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Research article

A characterization of $b-$generalized derivations on prime rings with involution

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Abstract: In this note, we characterize $b-$generalized derivations which are strong commutative preserving (SCP) on $\mathscr{R}$. Moreover, we also discuss and characterize $b-$generalized derivations involving certain $*-$differential/functional identities on rings possessing involution.

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1. Introduction

The algebra of derivations and generalized derivations play a crucial role in the study of functional identities and their applications. There are many generalizations of derivations viz., generalized derivations, multiplicative generalized derivations, skew generalized derivations, $b-$generalized derivations, etc. The notion of $b-$generalized derivation was first introduced by Koşan and Lee [17]. The most important and systematic research on the $b-$generalized derivations have been accomplished in [11, 17, 22, 26] and references therein. In this manuscript, we characterize $b-$generalized derivations which are strong commutative preserving (SCP) on $\mathscr{R}$. Moreover, we also discuss and characterize $b-$generalized derivations involving certain $*-$differential/functional identities on rings possessing involution.

In the early nineties, after a memorable work on the structure theory of rings, a tremendous work has been established by Brešar considering centralizing mappings, commuting mappings, commutativity preserving (CP) mappings and strong commutativity preserving (SCP) mappings on some appropriate subset of rings. Since then it became a tempting research idea in the matrix theory/operator theory/ring theory for algebraists. Commutativity preserving (CP) maps were introduced and studied by Watkins [32] and further extended to SCP by Bell and Mason [6]. Inspired by the concept of SCP maps, Bell and Daif [5] demonstrated the commutativity of (semi-)prime rings possessing derivations and endomorphisms on right ideals (see also [27] and references therein). In [12], Deng and Ashraf studied strong commutativity preserving maps in more general context as follows: Let $\mathscr{R}$ be a semiprime ring. If $\mathscr{R}$ admits a mapping ψ and a derivation δ on $\mathscr{R}$ such that $[\psi(\mathtt{r}), \delta(\mathtt{v})] = [\mathtt{r}, \mathtt{v}]$ for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$, then $\mathscr{R}$ is commutative. In 1994, Brešar and Miers [7] characterized an additive map $f : R \to R$ satisfying SCP on a semiprime ring R and showed that f is of the form $f(\mathtt{r}) = \lambda \mathtt{r} + \mu(\mathtt{r})$ for every $\mathtt{r} \in \mathscr{R}$, where $\lambda \in \mathscr{C}$, $\lambda^2 = 1$ and $\mu : \mathscr{R} \to \mathscr{R}$ is an additive map. Later, Lin and Liu [23] extended this result to Lie ideals of prime rings. Chasing to this, several techniques have been developed to investigate the behavior of strong commutativity preserving maps (SCP) using restrictions on polynomials, invoking derivations, generalized derivations, etcetera. An account of work has been done in the literature [3, 10, 12, 13, 15, 20, 21, 23–25, 27, 30, 31].

On the other hand, the study of additive maps on rings possessing involution was initiated by Brešar et al. [7] and they characterized the additive centralizing mappings on the skew-symmetric elements of prime rings possessing involution. In the same vein, Lin and Liu [24] describe SCP additive maps on skew-symmetric elements of prime rings possessing involution. Later, Liau et al. [21] improved the above mentioned result for non-additive SCP mappings. Interestingly, in 2015 Ali et al. [1] studied the SCP maps in different way on rings possessing involution. They established the commutativity of prime ring of charateristic not two possessing second kind of involution satisfying $[\delta(\mathtt{r}), \delta(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{r}^*] = 0$ for every $\mathtt{r} \in \mathscr{R}$, where δ is a nonzero derivation of $\mathscr{R}$. Recently, Khan and Dar [8] improved this result by studying the case of generalized derivations.

Motivated by the above presented work, in this manuscript we have characterized $b-$generalized derivations on prime rings possessing involution. Moreover, we also present some examples in support of our main theorems.

2. Preliminaries

Throughout the manuscript unless otherwise stated, $\mathscr{R}$ is a prime ring with center $\mathscr{Z}(\mathscr{R})$, $\mathscr{Q}$ is the maximal right ring of quotients, $\mathscr{C} = \mathscr{Z}(\mathscr{Q})$ is the center of $\mathscr{Q}$ usually known as the extended centroid of $\mathscr{R}$ and is a field. "A ring $\mathscr{R}$ is said to be $2-$ *torsion free* if $2\mathtt{r} = 0$ (where $\mathtt{r} \in \mathscr{R}$) implies $\mathtt{r} = 0$". The characteristic of $\mathscr{R}$ is represented by *char*($\mathscr{R}$). "A ring $\mathscr{R}$ is called a *prime ring* if $\mathtt{r}\mathscr{R}\mathtt{v} = (0)$ (where $\mathtt{r}, \mathtt{v} \in \mathscr{R}$) implies $\mathtt{r} = 0$ or $\mathtt{v} = 0$ and is called a *semiprime ring* in case $\mathtt{r}\mathscr{R}\mathtt{r} = (0)$ implies $\mathtt{r} = 0$". "An additive map $\mathtt{r} \mapsto \mathtt{r}^*$ of $\mathscr{R}$ into itself is called an *involution* if (i) $(\mathtt{r}\mathtt{v})^* = \mathtt{v}^*\mathtt{r}^*$ and (ii) $(\mathtt{r}^*)^* = \mathtt{r}$ hold for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. A ring equipped with an involution is known as a *ring with involution* or a $*-$*ring*. An element $\mathtt{r}$ in a ring with involution $*$ is said to be *hermitian/symmetric* if $\mathtt{r}^* = \mathtt{r}$ and *skew-hermitian/skew-symmetric* if $\mathtt{r}^* = -\mathtt{r}$". The sets of all hermitian and skew-hermitian elements of $\mathscr{R}$ will be denoted by $\mathscr{W}(\mathscr{R})$ and $\mathscr{K}(\mathscr{R})$, respectively. "If $\mathscr{R}$ is $2-$torsion free, then every $\mathtt{r} \in \mathscr{R}$ can be uniquely represented in the form $2\mathtt{r} = h + k$ where $h \in \mathscr{W}(\mathscr{R})$ and $k \in \mathscr{K}(\mathscr{R})$. Note that in this case $\mathtt{r}$ is normal, i.e., $\mathtt{r}\mathtt{r}^* = \mathtt{r}^*\mathtt{r}$, if and only if h and k commute. If all elements in $\mathscr{R}$ are normal, then $\mathscr{R}$ is called a *normal ring*". An example is the ring of quaternions. "The involution is said to be of the first kind if $\mathscr{Z}(\mathscr{R}) \subseteq \mathscr{W}(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case it is worthwhile to see that $\mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R}) \neq (0)$". We refer the reader to [4, 14] for justification and amplification for the above mentioned notations and key definitions.

For $\mathtt{r}, \mathtt{v} \in \mathscr{R}$, the commutator of $\mathtt{r}$ and $\mathtt{v}$ is defined as $[\mathtt{r}, \mathtt{v}] = \mathtt{r}\mathtt{v} - \mathtt{v}\mathtt{r}$. We say that a map

$f : \mathscr{R} \to \mathscr{R}$ preserves commutativity if $[f(\mathtt{r}), f(\mathtt{v})] = 0$ whenever $[\mathtt{r}, \mathtt{v}] = 0$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Following [7], "let $\mathscr{S}$ be a subset of $\mathscr{R}$, a map $f : \mathscr{R} \to \mathscr{S}$ is said to be strong commutativity preserving (SCP) on $\mathscr{S}$ if $[f(\mathtt{r}), f(\mathtt{v})] = [\mathtt{r}, \mathtt{v}]$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{S}$". Following [33], "an additive mapping $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ is said to be a *left (respectively right) centralizer (multiplier)* of $\mathscr{R}$ if $\mathscr{T}(\mathtt{rv}) = \mathscr{T}(\mathtt{r})\mathtt{v}$ (respectively $\mathscr{T}(\mathtt{rv}) = \mathtt{r}\mathscr{T}(\mathtt{v})$) for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. An additive mapping $\mathscr{T}$ is called a *centralizer* in case $\mathscr{T}$ is a left and a right centralizer of $\mathscr{R}$". In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ is a homomorphism of a ring module $\mathscr{R}$ into itself. For a prime ring $\mathscr{R}$ all such homomorphisms are of the form $\mathscr{T}(\mathtt{r}) = q\mathtt{r}$ for all $\mathtt{r} \in \mathscr{R}$, where $q \in \mathscr{Q}$ (see Chapter 2 in [4]). "An additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is said to be a *derivation* on $\mathscr{R}$ if $\delta(\mathtt{rv}) = \delta(\mathtt{r})\mathtt{v} + \mathtt{r}\delta(\mathtt{v})$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$". It is well-known that every derivation of $\mathscr{R}$ can be uniquely extended to a derivation of $\mathscr{Q}$. *"A derivation δ is said to be $\mathscr{Q}-$inner if there exists $\alpha \in \mathscr{Q}$ such that $\delta(\mathtt{r}) = \alpha\mathtt{r} - \mathtt{r}\alpha$ for all $\mathtt{r} \in \mathscr{R}$. Otherwise, it is called $\mathscr{Q}-$outer (δ is not inner)"*. "An additive map $\mathscr{H} : \mathscr{R} \to \mathscr{R}$ is called a *generalized derivation* of $\mathscr{R}$ if there exists a derivation δ of $\mathscr{R}$ such that $\mathscr{H}(\mathtt{rv}) = \mathscr{H}(\mathtt{r})\mathtt{v} + \mathtt{r}\delta(\mathtt{v})$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$". The derivation δ is uniquely determined by $\mathscr{H}$ and is called the associated derivation of $\mathscr{H}$. The concept of generalized derivations covers the both the concepts of a derivation and a left centralizer. We would like to point out that in [19] Lee proved that "every generalized derivation can be uniquely extended to a generalized derivation of $\mathscr{Q}$ and thus all generalized derivations of $\mathscr{R}$ will be implicitly assumed to be defined on the whole $\mathscr{Q}$".

The recent concept of generalized derivations were introduced by Koşan and Lee [17], namely, $b-$generalized derivations which was defined as follows: "An additive mapping $\mathscr{H} : \mathscr{R} \to \mathscr{Q}$ is called a *(left) $b-$generalized derivation* of $\mathscr{R}$ associated with δ, an additive map from $\mathscr{R}$ to $\mathscr{Q}$, if $\mathscr{H}(\mathtt{rv}) = \mathscr{H}(\mathtt{r})\mathtt{v} + b\mathtt{r}\delta(\mathtt{v})$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$, where $b \in \mathscr{Q}$". Also they proved that if "$\mathscr{R}$ is a prime ring and $0 \neq b \in \mathscr{Q}$, then the associated map δ is a derivation i.e., $\delta(\mathtt{rv}) = \delta(\mathtt{r})\mathtt{v} + \mathtt{r}\delta(\mathtt{v})$ for all $\mathtt{r}, \mathtt{v} \in \mathscr{R}$". It is easy to see that every generalized derivation is a $1-$generalized derivation. Also, the mapping $\mathtt{r} \in \mathscr{R} \to \alpha\mathtt{r} + b\mathtt{r}c \in \mathscr{Q}$ for $a, b, c \in \mathscr{Q}$ is a $b-$generalized derivation of $\mathscr{R}$, which is known as inner $b-$generalized derivation of $\mathscr{R}$. In spite of this, they also characterized $b-$generalized derivation. That is, every $b-$generalized derivation $\mathscr{H}$ on a semiprime ring $\mathscr{R}$ is of the form $\mathscr{H}(\mathtt{r}) = \alpha\mathtt{r} + b\delta(\mathtt{r})$ for all $\mathtt{r} \in \mathscr{R}$, where $a, b \in \mathscr{Q}$. Following important facts are frequently used in the proof of our results:

Fact 2.1 ([1, Lemma 2.1]). *"Let $\mathscr{R}$ be a prime ring with involution such that char($\mathscr{R}$) $\neq 2$. If $\mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R}) \neq (0)$ and $\mathscr{R}$ is normal, then $\mathscr{R}$ is commutative."*

Fact 2.2. *"Let $\mathscr{R}$ be a ring with involution such that char($\mathscr{R}$) $\neq 2$. Then every $\mathtt{r} \in \mathscr{R}$ can uniquely represented as $2\mathtt{r} = w + s$, where $w \in \mathscr{W}(\mathscr{R})$ and $s \in \mathscr{K}(\mathscr{R})$."*

Fact 2.3 ([17, Theorem 2.3]). *"Let $\mathscr{R}$ be a semiprime ring, $b \in \mathscr{Q}$, and let $\mathscr{H} : \mathscr{R} \to \mathscr{Q}$ be a $b-$generalized derivation associated with δ. Then δ is a derivation and there exists $\alpha \in \mathscr{Q}$ such that $\mathscr{H}(\mathtt{r}) = \alpha\mathtt{r} + b\delta(\mathtt{r})$ for every $\mathtt{r} \in \mathscr{R}$."*

Fact 2.4 ([8, Lemma 2.2]). *"Let $\mathscr{R}$ be a non-commutative prime ring with involution of the second kind such that char($\mathscr{R}$) $\neq 2$. If $\mathscr{R}$ admits a derivation $\delta : \mathscr{R} \to \mathscr{R}$ such that $[\delta(w), w] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$, then $\delta(\mathscr{Z}(\mathscr{R})) = (0)$."*

We need a well-known lemma due to Martindale [28], stated below in a form, convenient for our purpose.

Lemma 2.1 ([28, Theorem 2(a)]). *"Let $\mathscr{R}$ be a prime ring and $a_i, b_i, c_j, d_j \in \mathscr{Q}$. Suppose that $\sum_{i=1}^{m} a_i \mathtt{r} b_i + \sum_{j=1}^{n} c_j \mathtt{r} d_j = 0$ for all $\mathtt{r} \in \mathscr{R}$. If $a_1, \cdots, a_m$ are $\mathscr{C}-$independent, then each b_i is a $\mathscr{C}-$linear combination of $d_1, \cdots, d_n$. If $b_1, \cdots, b_m$ are $\mathscr{C}-$independent, then each a_i is a $\mathscr{C}-$linear combination of $c_1, \cdots, c_n$."*

We need Kharchenko's theorem for differential identities to prime rings [16]. The lemma below is a special case of [16, Lemma 2].

Lemma 2.2 ([16, Lemma 2]). *"Let $\mathscr{R}$ be a prime ring and $a_i, b_i, c_j, d_j \in \mathscr{Q}$ and δ a $\mathscr{Q}-$outer derivation of $\mathscr{R}$. Suppose that $\sum_{i=1}^{m} a_i \mathtt{r} b_i + \sum_{j=1}^{n} c_j \delta(\mathtt{r}) d_j = 0$ for all $\mathtt{r} \in \mathscr{R}$. Then $\sum_{i=1}^{m} a_i \mathtt{r} b_i = 0 = \sum_{j=1}^{n} c_j \mathtt{r} d_j$ for all $\mathtt{r} \in \mathscr{R}$."*

3. Results on $b-$generalized derivations

We begin with the following Lemma which is needed for developing the proof of our theorems:

Lemma 3.1. *Let $\mathscr{R}$ be a non-commutative prime ring of characteristic different from two with involuation of the second kind. If for any $\alpha \in \mathscr{R}$, $[\alpha\mathtt{r}, \alpha\mathtt{r}^*] - [\mathtt{r}, \mathtt{r}^*] = 0$ for every $\mathtt{r} \in \mathscr{R}$, then $\alpha \in \mathscr{C}$ and $\alpha^2 = 1$.*

Proof. For any $\mathtt{r} \in \mathscr{R}$, we have

$$[\alpha\mathtt{r}, \alpha\mathtt{r}^*] - [\mathtt{r}, \mathtt{r}^*] = 0. \tag{3.1}$$

This can also be written as

$$\alpha^2[\mathtt{r}, \mathtt{r}^*] + \alpha[\mathtt{r}, \alpha]\mathtt{r}^* + \alpha[\alpha, \mathtt{r}^*]\mathtt{r} - [\mathtt{r}, \mathtt{r}^*] = 0 \tag{3.2}$$

for every $\mathtt{r} \in \mathscr{R}$. Replace $\mathtt{r}$ by $\mathtt{r} + s'$ in above equation, where $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$, we get

$$\alpha^2[\mathtt{r}, \mathtt{r}^*] + \alpha[\mathtt{r}, \alpha]\mathtt{r}^* - \alpha[\mathtt{r}, \alpha]s' + \alpha[\alpha, \mathtt{r}^*]\mathtt{r} + \alpha[\alpha, \mathtt{r}^*]s' - [\mathtt{r}, \mathtt{r}^*] = 0 \tag{3.3}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. In view of (3.2), we have

$$\alpha[\alpha, \mathtt{r}]s' + \alpha[\alpha, \mathtt{r}^*]s' = \alpha[\alpha, \mathtt{r} + \mathtt{r}^*]s' = 0 \tag{3.4}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Since the involution is of the second kind, so we have

$$\alpha[\alpha, \mathtt{r} + \mathtt{r}^*] = 0 \tag{3.5}$$

for every $\mathtt{r} \in \mathscr{R}$. For $\mathtt{r} = w + s$, where $w \in \mathscr{W}(\mathscr{R})$ and $s \in \mathscr{K}(\mathscr{R})$, observe that

$$\alpha[\alpha, w] = 0 \tag{3.6}$$

for every $w \in \mathscr{W}(\mathscr{R})$. Substitute ss' for w in above expression, we get

$$\alpha[\alpha, s] = 0 \tag{3.7}$$

for every $s \in \mathscr{K}(\mathscr{R})$, since the involution is of the second kind. Observe from Fact 2.2 that $2\alpha[\alpha, \mathtt{r}] = \alpha[\alpha, 2\mathtt{r}] = \alpha[\alpha, w + s] = \alpha[\alpha, w] + \alpha[\alpha, s] = 0$. Thus we have $\alpha \in \mathscr{C}$ and, by our hypothesis, $(\alpha^2 - 1)[\mathtt{r}, \mathtt{r}^*] = 0$, for every $\mathtt{r} \in \mathscr{R}$. By the primeness of $\mathscr{R}$, it follows that either $\alpha^2 = 1$ or $[\mathtt{r}, \mathtt{r}^*] = 0$, for any $\mathtt{r} \in \mathscr{R}$. Thus we are led to the required conclusion by Fact 2.1. □

Theorem 3.1. *Let $\mathscr{R}$ be a non-commutative prime ring of characteristic different from two. If $\mathscr{H}$ is a non-zero $b-$generalized derivation on $\mathscr{R}$ associated with a derivation δ on $\mathscr{R}$ and ψ is a non-zero map on $\mathscr{R}$ satisfying $[\psi(\mathtt{r}), \mathscr{H}(\mathtt{v})] = [\mathtt{r}, \mathtt{v}]$ for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Then there exists $0 \neq \lambda \in \mathscr{C}$ and an additive map $\mu : \mathscr{R} \to \mathscr{C}$ such that $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$, $\psi(\mathtt{r}) = \lambda^{-1}\mathtt{r} + \mu(\mathtt{r})$, for any $\mathtt{r} \in \mathscr{R}$.*

Proof. Notice that if either $\delta = 0$ or $b = 0$, the map $\mathscr{H}$ reduces to a centralizer, that is $\mathscr{H}(\mathtt{v}) = \alpha\mathtt{v}$, for any $\mathtt{v} \in \mathscr{R}$. Then the conclusion follows as a reduced case of [25, Theorem 1.1]. Hence, in the rest of the proof we assume both $b \neq 0$ and $\delta \neq 0$. By Fact 2.3, there exists $\alpha \in \mathscr{H}$ such that $\mathscr{H}(\mathtt{r}) = \alpha\mathtt{r} + b\delta(\mathtt{r})$ for every $\mathtt{r} \in \mathscr{R}$. By the hypothesis

$$[\psi(\mathtt{r}), \alpha\mathtt{v} + b\delta(\mathtt{v})] = [\mathtt{r}, \mathtt{v}] \tag{3.8}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Taking of $\mathtt{v}z$ for $\mathtt{v}$ in above expression gives

$$(\alpha\mathtt{v} + b\delta(\mathtt{v}))[\psi(\mathtt{r}), z] + [\psi(\mathtt{r}), b\mathtt{v}\delta(z)] = \mathtt{v}[\mathtt{r}, z] \tag{3.9}$$

for every $\mathtt{r}, \mathtt{v}, z \in \mathscr{R}$.

Suppose firstly that δ is not an inner derivation of $\mathscr{R}$. In view of (3.9) and Lemma 2.2, we observe that

$$(\alpha\mathtt{v} + b\mathtt{v}')[\psi(\mathtt{r}), z] + [\psi(\mathtt{r}), b\mathtt{v}z'] = \mathtt{v}[\mathtt{r}, z] \tag{3.10}$$

for every $\mathtt{r}, \mathtt{v}, z, \mathtt{v}', z' \in \mathscr{R}$. In particular, for $\mathtt{v} = 0$ we have

$$b\mathtt{v}'[\psi(\mathtt{r}), z] = 0 \tag{3.11}$$

for every $\mathtt{r}, z, \mathtt{v}' \in \mathscr{R}$. By the primeness of $\mathscr{R}$ and since $b \neq 0$, it follows that $\psi(\mathtt{r}) \in \mathscr{Z}(\mathscr{R})$, for any $\mathtt{r} \in \mathscr{R}$. On the other hand, by $\psi(\mathtt{r}) \in \mathscr{Z}(\mathscr{R})$ and (3.8) one has that $[\mathtt{r}, \mathtt{v}] = 0$ for any $\mathtt{r}, \mathtt{v} \in \mathscr{R}$, which is a contradiction since $\mathscr{R}$ is not commutative.

Therefore, we have to consider the only case when there is $q \in \mathscr{H}$ such that $\delta(\mathtt{r}) = [\mathtt{r}, q]$, for every $\mathtt{r} \in \mathscr{R}$. Thus we rewrite (3.10) as follows

$$((\alpha - bq)\mathtt{v} + b\mathtt{v}q)[\psi(\mathtt{r}), z] + [\psi(\mathtt{r}), b\mathtt{v}zq - b\mathtt{v}qz] = \mathtt{v}[\mathtt{r}, z] \tag{3.12}$$

that is

$$\begin{aligned} &\{(\alpha - bq)\mathtt{v}\psi(\mathtt{r}) + b\mathtt{v}q\psi(\mathtt{r}) - \psi(\mathtt{r})b\mathtt{v}q - \mathtt{v}\mathtt{r}\}z \\ &+\{bq\mathtt{v} - \alpha\mathtt{v}\}z\psi(\mathtt{r}) + \psi(\mathtt{r})b\mathtt{v}zq - b\mathtt{v}zq\psi(\mathtt{r}) + \mathtt{v}z\mathtt{r} = 0 \end{aligned} \tag{3.13}$$

for every $\mathtt{r}, \mathtt{v}, z \in \mathscr{R}$.

Suppose there exists $\mathtt{v} \in \mathscr{R}$ such that $\{b\mathtt{v}, \mathtt{v}\}$ are linearly $\mathscr{C}$-independent. From relation (3.13) and Lemma 2.1, it follows that, for any $\mathtt{r} \in \mathscr{R}$, both $\mathtt{r}$ and $q\psi(\mathtt{r})$ are $\mathscr{C}$-linear combinations of $\{1, q, \psi(\mathtt{r})\}$. In other words, there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathscr{C}$, depending by the choice of $\mathtt{r} \in \mathscr{R}$, such that

$$\mathtt{r} = \alpha_1 + \alpha_2\psi(\mathtt{r}) + \alpha_3 q \tag{3.14}$$

and

$$q\psi(\mathtt{r}) = \beta_1 + \beta_2\psi(\mathtt{r}) + \beta_3 q. \tag{3.15}$$

Notice that, for $\alpha_2 = 0$, relation (3.14) implies that q commutes with element $\mathtt{r} \in \mathscr{R}$. On the other hand, in case $\alpha_2 \neq 0$, by (3.14) we have that

$$\psi(\mathtt{r}) = \alpha_2^{-1}(\mathtt{r} - \alpha_1 - \alpha_3 q). \tag{3.16}$$

Then, by using (3.16) in (3.15), it follows that

$$\beta_1 + \beta_3 q = \alpha_2^{-1}(q - \beta_2)(\mathtt{r} - \alpha_1 - \alpha_3 q). \tag{3.17}$$

So, by commuting (3.17) with q, we get $\alpha_2^{-1}(q - \beta_2)[\mathtt{r}, q] = 0$, implying that $[\mathtt{r}, q] = 0$ in any case.

Therefore q commutes with any element of $\mathscr{R}$ and this contradicts the fact that $\delta \neq 0$. Therefore, for any $\mathtt{v} \in \mathscr{R}$, $\{\mathtt{v}, b\mathtt{v}\}$ must be linearly $\mathscr{C}$-dependent. In this case a standard argument shows that $b \in \mathscr{C}$, which implies that $\mathscr{H}(\mathtt{r}) = (\alpha - bq)\mathtt{r} + \mathtt{r}(bq)$, for any $\mathtt{r} \in \mathscr{R}$. Hence $\mathscr{H}$ is a generalized derivation of $\mathscr{R}$ and once again the result follows from [25, Theorem 1.1]. □

The following theorem is a generalization of [8, Theorem 2.3].

Theorem 3.2. *Let $\mathscr{R}$ be a non-commutative prime ring with involution of the second kind of characteristic different from two. If $\mathscr{H}$ is a b−generalized derivation on $\mathscr{R}$ associated with a derivation δ on $\mathscr{R}$ such that $[\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{r}^*)] = [\mathtt{r}, \mathtt{r}^*]$ for every $\mathtt{r} \in \mathscr{R}$, then there exists $\lambda \in \mathscr{C}$ such that $\lambda^2 = 1$ and $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$ for every $\mathtt{r} \in \mathscr{R}$.*

Proof. By the given hypothesis, we have

$$[\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{r}^*] = 0 \text{ for all } \mathtt{r} \in \mathscr{R}. \tag{3.18}$$

Taking $\mathtt{r}$ as $\mathtt{r} + \mathtt{v}$ in (3.18) to get

$$[\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{v}^*)] + [\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{v}^*] - [\mathtt{v}, \mathtt{r}^*] = 0 \tag{3.19}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Substitute $\mathtt{v}s'$ for $\mathtt{v}$, where $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$, in above relation, we obtain

$$\begin{aligned} 0 = &- [\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{v}^*)]s' - [\mathscr{H}(\mathtt{r}), b\mathtt{v}^*]\delta(s') + [\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)]s' \\ &+ [b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)]\delta(s') + [\mathtt{r}, \mathtt{v}^*]s' - [\mathtt{v}, \mathtt{r}^*]s' \end{aligned} \tag{3.20}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Multiply (3.19) with s' and combine with (3.20) to get

$$2[\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)]s' - 2[\mathtt{v}, \mathtt{r}^*]s' - [\mathscr{H}(\mathtt{r}), b\mathtt{v}^*]\delta(s') + [b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)]\delta(s') = 0 \tag{3.21}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Again substitute $\mathtt{v}$ as $\mathtt{v}s'$ in (3.21), we get

$$\begin{aligned} 0 =&2[\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)]s'^2 + 2[b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)]\delta(s')s' - 2[\mathtt{v}, \mathtt{r}^*]s'^2 \\ &+ [\mathscr{H}(\mathtt{r}), b\mathtt{v}^*]\delta(s')s' + [b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)]\delta(s')s' \end{aligned} \tag{3.22}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. In view of (3.21), we have

$$2[b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)]\delta(s')s' + 2[\mathscr{H}(\mathtt{r}), b\mathtt{v}^*]\delta(s')s' = 0 \tag{3.23}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Since $char(\mathscr{R}) \neq 2$ and the involution is of the second kind, so $[b\mathtt{v}, \mathscr{H}(\mathtt{r}^*)] + [\mathscr{H}(\mathtt{r}), b\mathtt{v}^*] = [\mathscr{H}(\mathtt{r}), b\mathtt{v}^*] - [\mathscr{H}(\mathtt{r}^*), b\mathtt{v}] = 0$ for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ or $\delta(s')s' = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Observe that $s' = 0$ is also implies $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Assume that $\delta(s') \neq 0$, therefore we have

$$[\mathscr{H}(\mathtt{r}), b\mathtt{v}^*] - [\mathscr{H}(\mathtt{r}^*), b\mathtt{v}] = 0 \tag{3.24}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Taking $\mathtt{r} = \mathtt{v} = w + s$ in above expression, we obtain

$$[\mathscr{H}(s), bw] - [\mathscr{H}(w), bs] = 0 \tag{3.25}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s \in \mathscr{K}(\mathscr{R})$. Replace s by s' in (3.25), we get

$$[\mathscr{H}(s'), bw] - [\mathscr{H}(w), b]s' = 0 \tag{3.26}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Substitute ss' for w in last expression, we get

$$[\mathscr{H}(s'), bs]s' - [\mathscr{H}(s), b]s'^2 + b[b, s]\delta(s')s' = 0 \tag{3.27}$$

for every $s \in \mathscr{K}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. One can see from (3.24) that $[\mathscr{H}(s'), bs] = 0$ and $[\mathscr{H}(s), b]s' = 0$ for every $s \in \mathscr{K}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. This reduces (3.27) into

$$b[b, s]\delta(s')s' = 0 \tag{3.28}$$

for every $s \in \mathscr{K}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. This implies either $b[b, s] = 0$ for every $s \in \mathscr{K}(\mathscr{R})$ or $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Suppose $b[b, s] = 0$ for every $s \in \mathscr{K}(\mathscr{R})$. Take $s = w_0 s'$ and use the fact that the involution is of the second kind, we get $b[b, w_0] = 0$ for every $w_0 \in \mathscr{W}(\mathscr{R})$. An application of Fact 2.2 yields $b \in \mathscr{C}$. One can see from (3.27) that $b[\mathscr{H}(s'), s] = 0$ for every $s \in \mathscr{K}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. If $b = 0$ and in light of Fact 2.3, $\mathscr{H}$ has the following form: $\mathscr{H}(\mathtt{r}) = \alpha\mathtt{r}$, for some fixed element $\alpha \in \mathscr{H}$. Thus, by Lemma 3.1 and since $\mathscr{R}$ is not commutative, we get the required conclusion $\alpha \in \mathscr{C}$ and $\alpha^2 = 1$.

So we assume $b \neq 0$ and $[\mathscr{H}(s'), s] = 0$ for every $s \in \mathscr{K}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Again taking s as $w_0 s'$ and making use of Fact 2.2 in last relation gives $\mathscr{H}(s') \in \mathscr{Z}(\mathscr{R})$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Next, take $\mathtt{r} = w$ and $\mathtt{v} = s$ in (3.19), we get

$$[\mathscr{H}(s), \mathscr{H}(w)] + [w, s] = 0 \tag{3.29}$$

for every $s \in \mathscr{K}(\mathscr{R})$ and $w \in \mathscr{W}(\mathscr{R})$. Substitute $w_0 s'$ for s in above relation, we get

$$[\mathscr{H}(w_0 s'), \mathscr{H}(w)] + [w, w_0]s' = 0 \tag{3.30}$$

for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ and $w, w_0 \in \mathscr{W}(\mathscr{R})$. It follows from the hypothesis that

$$[\mathscr{H}(w_0), \mathscr{H}(w)]s' + [bw_0, \mathscr{H}(w)]\delta(s') + [w, w_0]s' = 0 \tag{3.31}$$

for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ and $w, w_0 \in \mathscr{W}(\mathscr{R})$. For $w_0 = w$, we have

$$b[\mathscr{H}(w), w]\delta(s') = 0 \tag{3.32}$$

for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ and $w \in \mathscr{W}(\mathscr{R})$. Since $\delta(s') \neq 0$ and $b \neq 0$, so $[\mathscr{H}(w), w] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$. Since $b \in \mathscr{C}$, so we observe from (3.25) that

$$[\mathscr{H}(s), w] - [\mathscr{H}(w), s] = 0 \tag{3.33}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s \in \mathscr{K}(\mathscr{R})$. Repalcement of s by $s'h$ in above expression and making use of $\mathscr{H}(s') \in \mathscr{Z}(\mathscr{R})$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ and $b \neq 0$ yields $[\delta(w), w] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$. In light of Fact 2.4, we have $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Finally, we suppose $\delta(s') = 0$ and substitute $\mathtt{v}$ by $\mathtt{v}s'$ in (3.19), we obtain

$$-[\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{v}^*)]s' + [\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)]s' + [\mathtt{r}, \mathtt{v}^*]s' - [\mathtt{v}, \mathtt{r}^*]s' = 0 \tag{3.34}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Combination of (3.19) and (3.34) gives

$$([\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)] - [\mathtt{v}, \mathtt{r}^*])s' = 0 \tag{3.35}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. This implies that

$$[\mathscr{H}(\mathtt{v}), \mathscr{H}(\mathtt{r}^*)] - [\mathtt{v}, \mathtt{r}^*] = 0 \tag{3.36}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. In particular

$$[\mathscr{H}(\mathtt{r}), \mathscr{H}(\mathtt{v})] - [\mathtt{r}, \mathtt{v}] = 0 \tag{3.37}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. As a special case of Theorem 3.1, $\mathscr{H}$ is of the form $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$, where $\lambda \in \mathscr{C}$ and $\lambda^2 = 1$. □

The following theorem is a generalization of [8, Theorem 2.4].

Theorem 3.3. *Let $\mathscr{R}$ be a non-commutative prime ring with involution of the second kind of characteristic different from two. If $\mathscr{H}$ is a $b-$generalized derivation on $\mathscr{R}$ associated with a derivation δ on $\mathscr{R}$ such that $[\mathscr{H}(\mathtt{r}), \delta(\mathtt{r}^*)] = [\mathtt{r}, \mathtt{r}^*]$ for every $\mathtt{r} \in \mathscr{R}$, then there exists $\lambda \in \mathscr{C}$ such that $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$ for every $\mathtt{r} \in \mathscr{R}$.*

Proof. By the hypothesis, we have

$$[\mathscr{H}(\mathtt{r}), \delta(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{r}^*] = 0 \tag{3.38}$$

for every $\mathtt{r} \in \mathscr{R}$. The derivation δ satisfies $\delta(\mathscr{R}) \nsubseteq \mathscr{Z}(\mathscr{R})$, otherwise the hypothesis $[\mathscr{H}(\mathtt{r}), \delta(\mathtt{r}^*)] = [\mathtt{r}, \mathtt{r}^*]$ for every $\mathtt{r} \in \mathscr{R}$, would reduce to $[\mathtt{r}, \mathtt{r}^*] = 0$ for all $\mathtt{r} \in \mathscr{R}$, and, therefore $\mathscr{R}$ would be commutative, by Fact 2.1. A linearization of (3.38) yields

$$[\mathscr{H}(\mathtt{r}), \delta(\mathtt{v}^*)] + [\mathscr{H}(\mathtt{v}), \delta(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{v}^*] - [\mathtt{v}, \mathtt{r}^*] = 0 \tag{3.39}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Replace $\mathtt{r}$ by $z \in \mathscr{Z}(\mathscr{R})$ in (3.39), we obtain $[\mathscr{H}(z), \delta(\mathtt{v})] = 0$ for every $\mathtt{v} \in \mathscr{R}$ and $z \in \mathscr{Z}(\mathscr{R})$. Observe from [18, Theorem 2] that $\mathscr{H}(z) \in \mathscr{Z}(\mathscr{R})$ for every $z \in \mathscr{Z}(\mathscr{R})$. Now take $\mathtt{r}$ as $\mathtt{r}s'$ in (3.38) and suppose $\delta(s') \neq 0$, we get

$$0 = -[\mathscr{H}(\mathtt{r}), \delta(\mathtt{r}^*)]s'^2 - [\mathscr{H}(\mathtt{r}), \mathtt{r}^*]\delta(s')s' - b[\mathtt{r}, \delta(\mathtt{r}^*)]\delta(s')s' \\ - [b, \delta(\mathtt{r}^*)]\mathtt{r}\delta(s')s' - b[\mathtt{r}, \mathtt{r}^*]\delta(s')^2 \\ - [b, \mathtt{r}^*]\mathtt{r}\delta(s')^2 + [\mathtt{r}, \mathtt{r}^*]s'^2 \tag{3.40}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. In view of (3.38), we have

$$0 = -[\mathscr{H}(\mathtt{r}), \mathtt{r}^*]\delta(s')s' - b[\mathtt{r}, \delta(\mathtt{r}^*)]\delta(s')s' \\ - [b, \delta(\mathtt{r}^*)]\mathtt{r}\delta(s')s' - b[\mathtt{r}, \mathtt{r}^*]\delta(s')^2 - [b, \mathtt{r}^*]\mathtt{r}\delta(s')^2 \tag{3.41}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Replace $\mathtt{r}$ by $\mathtt{r} + s'$ in last expression and use the fact that $\mathscr{H}(z) \in \mathscr{Z}(\mathscr{R})$ for every $z \in \mathscr{Z}(\mathscr{R})$, we get

$$0 = -[\mathscr{H}(\mathtt{r}), \mathtt{r}^*]\delta(s')s' - b[\mathtt{r}, \delta(\mathtt{r}^*)]\delta(s')s' \\ - [b, \delta(\mathtt{r}^*)]\mathtt{r}\delta(s')s' - b[\mathtt{r}, \mathtt{r}^*]\delta(s')^2 - [b, \mathtt{r}^*]\mathtt{r}\delta(s')^2 \\ - [b, \delta(\mathtt{r}^*)]\delta(s')s'^2 - [b, \mathtt{r}^*]\delta(s')^2 s' \tag{3.42}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Observe from (3.41)

$$[b, \delta(\mathtt{r}^*)]\delta(s')s'^2 + [b, \mathtt{r}^*]\delta(s')^2 s' = 0 \tag{3.43}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Replace $\mathtt{r}$ by $\mathtt{r}s'$ and use (3.43), we get $[b, \mathtt{r}^*]\delta(s')^2 s' = 0$ This forces that $[b, \mathtt{r}^*] = 0$ for every $\mathtt{r} \in \mathscr{R}$ since $\delta(s') \neq 0$. One can easily obtain from last relation that $b \in \mathscr{C}$. On the other hand

$$0 = -\mathscr{H}(s')[\mathtt{r}, \delta(\mathtt{r}^*)]s' - b[\delta(\mathtt{r}), \delta(\mathtt{r}^*)]s'^2 - [b, \delta(\mathtt{r}^*)]\delta(\mathtt{r})s'^2 \\ - \mathscr{H}(s')[\mathtt{r}, \mathtt{r}^*]\delta(s') - b[\delta(\mathtt{r}), \mathtt{r}^*]\delta(s')s' - [b, \mathtt{r}^*]\delta(\mathtt{r})\delta(s')s' \\ + [\mathtt{r}, \mathtt{r}^*]s'^2 \tag{3.44}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Since $b \in \mathscr{C}$, so we have

$$0 = -\mathscr{H}(s')[\mathtt{r}, \delta(\mathtt{r}^*)]s' - b[\delta(\mathtt{r}), \delta(\mathtt{r}^*)]s'^2 - \mathscr{H}(s')[\mathtt{r}, \mathtt{r}^*]\delta(s') \\ - b[\delta(\mathtt{r}), \mathtt{r}^*]\delta(s')s' + [\mathtt{r}, \mathtt{r}^*]s'^2 \tag{3.45}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Now substitute w and s for $\mathtt{r}$ and $\mathtt{v}$ in (3.39), respectively. This yields

$$[\mathscr{H}(s), \delta(w)] - [\mathscr{H}(w), \delta(s)] + 2[s, w] = 0 \tag{3.46}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s \in \mathscr{K}(\mathscr{R})$. Take s as $s'w$ in last relation and use $b \in \mathscr{C}$, we see that

$$\mathscr{H}(s')[w, \delta(w)] - [\mathscr{H}(w), w]\delta(s') = 0 \tag{3.47}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. On the other hand

$$b[h, \delta(w)]\delta(s') - [\mathscr{H}(w), w]\delta(s') = 0 \tag{3.48}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. In view of (3.47) and (3.48), we get

$$(\mathscr{H}(s') + b\delta(s'))[w, \delta(w)] = 0 \tag{3.49}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Since $\mathscr{H}(s') \in \mathscr{Z}(\mathscr{R})$, $\delta(s') \in \mathscr{Z}(\mathscr{R})$ and $b \in \mathscr{Z}(\mathscr{R})$, so either $\mathscr{H}(s') + b\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ or $[w, \delta(w)] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$. If $[w, \delta(w)] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$, then $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ from Fact 2.4. Therefore consider $\mathscr{H}(s') = -b\delta(s')$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ and use it in (3.45), we obtain

$$0 = b[\mathtt{r}, \delta(\mathtt{r}^*)]\delta(s')s' - b[\delta(\mathtt{r}), \delta(\mathtt{r}^*)]s'^2 + b[\mathtt{r}, \mathtt{r}^*]\delta(s')^2 - b[\delta(\mathtt{r}), \mathtt{r}^*]\delta(s')s' + [\mathtt{r}, \mathtt{r}^*]s'^2 \tag{3.50}$$

for every $\mathtt{r} \in \mathscr{R}$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. For $\mathtt{r} = w$, we have

$$2b[h, \delta(h)]\delta(s')s' = 0 \tag{3.51}$$

for every $w \in \mathscr{W}(\mathscr{R})$ and $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Thus $b = 0$ or $[w, \delta(w)] = 0$ for every $w \in \mathscr{W}(\mathscr{R})$ or $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. If $b = 0$ and since s' is not a zero-divisor, the relation (3.50) reduces to $[\mathtt{r}, \mathtt{r}^*] = 0$, for every $\mathtt{r} \in \mathscr{R}$. Thus the commutativity of $\mathscr{R}$ follows from Fact 2.1, a contradiction. The rest of two relations yields $\delta(s') = 0$ for every $s' \in \mathscr{K}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. Finally consider $\delta(s') = 0$ and replace $\mathtt{r}$ by $\mathtt{r}s'$ in (3.38) and use the facts that the involution is of the second kind, we see that

$$[\mathscr{H}(\mathtt{r}), \delta(\mathtt{v}^*)] - [\mathscr{H}(\mathtt{v}), \delta(\mathtt{r}^*)] - [\mathtt{r}, \mathtt{v}^*] + [\mathtt{v}, \mathtt{r}^*] = 0 \tag{3.52}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. Combination of (3.39) and (3.52) yields

$$[\mathscr{H}(\mathtt{r}), \delta(\mathtt{v}^*)] - [\mathtt{r}, \mathtt{v}^*] = 0 \tag{3.53}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. In particular,

$$[\mathscr{H}(\mathtt{r}), \delta(\mathtt{v})] - [\mathtt{r}, \mathtt{v}] = 0 \tag{3.54}$$

for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$. In view of Theorem 3.1, there exist $0 \neq \lambda \in \mathscr{C}$ and an additive map $\mu : \mathscr{R} \to \mathscr{C}$ such that $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$ and $\delta'(\mathtt{r}) = \lambda^{-1}\mathtt{r} + \mu(\mathtt{r})$ for every $\mathtt{r} \in \mathscr{R}$, where $\delta' = -\delta$. Commute the latter case with $\mathtt{r}$, we get $[\delta'(\mathtt{r}), \mathtt{r}] = 0$ for every $\mathtt{r} \in \mathscr{R}$. Since $\delta' = -\delta \neq 0$, so $\mathscr{R}$ is commutative from [29, Lemma 3], this leads to again a contradiction. This completes the proof of the theorem. □

4. Examples

The following example shows that the condition of the second kind involution is essential in Theorems 3.2 and 3.3. This example collected from [2, Example 1].

Example 4.1. *Let*

$$\mathscr{R} = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \middle| \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\},$$

which is of course a prime ring with ususal addition and multiplication of matrices, where $\mathbb{Z}$ is the set of integers. Define mappings $\mathscr{H}, \delta, * : \mathscr{R} \to \mathscr{R}$ such that

$$\mathscr{H}\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix},$$

$$\delta\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix},$$

and a fixed element

$$b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}.$$

Obviously,

$$\mathscr{Z}(\mathscr{R}) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \middle| \beta_1 \in \mathbb{Z} \right\}.$$

Then $\mathtt{r}^* = \mathtt{r}$ for every $\mathtt{r} \in \mathscr{Z}(\mathscr{R})$, and hence $\mathscr{Z}(\mathscr{R}) \subseteq \mathscr{W}(\mathscr{R})$, which shows that the involution $*$ is of the first kind. Moreover, $\mathscr{H}$, δ are nonzero $b-$generalized derivation and associated derivation with fixed element b defined as above, such that the hypotheses in Theorems 3.2 and 3.3 are satisfied but $\mathscr{H}$ is not in the form $\mathscr{H}(\mathtt{r}) = \lambda\mathtt{r}$ for every $\mathtt{r} \in \mathscr{R}$. Thus, the hypothesis of the second kind involution is crucial in our results.

We conclude the manuscript with the following example which reveals that Theorems 3.2 and 3.3 cannot be extended to semiprime rings.

Example 4.2. *Let $(\mathscr{R}, *)$ be a ring with involution as defined above, which admits a $b-$generalized derivation $\mathscr{H}$, where δ is an associated nonzero derivation same as above and $\mathscr{R}_1 = \mathbb{C}$ with the usual conjugation involution $^\diamond$. Next, let $\mathscr{S} = \mathscr{R} \times \mathscr{R}_1$ and define a $b-$generalized derivation $\mathscr{G}$ on $\mathscr{S}$ by $\mathscr{G}(\mathtt{r}, \mathtt{v}) = (\mathscr{H}(\mathtt{r}), 0)$ associated with a derivation $\mathscr{D}$ defined by $\mathscr{D}(\mathtt{r}, \mathtt{v}) = (\delta(\mathtt{r}), 0)$. Obviously, (S, τ) is a semiprime ring with involution of the second kind such that $\tau(\mathtt{r}, \mathtt{v}) = (\mathtt{r}^*, \mathtt{v}^\diamond)$. Then the $b-$generalized derivation $\mathscr{G}$ satisfies the requirements of Theorems 3.2 and 3.3, but $\mathscr{G}$ is not in the form $\mathscr{G}(\mathtt{r}) = \lambda\mathtt{r}$ for every $\mathtt{r} \in \mathscr{R}$, and $\mathscr{R}$ is not commutative. Hence, the primeness hypotheses in our results is not superfluous.*

5. Problems based on $X-$generalized skew derivations

We recall that "a generalized skew derivation is an additive mapping $\mathscr{G} : \mathscr{R} \to \mathscr{R}$ satisfying the rule $\mathscr{G}(\mathtt{rv}) = \mathscr{G}(\mathtt{r})\mathtt{v} + \zeta(\mathtt{r})\partial(\mathtt{v})$ for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$, where ∂ is an associated skew derivation of $\mathscr{R}$ and ζ is an automorphism of $\mathscr{R}$". Following [9], De Filippis proposed the new concept for further research and he defined the following: "Let $\mathscr{R}$ be an associative algebra, $b \in \mathscr{Q}$, ∂ be a linear mapping from $\mathscr{R}$ to itself, and ζ be an automorphism of $\mathscr{R}$. A linear mapping $\mathscr{G} : \mathscr{R} \to \mathscr{R}$ is called an $X-$generalized skew derivation of $\mathscr{R}$, with associated term (b, ζ, ∂) if $\mathscr{G}(\mathtt{rv}) = \mathscr{G}(\mathtt{r})\mathtt{v} + b\zeta(\mathtt{r})\partial(\mathtt{v})$ for every $\mathtt{r}, \mathtt{v} \in \mathscr{R}$". It is clear from both definitions, the notions of $X-$generalized skew derivation, generalize both generalized skew derivations and skew derivations. Hence, every $X-$generalized skew derivation is a generalized skew derivation as well as a skew derivation, but the converse statement is not true in general.

Actuated by the concept specified by De Filippis [9] and having regard to our main theorems, the following are natural problems.

Question 5.1. *Let $\mathscr{R}$ be a (semi)-prime ring and $\mathscr{L}$ be a Lie ideal of $\mathscr{R}$. Next, let $\mathscr{F}$ and $\mathscr{G}$ be two $X-$generalized skew derivation with an associated skew derivation ∂ of $\mathscr{R}$ such that*

$$[\mathscr{F}(\mathrm{r}), \mathscr{G}(\mathrm{v})] = [\mathrm{r}, \mathrm{v}], \textit{ for every } \mathrm{r}, \mathrm{v} \in \mathscr{L}.$$

Then what we can say about the behaviour of $\mathscr{F}$ and $\mathscr{G}$ or about the structure of $\mathscr{R}$?

Question 5.2. *Let $\mathscr{R}$ be a prime ring possessing second kind involution with suitable torsion restrictions and $\mathscr{L}$ be a Lie ideal of $\mathscr{R}$. Next, let $\mathscr{F}$ and $\mathscr{G}$ be two $X-$generalized skew derivation with an associated skew derivation ∂ of $\mathscr{R}$ such that*

$$[\mathscr{F}(\mathrm{r}), \mathscr{G}(\mathrm{r}^*)] = [\mathrm{r}, \mathrm{r}^*], \textit{ for every } \mathrm{r} \in \mathscr{L}.$$

Then what we can say about the behaviour of $\mathscr{F}$ and $\mathscr{G}$ or about the structure of $\mathscr{R}$?

6. Conclusions

The characterization of strong commutative preserving (SCP) $b-$generalized derivations has been discussed in non-commutative prime rings. In addition, the behavior of $b-$generalized derivations with $*-$differential/functional identities on prime rings with involution was investigated. Besides, we present some problems for $X-$generalized skew derivations on rings with involution.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Ali, N. A. Dar, On $*-$centralizing mappings in rings with involution, *Georgian Math. J.*, **21** (2014), 25–28. doi: 10.1515/gmj-2014-0006.
2. S. Ali, N. Dar, A. N. Khan, On strong commutativity preserving like maps in rings with involution, *Miskolc Math. Notes*, **16** (2015), 17–24. doi: 10.18514/MMN.2015.1297.

3. M. Ashraf, M. A. Raza, S. A. Parry, Commutators having idempotent values with automorphisms in semi-prime rings, *Math. Rep.*, **20** (2018), 51–57.
4. K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities*, Monographs and Textbooks in Pure and Applied Mathematics, New York: Marcel Dekker, Inc., 1996.
5. H. E. Bell, M. N. Daif, On commutativity and strong commutativity-preserving maps, *Can. Math. Bull.*, **37** (1994), 443–447. doi: 10.4153/CMB-1994-064-x.
6. H. E. Bell, G. Mason, On derivations in near-rings and rings, *Math. J. Okayama Univ.*, **34** (1992), 135–144.
7. M. Brešar, W. S. Martindale III, C. R. Miers, Centralizing maps in prime rings with involution, *J. Algebra*, **161** (1993), 342–357. doi: 10.1006/jabr.1993.1223.
8. N. A. Dar, A. N. Khan, Generalized derivations in rings with involution, *Algebra Colloq.*, **24** (2017), 393–399. doi: 10.1142/S1005386717000244.
9. V. De Filippis, Annihilators and power values of generalized skew derivations on Lie ideals, *Can. Math. Bull.*, **59** (2016), 258–270. doi: 10.4153/CMB-2015-077-x.
10. V. De Filippis, N. Rehman, M. A. Raza, Strong commutativity preserving skew derivations in semiprime rings, *Bull. Malays. Math. Sci. Soc.*, **41** (2018), 1819–1834. doi: 10.1007/s40840-016-0429-9.
11. V. De Filippis, F. Wei, $b-$generalized skew derivations on Lie ideals, *Mediterr. J. Math.*, **15** (2018), 65. doi: 10.1007/s00009-018-1103-2.
12. Q. Deng, M. Ashraf, On strong commutativity preserving maps, *Results Math.*, **30** (1996), 259–263.
13. B. Dhara, N. Rehman, M. A. Raza, Lie ideals and action of generalized derivations in rings, *Miskolc Math. Notes*, **16** (2015), 769–779.
14. I. N. Herstein, *Rings with involution*, Chicago: The University of Chicago Press, 1976.
15. A. N. Khan, M. A. Raza, H. Alhazmi, Strong commutativity preserving skew derivations on Banach algebras, *J. Taibah University Sci.*, **13** (2019), 478–480. doi: 10.1080/16583655.2019.1596377.
16. V. K. Kharchenko, Differential identity of prime rings, *Algebra Logic*, **17** (1978), 155–168. doi: 10.1007/BF01670115.
17. M. T. Kọsan, T. K. Lee, $b-$generalized derivations of semiprime rings having nilpotent values, *J. Aust. Math. Soc.*, **96** (2014), 326–337. doi: 10.1017/S1446788713000670.
18. P. H. Lee, T. K. Lee, Lie ideals of prime rings with derivations, *Bull. Inst. Math. Acad. Sinica*, **11** (1983), 75–80.
19. T. K. Lee, Generalized derivations of left faithful rings, *Commun. Algebra*, **27** (1999), 4057–4073. doi: 10.1080/00927879908826682.
20. T. K. Lee, T. L. Wong, Nonadditive strong commutativity preserving maps, *Commun. Algebra*, **40** (2012), 2213–2218. doi: 10.1080/00927872.2011.578287.
21. P. K. Liau, W. L. Huang, C. K. Liu, Nonlinear strong commutativity preserving maps on skew elements of prime rings with involution, *Linear Algebra Appl.*, **436** (2012), 3099–3108. doi: 10.1016/j.laa.2011.10.014.

22. P. K. Liau, C. K. Liu, An Engel condition with $b-$generalized derivations for Lie ideals, *J. Algebra Appl.*, **17** (2018), 1850046. doi: 10.1142/S0219498818500469.

23. J. S. Lin, C. K. Liu, Strong commutativity preserving maps on Lie ideals, *Linear Algebra Appl.*, **428** (2008), 1601–1609 doi: 10.1016/j.laa.2007.10.006.

24. J. S. Lin, C. K. Liu, Strong commutativity preserving maps in prime rings with involution, *Linear Algebra Appl.*, **432** (2010), 14–23. doi: 10.1016/j.laa.2009.06.036.

25. C. K. Liu, Strong commutativity preserving generalized derivations on right ideals, *Monatsh. Math.*, **166** (2012), 453–465. doi: 10.1007/s00605-010-0281-1.

26. C. K. Liu, An Engel condition with $b-$generalized derivations, *Linear Multilinear Algebra*, **65** (2017), 300–312. doi: 10.1080/03081087.2016.1183560.

27. J. Ma, X. W. Xu, F. W. Niu, Strong commutativity-preserving generalized derivations on semiprime rings, *Acta Math. Sin. (Engl. Ser.)*, **24** (2008), 1835–1842. doi: 10.1007/s10114-008-7445-0.

28. W. S. Martindale III, Prime rings satisfying a generalized polynomial identity, *J. Algebra*, **12** (1969), 576–584. doi: 10.1016/0021-8693(69)90029-5.

29. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093–1100. doi: 10.1090/S0002-9939-1957-0095863-0.

30. M. A. Raza, On automorphisms in prime rings with applications, *Commun. Korean Math. Soc.*, **36** (2021), 641–650. doi: 10.4134/CKMS.c200341.

31. N. Rehman, M. A. Raza, On m-commuting mappings with skew derivations in prime rings, *St. Petersburg Math. J.*, **27** (2016), 641–650. doi: 10.1090/spmj/1411.

32. W. Watkins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.*, **14** (1976), 29–35. doi: 10.1016/0024-3795(76)90060-4.

33. B. Zalar, On centralizers of semiprime rings, *Commentat. Math. Univ. Carol.*, **32** (1991), 609–614.

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