpha)}$$

and

$$|a_3| \leqq \frac{\lambda}{(6-4\alpha)} \left(\frac{(\lambda+1)(2-\alpha)^2 - (6\alpha - 8)\lambda}{(2-\alpha)^2} 2t^2 - 1 \right),$$

provided that

$$0 \leqq \alpha \leqq 1, \quad \lambda \geqq 0 \quad \textit{and} \quad t \in \left(\frac{1}{\sqrt{2}}, 1\right].$$

V. If we set $\alpha = \beta = \gamma = 0$ in Theorem 1, we obtain the following corollary .

Corollary 5. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(0,0)}\Big(0, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big).$$

Then

$$|a_2| \leqq \frac{2\lambda t}{(2-\mu)}$$

and

$$|a_3| \leqq \frac{\lambda}{(6-4\mu)}\left(\left[\frac{(2-\mu)^2 + (12-9\mu)\lambda}{(2-\mu)^2}\right]2t^2 - 1\right),$$

provided that

$$0 \leqq \mu \leqq 1, \quad \lambda \geqq 0 \quad \textit{and} \quad t \in \left(\frac{1}{\sqrt{2}}, 1\right].$$

3. Fekete-Szegö inequality for the function class $\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big)$

In this section, we find the sharp bounds of the Fekete-Szegö functional $a_3 - \xi a_2^2$ defined for functions $f \in \mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big)$, which are given by (1.1).

Theorem 2. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big).$$

Then, for some $\xi \in \mathbb{R}$,

$$\left|a_3 - \xi a_2^2\right| \leqq \begin{cases} \frac{2\lambda t}{K} & (\xi \in [\xi_1, \xi_2]) \\ \frac{2\lambda t}{K}\left|\frac{2\lambda(\lambda+1)t^2 - \lambda}{2\lambda t} - \frac{R\lambda t}{B} - \xi \frac{2\lambda t K}{B(\nabla+1)^2}\right| & (\xi \notin [\xi_1, \xi_2]), \end{cases} \tag{3.1}$$

where

$$\xi_1 = \left(\frac{2\left[(\lambda+1)B - \lambda R\right]t^2 - (1+2t)B}{4\lambda K t^2}\right)(\nabla+1)^2$$

and

$$\xi_2 = \left(\frac{2\left[(\lambda+1)B - \lambda R\right]t^2 - (1-2t)B}{4\lambda K t^2}\right)(\nabla+1)^2$$

such that

$$[2\alpha\delta + 2(1-\alpha)(3-2\mu)]\,[2(\nabla + \gamma\beta) + 1] =: K,$$
$$[\alpha\delta + (1-\alpha)(2-\mu)]^2 =: B$$

and

$$\alpha\left(\delta^2 - 3\delta\right) + (1-\alpha)\left(\mu^2 + 5\mu - 8\right) =: R,$$

δ *being given by*

$$\delta := 2\gamma\beta + \gamma - \beta.$$

Proof. If the above expressions for K, B and R are used for those in the Eqs (2.1) and (2.2), we get

$$\begin{aligned}
\left[\alpha\delta + (1-\alpha)(2-\mu)\right](\nabla + 1)\, a_2 &= C_1^{(\lambda)}(t)\, p_1 \\
\Longrightarrow a_2 &= \frac{C_1^{(\lambda)}(t)\, p_1}{\left[\alpha\delta + (1-\alpha)(2-\mu)\right](\nabla+1)} \\
\Longrightarrow a_2^2 &= \frac{\left[C_1^{(\lambda)}(t)\right]^2 p_1^2}{\left[\alpha\delta + (1-\alpha)(2-\mu)\right]^2(\nabla+1)^2} \\
\Longrightarrow a_2^2 &= \frac{\left[C_1^{(\lambda)}(t)\right]^2 p_1^2}{B(\nabla+1)^2}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\left[2\alpha\delta + 2(1-\alpha)(3-2\mu)\right]\left[2(\nabla+\gamma\beta)+1\right] a_3 \\
&\quad = C_1^{(\lambda)}(t)\, p_2 + C_2^{(\lambda)}(t)\, p_1^2 - \left[\frac{\alpha}{2}\left(\delta^2 - 3\delta\right) + \frac{1-\alpha}{2}\left(\mu^2 + 5\mu - 8\right)\right] \\
&\quad\quad \cdot (\nabla+1)^2 \left[\frac{C_1^{(\lambda)}(t)\, p_1}{\left[\alpha\delta + (1-\alpha)(2-\mu)\right](\nabla+1)}\right]^2 \\
&\Longrightarrow K a_3 = C_1^{(\lambda)}(t)\, p_2 + C_2^{(\lambda)}(t)\, p_1^2 - \left(\frac{R}{2B}\left[C_1^{(\lambda)}(t)\right]^2\right) p_1^2 \\
&\Longrightarrow a_3 = \frac{C_1^{(\lambda)}(t)}{K} p_2 + \frac{C_2^{(\lambda)}(t)}{K} p_1^2 - \left(\frac{R}{2BK}\left[C_1^{(\lambda)}(t)\right]^2\right) p_1^2.
\end{aligned} \tag{3.3}$$

Now, from (3.2) and (3.3), we can easily see that

$$\begin{aligned}
a_3 - \xi a_2^2 &= \frac{C_1^{(\lambda)}(t)}{K} p_2 + \frac{C_2^{(\lambda)}(t)}{K} p_1^2 - \left(\frac{R}{2BK}\left[C_1^{(\lambda)}(t)\right]^2\right) p_1^2 - \xi \frac{\left[C_1^{(\lambda)}(t)\right]^2 p_1^2}{B(\nabla+1)^2} \\
\Longrightarrow a_3 - \xi a_2^2 &= \frac{C_1^{(\lambda)}(t)}{K} p_2 + \left(\frac{C_2^{(\lambda)}(t)}{K} - \left(\frac{R}{2BK}\left[C_1^{(\lambda)}(t)\right]^2\right) - \xi \frac{\left[C_1^{(\lambda)}(t)\right]^2}{B(\nabla+1)^2}\right) p_1^2 \\
\Longrightarrow a_3 - \xi a_2^2 &= \frac{C_1^{(\lambda)}(t)}{K} \left\{ p_2 + \left[\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)} - \frac{R \cdot C_1^{(\lambda)}(t)}{2B} - \xi \frac{K \cdot C_1^{(\lambda)}(t)}{B(\nabla+1)^2}\right] p_1^2 \right)
\end{aligned}$$

and

$$\left|a_3 - \xi a_2^2\right| = \frac{C_1^{(\lambda)}(t)}{K} \left| p_2 + \left[\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)} - \frac{R \cdot C_1^{(\lambda)}(t)}{2B} - \xi \frac{K \cdot C_1^{(\lambda)}(t)}{B(\nabla+1)^2}\right] p_1^2 \right|.$$

Therefore, in view of (2.6), we conclude that

$$\left|a_3 - \xi a_2^2\right| \leqq \frac{C_1^{(\lambda)}(t)}{K} \max\left\{1, \left|\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)} - \frac{R \cdot C_1^{(\lambda)}(t)}{2B} - \xi \frac{K \cdot C_1^{(\lambda)}(t)}{B(\nabla+1)^2}\right|\right\}. \tag{3.4}$$

Finally, by using the generating function (1.11) in (3.4), we get

$$\left|a_3 - \xi a_2^2\right| \leqq \frac{2\lambda t}{K} \max\left\{1, \left|\frac{2(\lambda+1)t^2 - 1}{2t} - \frac{R\lambda t}{B} - \xi \frac{2\lambda t K}{B(\nabla+1)^2}\right|\right\}.$$

Moreover, since $t > 0$, we have

$$\begin{aligned}
&\left|\frac{2(\lambda+1)t^2-1}{2t}-\frac{R\lambda t}{B}-\xi\frac{2\lambda tK}{B(\nabla+1)^2}\right| \leqq 1\\
&\Longleftrightarrow -1-\frac{2(\lambda+1)t^2-1}{2t}+\frac{R\lambda t}{B}\\
&\qquad \leqq -\xi\frac{2\lambda tK}{B(\nabla+1)^2}\\
&\qquad \leqq 1-\frac{2(\lambda+1)t^2-1}{2t}+\frac{R\lambda t}{B}\\
&\Longleftrightarrow \frac{(\nabla+1)^2}{2\lambda tK}\left(\frac{2[(\lambda+1)B-\lambda R]t^2-(1+2t)B}{2t}\right) \leqq \xi\\
&\qquad \leqq \frac{(\nabla+1)^2}{2\lambda tK}\left(\frac{2[(\lambda+1)B-\lambda R]t^2-(1-2t)B}{2t}\right)\\
&\Longleftrightarrow \left(\frac{2[(\lambda+1)B-\lambda R]t^2-(1+2t)B}{4\lambda Kt^2}\right)(\nabla+1)^2 \leqq \xi\\
&\qquad \leqq \left(\frac{2[(\lambda+1)B-\lambda R]t^2-(1-2t)B}{4\lambda Kt^2}\right)(\nabla+1)^2\\
&\Longleftrightarrow \xi_1 \leqq \xi \leqq \xi_2,
\end{aligned}$$

which evidently completes the proof of Theorem 2. □

Just as we deduced several consequences of Theorem 1 in the preceding section, here we deduce the following analogous corollaries of Theorem 2.

I. Taking $\alpha = \delta = 1$ or $\alpha = \mu - 1 = 0$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let the function $f(z)$ given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(\beta,\gamma)}\Big(1,1,\mu,H(z,C_n^{(\lambda)}(t))\Big) \equiv \mathcal{F}_{(\beta,\gamma)}\Big(0,\delta,1,H(z,C_n^{(\lambda)}(t))\Big).$$

Then, for some $\xi \in \mathbb{R}$,

$$\left|a_3-\xi a_2^2\right| \leqq \begin{cases} \frac{\lambda t}{[2(\nabla+\gamma\beta)+1]} & (\xi \in [\xi_1,\xi_2]) \\ \\ \frac{\lambda t}{[2(\nabla+\gamma\beta)+1]}\left|\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+2\lambda t-\xi\frac{4\lambda t[2(\nabla+\gamma\beta)+1]}{(\nabla+1)^2}\right| & (\xi \notin [\xi_1,\xi_2]), \end{cases}$$

where

$$\xi_1 = \left(\frac{2(3\lambda+1)t^2-(1+2t)}{8\lambda[2(\nabla+\gamma\beta)+1]t^2}\right)(\nabla+1)^2$$

and

$$\xi_2 = \left(\frac{2(3\lambda+1)t^2-(1-2t)}{8\lambda[2(\nabla+\gamma\beta)+1]t^2}\right)(\nabla+1)^2,$$

δ being given by

$$\delta := 2\gamma\beta+\gamma-\beta.$$

II. Upon setting $\beta = \gamma = 0$ in Theorem 2, we are led to the following corollary.

Corollary 7. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(0,0)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big).$$

Then, for some $\xi \in \mathbb{R}$,

$$\left|a_3 - \xi a_2^2\right| \leqq \begin{cases} \frac{2\lambda t}{K_1} & (\xi \in [\xi_1, \xi_2]) \\ \frac{2\lambda t}{K_1}\left|\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t} - \frac{R\lambda t}{B} - \xi \frac{2\lambda t K_1}{B}\right| & (\xi \notin [\xi_1, \xi_2]), \end{cases}$$

where

$$\xi_1 = \frac{2\left[(\lambda+1) B - \lambda R\right] t^2 - (1+2t) B}{4\lambda K_1 t^2}$$

and

$$\xi_2 = \frac{2\left[(\lambda+1) B - \lambda R\right] t^2 - (1-2t) B}{4\lambda K_1 t^2}$$

such that

$$[2\alpha\delta + 2(1-\alpha)(3-2\mu)] =: K_1,$$
$$[\alpha\delta + (1-\alpha)(2-\mu)]^2 =: B$$

and

$$\alpha\left(\delta^2 - 3\delta\right) + (1-\alpha)\left(\mu^2 + 5\mu - 8\right) =: R.$$

III. Putting $\delta - 1 = \mu = 0$ in Theorem 2, we get the following corollary.

Corollary 8. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, 1, 0, H(z, C_n^{(\lambda)}(t))\Big).$$

Then, for some $\xi \in \mathbb{R}$,

$$\left|a_3 - \xi a_2^2\right| \leqq \begin{cases} \frac{2\lambda t}{K_2} & (\xi \in [\xi_1, \xi_2]) \\ \frac{2\lambda t}{K_2}\left|\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t} - \frac{R_1\lambda t}{B_1} - \xi \frac{2\lambda t K_2}{B_1(\nabla+1)^2}\right| & (\xi \notin [\xi_1, \xi_2]), \end{cases}$$

where

$$\xi_1 = \left(\frac{2\left[(\lambda+1) B_1 - \lambda R_1\right] t^2 - (1+2t) B_1}{4\lambda K_2 t^2}\right)(\nabla+1)^2$$

and

$$\xi_2 = \left(\frac{2\left[(\lambda+1) B_1 - \lambda R_1\right] t^2 - (1-2t) B_1}{4\lambda K_2 t^2}\right)(\nabla+1)^2$$

such that

$$\nabla := 2\gamma\beta + \gamma - \beta, \qquad (6-4\alpha)\left[2(\nabla + \gamma\beta) + 1\right] =: K_2,$$
$$(2-\alpha)^2 =: B_1 \qquad \textit{and} \qquad 6\alpha - 8 =: R_1.$$

4. Applications associated with the Legendre and Chebyshev polynomials

In order to apply our main results in Section 2 and Section 2 to the corresponding function classes associated with the Legendre polynomials $P_n(t)$, the Chebyshev polynomials $T_n(t)$ of the first kind and the Chebyshev polynomials $U_n(t)$ of the second kind, we can make use of their relationships in (1.13) with the Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(t)$. For example, if we set $\lambda = \frac{1}{2}$, Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8, would readily yield the corresponding results for the function classes associated with the Legendre polynomials $P_n(t)$. In a similar manner, upon setting $\lambda = 1$, we can easily derive the corresponding results for the function classes associated with the Chebyshev polynomials $U_n(t)$ of the second kind. The analogous derivations in respect of the Chebyshev polynomials $T_n(t)$ of the first kind would obviously involve limit processes. Thus, except possibly in the case of the function class associated with the Chebyshev polynomials $T_n(t)$ of the first kind, it is fairly straightforward to set $\lambda = \frac{1}{2}$ and $\lambda = 1$ in Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8, in order to deduce the corresponding assertions for the function classes associated, respectively, with the Legendre polynomials $P_n(t)$ and the Chebyshev polynomials $U_n(t)$ of the second kind. We, therefore, choose to leave all such applications of Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8, as an exercise for the interested reader.

Some of the known special cases of Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8, are being listed below.

I. The special case of Theorem 1 when $\lambda = 1$ was given in [7].
II. If we further put $\lambda = 1$ in Corollary 2, we can derive a known result (see [24]).
III. Upon setting $\lambda = 1$ in Corollary 5, we are led to a known result (see [2]).
IV. In the special case of Theorem 1 when $\lambda = \alpha = \delta = 1$, we get a known result (see [4]).
V. The special case of Theorem 2 when $\lambda = 1$ yields a known result (see [7]).
VI. For $\lambda = 1$, Corollary 7 yields a known result (see [24]).
VII. In its special case when $\lambda = 1$, if we further set $\alpha = 0$, we are led to a known result (see [2]).
VIII. For $\lambda = \alpha = \delta = 1$, Theorem 2 reduces to a known result (see [4]).

Other (known or new) special cases and consequences of our main results asserted by Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8, can be deduced fairly easily. We omit the details involved in these derivations.

5. Conclusions and observations

Motivated by several interesting developments on the subjects, here we have introduced and investigated the following new subclass of normalized analytic functions in the open unit disk $\mathbb{U}$:

$$\mathcal{F}_{(\beta,\gamma)}\Big(\alpha, \delta, \mu, H(z, C_n^{(\lambda)}(t))\Big),$$

which satisfy a certain subordination condition and are associated with the Gegenbauer (or ultrospherica) polynomials $C_n^{(\lambda)}(t)$ of order λ and degree n in t. For functions belonging to this function class, we have derived the estimates for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and we have also examined the Fekete-Szegö functional. Our main results are asserted by Theorem 1 and its Corollaries 1 to 5, as well as Theorem 2 and its Corollaries 6 to 8. It is also shown how some of these main results can be applied to (known or new) subclasses of normalized analytic functions satisfying the corresponding subordination condition and associated with the Legendre polynomials $P_n(t)$, the Chebyshev polynomials $T_n(t)$ of the first kind, and the Chebyshev polynomials $U_n(t)$ of the first kind.

In several recent developments on the Taylor-Maclaurin coefficient estimate problem and the Fekete-Szegö coefficient inequality problem, use has been made successfully of the Horadam polynomials $h_n(t)$ which are given by the following recurrence relation:

$$h_n(t) = \mathfrak{p}th_{n-1}(t) + \mathfrak{q}h_{n-2}(t) \qquad (t \in \mathbb{R})$$

with

$$h_1(t) = a \quad \text{and} \quad h_2(t) = bt,$$

for some real constants $a, b, \mathfrak{p}$ and $\mathfrak{q}$ (see, for details, [16, 22, 23]; see also the references to the earlier works which are cited in each of these references). Indeed, as its special cases, the Horadam polynomials $h_n(t)$ contain a remarkably large number of other relatively more familiar polynomials including (for example) the Fibonacci polynomials, the Lucas polynomials, and the Pell-Lucas polynomials, as well as the Chebyshev polynomials $T_n(t)$ of the first kind and the Chebyshev polynomials $U_n(t)$ of the first kind. Most (if not all) of these recent developments also apply the basic or quantum (or q-) calculus as well. A possible presumably open problem for future researches emerging from our present investigation would involve the analogous usage of the Horadam polynomials $h_n(t)$ instead of the Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(t)$ which we have used in our investigation.

In concluding this paper, we recall a recently-published survey-cum-expository review article in which Srivastava [14] explored the mathematical applications of the q-calculus, the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory of Complex Analysis, especially in the study of Fekete-Szegö functional. Srivastava [14] also exposed the not-yet-widely-understood fact that the so-called (p, q)-variation of the classical q-calculus is, in fact, a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [14, p. 340]; see also [15, pp. 1511–1512]).

Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. Ş. Altınkaya, S. Yalçin, On the Chebyshev polynomial bounds for classes of univalent functions, *Khayyam J. Math.*, **2** (2016), 1–5. doi: 10.22034/kjm.2016.13993.
2. Ş. Altınkaya, S. Yalçin, On the Chebyshev coefficients for a general subclass of univalent functions, *Turkish J. Math.*, **42** (2018), 2885–2890. doi: 10.3906/mat-1510-53.

3. S. Bulut, N. Magesh, V. K. Balaji, Certain subclasses of analytic functions associated with the Chebyshev polynomials, *Honam Math. J.*, **40** (2018), 611–619.
4. M. Çağlar, H. Orhan, M. Kamali, Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials, *Bol. Soc. Paran. Mat.* (*BSPM*), in press.
5. J. Dziok, R. K. Raina, J. Sókoł, Application of Chebyshev polynomials to classes of analytic functions, *C. R. Math.*, **353** (2015), 433–438. doi: 10.1016/j.crma.2015.02.001.
6. M. Fekete, G. Szegö, Eine Bemerkung Über ungerade schlichte Funktionen, *J. London Math. Soc.*, **s1-8** (1933), 85–89. doi: 10.1112/jlms/s1-8.2.85.
7. M. Kamali, M. Çağlar, E. Deniz, M. Turabaev, Fekete Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials, *Turkish J. Math.*, **45** (2021), 1195–1208.
8. B. Kowalczyk. A. Lecko, H. M. Srivastava, A note on the Fekete-Szegö problem for close-to-convex functions with respect to convex functions, *Publ. Inst. Math.* (*Beograd*) (*Nouvelle Sér.*), **101** (2017), 143–149. doi: 10.2298/PIM1715143K.
9. Z. Lewandowski, S. S. Miller, E. Złotkiewicz, Generating functions for some classes of univalent functions, *Proc. Amer. Math. Soc.*, **56** (1976), 111–117. doi:10.1090/S0002-9939-1976-0399438-7.
10. N. Magesh, S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afrika Mat.*, **29** (2018), 203–209. doi: 10.1007/s13370-017-0535-3.
11. G. V. Milovanović, M. Th. Rassias, *Analytic number theory*, *approximation theory*, *and special functions*: *In honor of Hari M. Srivastava*, New York: Springer, 2014. doi: 10.1007/978-1-4939-0258-3.
12. P. T. Mocanu, Une proprieté de convexité généralisée dans la théorie de la représentation conforme, *Mathematica* (*Cluj*), **11** (1969), 127–133.
13. C. Ramachandran, K. Dhanalaksmi, Fekete-Szegö inequality for the subclasses of analytic functions bounded by Chebyshev polynomial, *Global J. Pure Appl. Math.*, **13** (2017), 4953–4958.
14. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A*: *Sci.*, **44** (2020), 327–344. doi: 10.1007/s40995-019-00815-0.
15. H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, *J. Nonlinear Convex Anal.*, **22** (2021), 1501–1520.
16. H. M. Srivastava, Ş. Altınkaya, S. Yalçin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, *Iran. J. Sci. Technol. Trans. A*: *Sci.*, **43** (2019), 1873–1879. doi: 10.1007/s40995-018-0647-0.
17. H. M. Srivastava, S. Hussain, A. Raziq, M. Raza, The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination, *Carpathian J. Math.*, **34** (2018), 103–113.
18. H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

19. H. M. Srivastava, A. K. Mishra, M. K. Das, The Fekete-Szegö problem for a subclass of close-to-convex functions, *Complex Variables Theory Appl.*, **44** (2001), 145–163. doi: 10.1080/17476930108815351.

20. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), 1188–1192. doi: 10.1016/j.aml.2010.05.009.

21. H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava, M. H. AbuJarad, Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions, *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (*RACSAM*), **113** (2019), 3563–3584. doi: 10.1007/s13398-019-00713-5.

22. H. M. Srivastava, A. K. Wanas, G. Murugusundaramoorthy, Certain family of bi-univalent functions associated with Pascal distribution series based on Horadam polynomials, *Surveys Math. Appl.*, **16** (2021), 193–205.

23. H. M. Srivastava, A. K. Wanas, R. Srivastava, Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials, *Symmetry*, **13** (2021), 1–14. doi: 10.3390/sym13071230 .

24. E. Szatmari, Ş. Altınkaya, Coefficient estimates and Fekete-Szegö inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials, *Acta Univ. Sapientiae Math.*, **11** (2019), 430–436. doi: 10.2478/ausm-2019-0031.

25. J. Szynal, An extension of typically real functions, *Ann. Univ. Mariae Curie-Skołodowska Sect. A*, **48** (1994), 193–201.