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Research article

Existence of local and global solutions to fractional order fuzzy delay differential equation with non-instantaneous impulses

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Abstract: The main concern of this manuscript is to examine some sufficient conditions under which the fractional order fuzzy delay differential system with the non-instantaneous impulsive condition has a unique solution. We also study the existence of a global solution for the considered system. Fuzzy set theory, Banach fixed point theorem and Non-linear functional analysis are the major tools to demonstrate our results. In last, an example is given to illustrate these analytical results.

Keywords: fuzzy delay differential equations; Atangana-Baleanu fractional derivative; fixed point theorem; non-instantaneous impulses
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1. Introduction

The dynamics of many evolutionary processes are characterized by the fact that at a specific moment of time they experience a sudden change in their state, such as harvesting, natural disasters and shocks, etc. These processes are subject to short-term perturbations, whose time period is minimal in analogy with the whole evolution. In dynamical systems associated with such sudden changes, we assume these changes in the form of impulses. Therefore, impulsive differential equations have been developed to model these types of situations. In literature, there are two types of impulses, one is instantaneous and another one is non-instantaneous impulses. For more detail, one can see [1–4].

The theory of fractional calculus deals with the integral and derivative of any arbitrary (real or complex) order. It was first proposed in the works by mathematicians Leibniz, Abel, L'Hopital, Riemann, Liouville [5, 6]. Because of the nonlocality and inherent properties of numerous complex systems, fractional calculus is important to model many physical applications in dissimilar branches of science and engineering. Memory and hereditary are additionally significant properties of various materials and processes in biomechanics, electrical circuits, electrochemistry, biology, electromagnetic processes, control and porous media, which are widely recognized to be well predicted by using fractional differential operators [7–9]. In the existing literature, there are numerous definitions for fractional operators, for example, Riemann-Liouville, Grunwald-Letnikov, Hadamard, Caputo, Riesz-Caputo and so on.

More recently, in 2015, Caputo and Fabrizio introduced a new fractional derivative known as Caputo-Fabrizio fractional derivative which is given by

$$^{CF}D^{\varsigma}_{a+}\mathcal{P}(\xi) = \frac{M(\varsigma)}{1-\varsigma}\int_a^{\xi} exp\Big[-\frac{\varsigma}{1-\varsigma}(\xi-\vartheta)\Big]\mathcal{P}'(\vartheta)d\vartheta,$$

where $\varsigma \in \mathbb{R}$ is the order of the derivative [10,11]. A year after, Atangana and Baleanu proposed another definition of nonlocal derivatives with non-singular kernel relied on the Mittag-Leffler function

$$^{ABC}D^{\varsigma}_{a+}\mathcal{P}(\xi) = \frac{B(\varsigma)}{1-\varsigma}\int_a^{\xi} E_{\varsigma}\Big[-\frac{\varsigma}{1-\varsigma}(\xi-\vartheta)^{\varsigma}\Big]\mathcal{P}'(\vartheta)d\vartheta,$$

where $\mathcal{P} \in C^F(I) \cap L^F(I)$ is the fuzzy function and $B(\varsigma) = (1-\varsigma) + \frac{\varsigma}{\Gamma(\varsigma)}$ is known as a normalization function which satisfies $B(0) = B(1) = 1$ and $0 < \varsigma < 1$. This definition upheld the Caputo-Fabrizio's one relied on the exponential function. It expand the profundity of the connection between the Mittag-Leffler function and fractional calculus which leads to the significant applications such as thermal physics, population dynamics, control problems and so on [12–15].

On the other hand, in many real applications there is some uncertainty that occurs in the system due to that the behaviour of the system is affected. Therefore, to overcome this type of issue, Zadeh [16] in 1965, presented the theory of fuzzy set by using the membership function. The theory of fuzzy set is a well-built tool for modelling the uncertainty, ambiguity and vague information such as particle system, medicine, quantum optics, civil engineering, computational biology, bioinformatics and hydraulic process [17–20] etc. Moreover, the nonlocal effects, as well as uncertainty behaviours, represent interesting phenomena and hence nowadays many researchers working on the fuzzy fractional operator which combine fractional calculus with the fuzzy set theory. In the last few decays, many authors established some results on the existence, uniqueness and stability of the solution for fuzzy differential equations of an integer as well as fractional order [21–28]. In particular, S. Seikkala [21], established the existence and uniqueness of the solution for a fuzzy initial value problem. In [25], authors investigated the existence and uniqueness of solution for the fractional fuzzy differential equation. In [26], authors examined the uniqueness of the solution for a nonlinear impulsive fuzzy integro-differential equation. In [27] authors examined the existence and uniqueness of a mild solution to a nonlinear fuzzy differential equation with time delay. Moreover, there are only a few papers that established the existence and uniqueness results for ABC fuzzy fractional differential equation [29,30]. For instance, in [30] authors considered the ABC fractional fuzzy differential equation and establish the existence and uniqueness of the solution. As per the author's knowledge, there is not a single paper that established the existence of local and global solutions for the ABC fractional fuzzy delay differential system with impulsive effects.

Therefore, motivated by the above facts, in this paper, we will study the existence of local and global solution for fractional order fuzzy delay differential equation with non-instantaneous impulsive condition of the form

$$\begin{aligned} {}^{ABC}{}_{0}D_{\xi}^{\varsigma}\mathcal{P}(\xi) &= \mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)), && \xi\in(k_j,\xi_{j+1}],\ j=0,1,\cdots,n,\\ \mathcal{P}(\xi) &= h_j(\xi,\mathcal{P}(\xi_j^-)), && \xi\in(\xi_j,k_j], \quad j=1,2,\cdots,n,\\ \mathcal{P}(\xi) &= \Psi(\xi), && \xi\in[-\delta,0], \end{aligned} \tag{1.1}$$

where $\mathcal{P}$ is a state fuzzy function and ${}^{ABC}{}_{0}D_{\xi}^{\varsigma}$ is the ABC derivative of order $\varsigma\in(0,1)$. The points k_j and ξ_j satisfies the sequence $0=k_0=\xi_0<\xi_1<k_1<\xi_2<\cdots<\xi_n<k_n<\xi_{n+1}=T<\infty$. $\mathcal{H}$ is the nonlinear function which is defined from $(k_j,\xi_{j+1}]\times\mathbb{K}^n\times\mathbb{K}^n$ into $\mathbb{K}^n$, where $\mathbb{K}^n$ denotes the set of all upper semi continuous convex normal fuzzy number with bounded ι-level intervals and $j=0,1,\cdots,n$ (which will be specified in Definition 2.9). The functions $h_j(\xi,\mathcal{P}(\xi_j^-))$ represent non-instantaneous impulses during the interval $(\xi_j,k_j],\ j=1,2,\cdots,n$. $\mathcal{P}(\xi_j^-)$, $\mathcal{P}(\xi_j^+)$ represent the left and right limit of the state fuzzy function $\mathcal{P}$ at ξ_j. For $\delta>0$, $\Psi:[-\delta,0]\to\mathbb{K}^n$ is a continuous function.

The structure of the manuscript is as follows: In Section 2, we have given the fundamental definitions and some important lemmas. In Section 3, we establish the local existence and uniqueness results for the solution. Section 4 is devoted to establishing the global solution for the considered system and in the last Section 5, an example is given to validate the obtained analytical results.

2. Preliminaries

In this section, we briefly describe some notations, fundamental definitions and important lemmas which are useful to prove the main results. $C^F(I=[0,T])$ denote the set of all continuous fuzzy valued functions on I and $L^F(I)$ denote the set of all Lebesgue integrable fuzzy valued functions on I. Also we define $PC(J;\mathbb{K}^n)$ as: $PC(J;\mathbb{K}^n)=\{\mathcal{P}:J\to\mathbb{K}^n:\mathcal{P}\in C^F([-\delta,0);\ \mathbb{K}^n)\cup C^F((\xi_j,\xi_{j+1}];\ \mathbb{K}^n),\ j=0,1,\cdots,n$ and there exists $\mathcal{P}(\xi_j^-)$ and $\mathcal{P}(\xi_j^+), j=1,2,\cdots,n$, with $\mathcal{P}(\xi_j^-)=\mathcal{P}(\xi_j)\}$ for the space of piecewise continuous functions, where $J=[-\delta,0)\cup[0,T]$.

Next, we define the definition of ABC derivative and integrals.

Definition 2.1. *[31] The Atangana-Baleanu fractional integral of order $\varsigma\in(0,1)$, is given by*

$$ {}^{AB}{}_{a}I^{\varsigma}\mathcal{P}(\xi)=\frac{1-\varsigma}{B(\varsigma)}\mathcal{P}(\xi)+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_a^{\xi}(\xi-\vartheta)^{(\varsigma-1)}\mathcal{P}(\vartheta)d\vartheta,\ \mathcal{P}\geq a,$$

where $\mathcal{P}\in C^F(I)\cap L^F(I)$ is the fuzzy function and $B(\varsigma)=(1-\varsigma)+\frac{\varsigma}{\Gamma(\varsigma)}$ is known as the normalization function which satisfies $B(0)=B(1)=1$.

Definition 2.2. *[31] The Atangana-Baleanu fractional fuzzy derivative in Caputo sense is defined by*

$$ {}^{ABC}D_{a+}^{\varsigma}\mathcal{P}(\xi)=\frac{B(\varsigma)}{1-\varsigma}\int_a^{\xi}E_{\varsigma}\Big[-\frac{\varsigma}{1-\varsigma}(\xi-\vartheta)^{\varsigma}\Big]\mathcal{P}'(\vartheta)d\vartheta,$$

where $\mathcal{P}\in C^F(I)\cap L^F(I)$ is the fuzzy function and E_{ς} is the Mittag-Leffler function.

Definition 2.3. *For ABC derivative, we have the following important properties of Laplace transformation*

1) $L({}^{ABC}D^{\varsigma}_{a+}\mathcal{P}(\xi)) = \frac{B(\varsigma)}{1-\varsigma}\frac{s^{\varsigma-1}}{s^{\varsigma}+\frac{\varsigma}{1-\varsigma}}(s\mathcal{P}(s) - \mathcal{P}(0))$,
2) $L(\xi^{\varsigma}) = \frac{\varsigma}{s^{\varsigma+1}}$,
3) $L(\mathcal{H}^{n}(\xi)) = s^{n}L(\mathcal{H}(\xi)) - s^{n-1}\mathcal{H}(0)\cdots - \mathcal{H}^{(n-1)}(0)$,
4) $L(u(\xi) * \mathcal{P}(\xi)) = L(u(\xi))L(\mathcal{P}(\xi))$.

Now, we define some important definitions and lemmas which are often used.

Definition 2.4. *[32] Let $\varkappa$ and α be two nonempty bounded subsets of $\mathbb{R}^n$ then we define*

$$d_{\mathcal{H}}(\varkappa, \alpha) = max\{\sup_{\hat{\varkappa}\in\varkappa}\inf_{\hat{\alpha}\in\alpha}\|\hat{\varkappa} - \hat{\alpha}\|, \sup_{\hat{\alpha}\in\alpha}\inf_{\hat{\varkappa}\in\varkappa}\|\hat{\varkappa} - \hat{\alpha}\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Clearly, $d_{\mathcal{H}}(\varkappa, \alpha) = d_{\mathcal{H}}(\alpha, \varkappa)$, i.e., it is symmetric on $\varkappa$ and α.

Consequently, for any nonemptey subsets $\varkappa, \alpha$ and Θ of $\mathbb{R}^n$, we have

(1) $d_{\mathcal{H}}(\varkappa, \alpha) \geq 0$ with $d_{\mathcal{H}}(\varkappa, \alpha) = 0$ iff $\varkappa = \alpha$,
(2) $d_{\mathcal{H}}(\varkappa, \alpha) = d_{\mathcal{H}}(\alpha, \varkappa)$,
(3) $d_{\mathcal{H}}(\varkappa, \alpha) \leq d_{\mathcal{H}}(\varkappa, \Theta) + d_{\mathcal{H}}(\Theta, \alpha)$.

We denote $\mathbb{K}^n(\mathbb{R}^n)$ for the family of all nonempty subset of $\mathbb{R}^n$ which are convex and compact. The scalar multiplication and addition in $\mathbb{K}^n(\mathbb{R}^n)$ are defined as

$$\varkappa + \alpha = \{\hat{\varkappa} + \hat{\alpha} : \hat{\varkappa} \in \varkappa \text{ and } \hat{\alpha} \in \alpha\},$$
$$\varrho\varkappa = \{\varrho\hat{\varkappa} : \hat{\varkappa} \in \varkappa\}$$

for all $\varrho \geq 0$ and $\varkappa, \alpha \in \mathbb{K}^n(\mathbb{R}^n)$.

Definition 2.5. *[32] We define*

$$d_{\mathcal{H}}([z]^{\iota}, [y]^{\iota}) = \max\{d([z]^{\iota}, [y]^{\iota}), d([y]^{\iota}, [z]^{\iota}) : \iota \in (0, 1]\},\ z, y, \in \mathbb{K}^n.$$

Clearly, $(\mathbb{K}^n(\mathbb{R}^n), d_{\mathcal{H}})$ forms a complete metric space.

Definition 2.6. *[32] The supremium metric d_∞ on $\mathbb{K}^n$ is defined by*

$$d_\infty(z, y) = \sup\{d_{\mathcal{H}}([z]^{\iota}, [y]^{\iota}) : \iota \in (0, 1],\ \forall\ z, y \in \mathbb{K}^n\}.$$

Clearly, we can see that d_∞ is a metric in $\mathbb{K}^n$ and $(\mathbb{K}^n, d_\infty)$ forms a complete metric space.

Suppose that $J = [-\delta, T] \subset \mathbb{R}$ be a compact interval and $PC(J; \mathbb{K}^n)$ denotes tha space of all fuzzy functions which are piece-wise continuous from J to $\mathbb{K}^n$. We define the metric H_1 on $PC(J; \mathbb{K}^n)$ by

$$H_1(z, y) = \sup\{d_\infty(z(\xi), y(\xi))\ :\ \xi \in J,\ \forall\ z,\ y \in PC(J; \mathbb{K}^n)\}.$$

Clearly, $(PC(J; \mathbb{K}^n), H_1)$ is a complete metric space.

Definition 2.7. *[33] A membership function $\Omega_{\mathcal{D}} : \Xi \to [0, 1]$ of fuzzy set $\mathcal{D}$ satisfy the following:*

1) If $\Omega_{\mathcal{D}}(\rho) = 1$, then ρ is completely belongs to $\mathcal{D}$,
2) If $0 < \Omega_{\mathcal{D}}(\rho) < 1$, then ρ is partially belongs to $\mathcal{D}$,

3) If $\Omega_{\mathcal{D}}(\rho) = 0$, then $\rho \notin \mathcal{D}$.

Definition 2.8. *[33] A fuzzy set $\mathcal{D}$ is said to be fuzzy number if it satisfy the following properties:*

1) $\mathcal{D}$ is normal, i.e., $\exists\, \rho_0 \in \mathbb{R}$ with $\Omega_{\mathcal{D}}(\rho_0) = 1$.
2) $\mathcal{D}$ is fuzzy convex, i.e., $\Omega_{\mathcal{D}}(\xi\rho + (1-\xi)\hat{\rho}) \geq min\{\Omega_{\mathcal{D}}(\rho), \Omega_{\mathcal{D}}(\hat{\rho})\},\ \forall\, \xi \in [0,1], \rho, \hat{\rho} \in \mathbb{R}^n$.
3) $\mathcal{D}$ is upper semi continuous on $\mathbb{R}^n$, i.e., $\forall\, \epsilon > 0,\ \exists\, \tau > 0$ such that $\Omega_{\mathcal{D}}(\rho) - \Omega_{\mathcal{D}}(\rho_0) < \epsilon, |\rho - \rho_0| < \tau$.
4) $\mathcal{D}$ is compactly supported, i.e., $cl\{\rho \in \mathbb{R}^n; \Omega_{\mathcal{D}}(\rho) > 0\}$ is compact.

Definition 2.9. *[34] The ι - level set of fuzzy set $\mathcal{D}$ is defined by*

$$[\mathcal{D}]^{\iota} = \{\rho | \rho \in \Xi, \Omega_{\mathcal{D}}(\rho) \geq \iota\}, \quad \iota \in (0,1],$$

and for $\iota = 0$, we have

$$[\mathcal{D}]^{0} = cl\{\rho | \rho \in \Xi, \Omega_{\mathcal{D}}(\rho) \geq 0\}.$$

Definition 2.10. *[34] A fuzzy number $g \in \mathbb{R}$ is called positive if for two arbitrary fuzzy number g_1, g_2, it holds $0 < g_1 < g_2$ for the support $\psi_g = [g_1, g_2]$ of g, i.e., ψ_g is in the positive real line. Similarly, g is called negative if $g_1 \leq g_2 < 0$ and zero if $g_1 \leq 0 \leq g_2$.*

Lemma 2.1. *[35] If $g, h \in \mathbb{K}^n$, then for $\iota \in (0,1]$,*

$$\begin{aligned} &[g+h]^{\iota} = [g_a^{\iota} + h_a^{\iota}, g_b^{\iota} + h_b^{\iota}]. \\ &[g \times h]^{\iota} = [min\{h_i^{\iota} h_j^{\iota}\}, \max\{h_i^{\iota} h_j^{\iota}\}], \quad i, j = a, b. \\ &[g-h]^{\iota} = [g_a^{\iota} - h_b^{\iota}, g_b^{\iota} - h_a^{\iota}]. \end{aligned}$$

Definition 2.11. *[35] We define the fuzzy integral as follows*

$$\left[\int_a^b [\mathcal{P}(\xi) d\xi]\right]^{\iota} = \left[\int_a^b \mathcal{P}_q^{\iota}(\xi) d\xi, \int_a^b \mathcal{P}_r^{\iota}(\xi) d\xi\right], \quad \forall\, a, b \in I,$$

provided that the right side Lebesgue integrals in the above equation are exists. Also, the fuzzy integral is a fuzzy number.

Lemma 2.2. *[30] If $y(\xi) \in C^F(I) \cap L^F(I),\ 0 < \varsigma < 1$, then the unique solution of following problem*

$${}^{ABC}_{\ \ \ 0}D_{\xi}^{\varsigma} y(\xi) = u(\xi),$$

is given by

$$y(\xi) = \frac{1-\varsigma}{B(\varsigma)} u(\xi) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} u(\tau)(\xi - \tau)^{\varsigma - 1} d\tau.$$

Lemma 2.3. *A function $\mathcal{P} \in PC(J;\mathbb{K}^n)$ is the solution of the considered system (1.1) if $\mathcal{H}(0,\mathcal{P}(0),\mathcal{P}(-\delta)) = 0$ holds and solution is given by*

$$\mathcal{P}(\xi) = \begin{cases} \Psi(\xi), \quad \xi \in [-\delta, 0), \\ \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta, \\ \hfill \forall\ \xi \in [0,\xi_1], \\ h_j(\xi,\mathcal{P}(\xi_j^-)), \quad \xi \in (\xi_j, k_j], \quad j = 1,2,\cdots,n, \\ h_j(k_j,\mathcal{P}(\xi_j^-)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta \\ \quad + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)),\ \forall\ \xi \in (k_j,\xi_{j+1}],\ j = 1,2,\cdots,n. \end{cases} \tag{2.1}$$

Proof. From Lemma 2.2, for any $\xi \in [0,\xi_1]$, we have

$$\mathcal{P}(\xi) = \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta.$$

Now, for any $\xi \in (\xi_1, k_1]$,

$$\mathcal{P}(\xi) = h_1(k_1,\mathcal{P}(\xi_1^-)).$$

Also, for any $\xi \in (k_1, \xi_2]$,

$$\begin{aligned} \mathcal{P}(\xi) &= \mathcal{P}(\xi^-) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_1}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta \\ &\quad + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)), \\ &= h_1(k_1,\mathcal{P}(\xi_1^-)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_1}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta \\ &\quad + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)). \end{aligned}$$

Now, for any $\xi \in (\xi_2, k_2]$,

$$\mathcal{P}(\xi) = h_2(k_2,\mathcal{P}(\xi_2^-)).$$

Also, for any $\xi \in (k_2, \xi_3]$,

$$\begin{aligned} \mathcal{P}(\xi) &= \mathcal{P}(\xi_2^-) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_2}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta \\ &\quad + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)), \\ &= h_2(k_2,\mathcal{P}(\xi_2^-)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_2}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta \end{aligned}$$

$$+\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)).$$

By using the similar process, we will get for $\xi \in (k_j, \xi_{j+1}]$,

$$\mathcal{P}(\xi) = h_j(k_j, \mathcal{P}(\xi_j^-)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta$$
$$+\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta)),$$

which has the form (2.1). Hence, the result follows. For more detail on solution, please see [36]. □

3. Existence of local solution

In this section, we state and prove the existence and uniqueness of local solution to the system (1.1). For this purpose, the following assumptions are required:

(B1) The function $\mathcal{H} : (k_j, \xi_{j+1}] \times \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n,\ j = 0, 1, \cdots, n$, is continuous and there exists positive constant $M_{\mathcal{H}_1}$ and $M_{\mathcal{H}_2}$ such that

$$d_{\mathcal{H}}([\mathcal{H}(\xi,\eta_1,\eta_2)]^{\iota},[\mathcal{H}(\xi,\gamma_1,\gamma_2)]^{\iota}) \le M_{\mathcal{H}_1}d_{\mathcal{H}}([\eta_1]^{\iota},[\gamma_1]^{\iota}) + M_{\mathcal{H}_2}d_{\mathcal{H}}([\eta_2]^{\iota},[\gamma_2]^{\iota}),$$
$$\forall\ \eta_1,\gamma_1,\eta_2,\gamma_2 \in \mathbb{K}^n.$$

(B2) The functions $h_j : (\xi_j, k_j] \times \mathbb{K}^n \to \mathbb{K}^n,\ j = 1, 2, \cdots, n$, are continuous and there exists a positive constants $M_{h_j} < 1$ such that

$$d_{\mathcal{H}}([h_j(\xi,\eta_1)]^{\iota},[h_j(\xi,\gamma_1)]^{\iota}) \le M_{h_j}d_{\mathcal{H}}([\eta_1]^{\iota},[\gamma_1)]^{\iota}),\ \forall\ \eta_1,\gamma_1 \in \mathbb{K}^n, \xi \in (\xi_j,k_j].$$

For the convenience, we use the following notations throughout the manuscript $L = \max_{1\le j\le n}\{L_1, L_{2_j}\}$, where $L_1 = \left(\frac{1-\varsigma}{B(\varsigma)}(M_{\mathcal{H}_1}+M_{\mathcal{H}_2}) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}(M_{\mathcal{H}_1}+M_{\mathcal{H}_2})T^{\varsigma}\right)$ and $L_{2_j} = \left(M_{h_j} + \frac{1-\varsigma}{B(\varsigma)}(M_{\mathcal{H}_1}+M_{\mathcal{H}_2}) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}(M_{\mathcal{H}_1}+M_{\mathcal{H}_2})T^{\varsigma}\right),\ j = 1, 2, \cdots, n.$

Theorem 3.1. *If the assumptions* $(B1)$ *and* $(B2)$ *are satisfied then the problem (1.1) has a unique local solution on* J.

Proof. For each $\eta \in PC(J;\mathbb{K}^n)$, we define an operator $\Lambda : PC(J;\mathbb{K}^n) \to PC(J;\mathbb{K}^n)$ such that

$$(\Lambda\eta)(\xi) = \begin{cases} \Psi(\xi), \quad \xi \in [-\delta,0), \\ \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta, \\ \qquad\qquad \forall\ \xi \in [0,\xi_1], \\ h_j(\xi,\eta(\xi_j^-)),\ \xi \in (\xi_j,k_j], \qquad j = 1,2,\cdots,n, \\ h_j(k_j,\eta(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta)) \\ \qquad +\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta, \\ \qquad\qquad \forall\ \xi \in (k_j,\xi_{j+1}],\ j = 1,2,\cdots,n. \end{cases}$$

Here, we need to show that the operator Λ has a fixed point, which is the solution of our considered system (1.1). The proof of this theorem is divided into the following cases:

Case 1: For $\xi \in [-\delta, 0),\ \eta,\gamma \in PC(J;\mathbb{K}^n)$,

$$(\Lambda\eta)(\xi) = \Psi(\xi),$$

$$(\Lambda\gamma)(\xi) = \Psi(\xi).$$

Hence,

$$H_1((\Lambda\eta),(\Lambda\gamma)) = 0.$$

Case 2: For $\xi \in [0,\xi_1],\ \eta,\gamma \in PC(J;\mathbb{K}^n)$,

$$(\Lambda\eta)(\xi) = \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta,$$
$$(\Lambda\gamma)(\xi) = \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta.$$

Therefore,

$$\begin{aligned}
&d_{\mathcal{H}}([\Lambda\eta(\xi)]^\iota,[\Lambda\gamma(\xi)]^\iota)\\
&= d_{\mathcal{H}}([\Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta]^\iota,\\
&\qquad [\Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta]^\iota)\\
&= d_{\mathcal{H}}\Big([\Psi(0)]^\iota + \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta))\Big]^\iota + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta\Big]^\iota,\\
&\qquad [\Psi(0)]^\iota + \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta))\Big]^\iota + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta\Big]^\iota\Big)\\
&= d_{\mathcal{H}}\Big(\Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta))\Big]^\iota + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta\Big]^\iota,\\
&\qquad \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta))\Big]^\iota + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta\Big]^\iota\Big)\\
&\le d_{\mathcal{H}}\Big(\Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta))\Big]^\iota, \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta))\Big]^\iota\Big)\\
&\qquad + \ d_{\mathcal{H}}\Big(\Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta\Big]^\iota,\\
&\qquad\qquad \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta\Big]^\iota\Big)\\
&\le \frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_1}d_{\mathcal{H}}([\eta(\xi)]^\iota,[\gamma(\xi)]^\iota) + \frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_2}d_{\mathcal{H}}([\eta(\xi-\delta)]^\iota,[\gamma(\xi-\delta)]^\iota)\\
&\qquad\qquad + \ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi |(\xi-\vartheta)^{\varsigma-1}|(M_{\mathcal{H}_1}d_{\mathcal{H}}([\eta(\vartheta)]^\iota,[\gamma(\vartheta)]^\iota)\\
&\qquad\qquad + M_{\mathcal{H}_2}d_{\mathcal{H}}([\eta(\vartheta-\delta)]^\iota,[\gamma(\vartheta-\delta)]^\iota))d\vartheta.
\end{aligned}$$

Therefore,

$$d_\infty[\Lambda\eta(\xi),\Lambda\gamma(\xi)]$$

$$
\begin{aligned}
&= \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\Lambda\eta(\xi)]^{\iota}, [\Lambda\gamma(\xi)]^{\iota}) \\
&= \sup_{\iota\in(0,1]} \Big(\frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} d_{\mathcal{H}}([\eta(\xi)]^{\iota}, [\gamma(\xi)]^{\iota}) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} d_{\mathcal{H}}([\eta(\xi-\delta)]^{\iota}, [\gamma(\xi-\delta)]^{\iota}) \\
&\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| (M_{\mathcal{H}_1} d_{\mathcal{H}}([\eta(\vartheta)]^{\iota}, [\gamma(\vartheta)]^{\iota}) + M_{\mathcal{H}_2} d_{\mathcal{H}}([\eta(\vartheta-\delta)]^{\iota}, [\gamma(\vartheta-\delta)]^{\iota})) d\vartheta \Big) \\
&\leq \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\eta(\xi)]^{\iota}, [\gamma(\xi)]^{\iota}) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\eta(\xi-\delta)]^{\iota}, [\gamma(\xi-\delta)]^{\iota}) \\
&\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| \\
&\qquad (M_{\mathcal{H}_1} \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\eta(\vartheta)]^{\iota}, [\gamma(\vartheta)]^{\iota}) + M_{\mathcal{H}_2} \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\eta(\vartheta-\delta)]^{\iota}, [\gamma(\vartheta-\delta)]^{\iota})) d\vartheta \\
&\leq \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} d_{\infty}(\eta(\xi), \gamma(\xi)) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} d_{\infty}(\eta(\xi-\delta), \gamma(\xi-\delta)) \\
&\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| (M_{\mathcal{H}_1} d_{\infty}(\eta(\vartheta), \gamma(\vartheta)) + M_{\mathcal{H}_2} d_{\infty}(\eta(\vartheta-\delta), \gamma(\vartheta-\delta))) d\vartheta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&H_1((\Lambda\eta), (\Lambda\gamma)) \\
&\quad = \sup_{\xi\in[0,\xi_1]} d_{\infty}[\Lambda\eta(\xi), \Lambda\gamma(\xi)] \\
&\quad = \sup_{\xi\in[0,\xi_1]} \Big(\frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} d_{\infty}(\eta(\xi), \gamma(\xi)) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} d_{\infty}(\eta(\xi-\delta), \gamma(\xi-\delta)) \\
&\qquad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| (M_{\mathcal{H}_1} d_{\infty}(\eta(\vartheta), \gamma(\vartheta)) \\
&\qquad\qquad + M_{\mathcal{H}_2} d_{\infty}(\eta(\vartheta-\delta), \gamma(\vartheta-\delta))) d\vartheta \Big) \\
&\quad \leq \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} \sup_{\xi\in[0,\xi_1]} d_{\infty}(\eta(\xi), \gamma(\xi)) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} \sup_{\xi\in[0,\xi_1]} d_{\infty}(\eta(\xi-\delta), \gamma(\xi-\delta)) \\
&\qquad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| (M_{\mathcal{H}_1} \sup_{\xi\in[0,\xi_1]} d_{\infty}(\eta(\vartheta), \gamma(\vartheta)) \\
&\qquad\qquad + M_{\mathcal{H}_2} \sup_{\xi\in[0,\xi_1]} d_{\infty}(\eta(\vartheta-\delta), \gamma(\vartheta-\delta))) d\vartheta \\
&\quad \leq \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} H_1(\eta,\gamma) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} H_1(\eta,\gamma) \\
&\qquad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) H_1(\eta,\gamma) \int_0^{\xi} |(\xi-\vartheta)^{\varsigma-1}| d\vartheta \\
&\quad \leq L_1 H_1(\eta,\gamma).
\end{aligned}
$$

Case 3: For $\xi \in (\xi_j, k_j],\ j = 1, 2, \cdots, n$ and $\eta, \gamma \in PC(J; \mathbb{K}^n)$,

$$
(\Lambda\eta)(\xi) = h_j(\xi, \eta(\xi_j^-)),
$$

$$(\Lambda\gamma)(\xi) = h_j(\xi, \gamma(\xi_j^-)).$$

Therefore,

$$\begin{aligned} d_{\mathcal{H}}([\Lambda\eta(\xi)]^\iota, [\Lambda\gamma(\xi)]^\iota) &= d_{\mathcal{H}}([h_j(\xi, \eta(\xi_j^-))]^\iota, [h_j(\xi, \gamma(\xi_j^-)]^\iota) \\ &\leq M_{h_j} d_{\mathcal{H}}([\eta(\xi_j^-)]^\iota, [\gamma(\xi_j^-)]^\iota). \end{aligned}$$

Thus,

$$\begin{aligned} d_\infty[\Lambda\eta(\xi), \Lambda\gamma(\xi)] &= \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\Lambda\eta(\xi)]^\iota, [\Lambda\gamma(\xi)]^\iota) \\ &\leq M_{h_j} d_\infty(\eta(\xi_j^-), \gamma(\xi_j^-)). \end{aligned}$$

Hence,

$$\begin{aligned} H_1((\Lambda\eta), (\Lambda\gamma)) &= \sup_{\xi\in[\xi_j,k_j]} d_\infty[\Lambda\eta(\xi), \Lambda\gamma(\xi)] \\ H_1((\Lambda\eta), (\Lambda\gamma)) &\leq M_{h_j} H_1(\eta, \gamma). \end{aligned}$$

Case 4: For $\xi \in (k_j, \xi_{j+1}]$, $j = 1, 2, \cdots, n$ and $\eta, \gamma \in PC(J; \mathbb{K}^n)$,

$$\begin{aligned} (\Lambda\eta)(\xi) &= h_j(k_j, \eta(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \eta(\xi), \eta(\xi-\delta)) \\ &\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \eta(\vartheta), \eta(\vartheta-\delta))d\vartheta, \\ (\Lambda\gamma)(\xi) &= h_j(k_j, \gamma(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \gamma(\xi), \gamma(\xi-\delta)) \\ &\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \gamma(\vartheta), \gamma(\vartheta-\delta))d\vartheta. \end{aligned}$$

Therefore,

$$\begin{aligned} & d_{\mathcal{H}}([\Lambda\eta(\xi)]^\iota, [\Lambda\gamma(\xi)]^\iota) \\ &= d_{\mathcal{H}}([h_j(k_j, \eta(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \eta(\xi), \eta(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \eta(\vartheta), \eta(\vartheta-\delta))d\vartheta]^\iota, \\ &\qquad [h_j(k_j, \gamma(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \gamma(\xi), \gamma(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \gamma(\vartheta), \gamma(\vartheta-\delta))d\vartheta]^\iota) \\ &= d_{\mathcal{H}}\Big([h_j(k_j, \eta(\xi_j^-))]^\iota + \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \eta(\xi), \eta(\xi-\delta))\Big]^\iota \\ &\qquad + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \eta(\vartheta), \eta(\vartheta-\delta))d\vartheta\Big]^\iota,\ [h_j(k_j, \gamma(\xi_j^-))]^\iota \\ &\qquad + \Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \gamma(\xi), \gamma(\xi-\delta))\Big]^\iota + \Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \gamma(\vartheta), \gamma(\vartheta-\delta))d\vartheta\Big]^\iota\Big) \\ &\leq d_{\mathcal{H}}([h_j(k_j, \eta(\xi_j^-))]^\iota, [h_j(k_j, \gamma(\xi_j^-))]^\iota) \end{aligned}$$

$$
\begin{aligned}
&+\ d_{\mathcal{H}}\Big(\Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta))\Big]^{\iota},\Big[\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta))\Big]^{\iota}\Big)\\
&+\ d_{\mathcal{H}}\Big(\Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))d\vartheta\Big]^{\iota},\\
&\qquad\Big[\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))d\vartheta\Big]^{\iota}\Big)\\
\le\ & d_{\mathcal{H}}([h_j(k_j,\eta(\xi_j^-))]^{\iota},[h_j(k_j,\gamma(\xi_j^-))]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}d_{\mathcal{H}}([\mathcal{H}(\xi,\eta(\xi),\eta(\xi-\delta))]^{\iota},[\mathcal{H}(\xi,\gamma(\xi),\gamma(\xi-\delta))]^{\iota})\\
&+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|d_{\mathcal{H}}([\mathcal{H}(\vartheta,\eta(\vartheta),\eta(\vartheta-\delta))]^{\iota},[\mathcal{H}(\vartheta,\gamma(\vartheta),\gamma(\vartheta-\delta))]^{\iota})d\vartheta\\
\le\ & M_{h_j}d_{\mathcal{H}}([\eta(\xi_j^-)]^{\iota},[\gamma(\xi_j^-)]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_1}d_{\mathcal{H}}([\eta(\xi)]^{\iota},[\gamma(\xi)]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_2}d_{\mathcal{H}}([\eta(\xi-\delta)]^{\iota},[\gamma(\xi-\delta)]^{\iota})\\
&+\ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|(M_{\mathcal{H}_1}d_{\mathcal{H}}([\eta(\vartheta)]^{\iota},[\gamma(\vartheta)]^{\iota})+M_{\mathcal{H}_2}d_{\mathcal{H}}([\eta(\vartheta-\delta)]^{\iota},[\gamma(\vartheta-\delta)]^{\iota}))d\vartheta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&d_{\infty}[\Lambda\eta(\xi),\Lambda\gamma(\xi)]\\
&\quad=\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\Lambda\eta(\xi)]^{\iota},[\Lambda\gamma(\xi)]^{\iota})\\
&\quad\le M_{h_j}\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\eta(\xi_j^-)]^{\iota},[\gamma(\xi_j^-)]^{\iota})\\
&\qquad+\ \frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_1}\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\eta(\xi)]^{\iota},[\gamma(\xi)]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_2}\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\eta(\xi-\delta)]^{\iota},[\gamma(\xi-\delta)]^{\iota})\\
&\qquad+\ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|\\
&\qquad\qquad(M_{\mathcal{H}_1}\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\eta(\vartheta)]^{\iota},[\gamma(\vartheta)]^{\iota})+M_{\mathcal{H}_2}\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\eta(\vartheta-\delta)]^{\iota},[\gamma(\vartheta-\delta)]^{\iota}))d\vartheta\\
&\quad\le M_{h_j}d_{\infty}(\eta(\xi_j^-),\gamma(\xi_j^-))+\ \frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_1}d_{\infty}(\eta(\xi),\gamma(\xi))+\frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_2}d_{\infty}(\eta(\xi-\delta),\gamma(\xi-\delta))\\
&\qquad+\ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|(M_{\mathcal{H}_1}d_{\infty}(\eta(\vartheta),\gamma(\vartheta))+M_{\mathcal{H}_2}d_{\infty}(\eta(\vartheta-\delta),\gamma(\vartheta-\delta)))d\vartheta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&H_1((\Lambda\eta),(\Lambda\gamma))\\
&\qquad=\sup_{\xi\in[k_j,\xi_{j+1}]}d_{\infty}[\Lambda\eta(\xi),\Lambda\gamma(\xi)]\\
&\qquad=\sup_{\xi\in[k_j,\xi_{j+1}]}(M_{h_j}d_{\infty}(\eta(\xi_j^-),\gamma(\xi_j^-))+\ \frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_1}d_{\infty}(\eta(\xi),\gamma(\xi))\\
&\qquad\quad+\frac{1-\varsigma}{B(\varsigma)}M_{\mathcal{H}_2}d_{\infty}(\eta(\xi-\delta),\gamma(\xi-\delta))+\ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|(M_{\mathcal{H}_1}d_{\infty}(\eta(\vartheta),\gamma(\vartheta))\\
&\qquad\qquad+M_{\mathcal{H}_2}d_{\infty}(\eta(\vartheta-\delta),\gamma(\vartheta-\delta)))d\vartheta)
\end{aligned}
$$

$$
\begin{aligned}
&= M_{h_j} \sup_{\xi\in[k_j,\xi_{j+1}]} d_\infty(\eta(\xi_j^-),\gamma(\xi_j^-)) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} \sup_{\xi\in[k_j,\xi_{j+1}]} d_\infty(\eta(\xi),\gamma(\xi)) \\
&\quad + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} \sup_{\xi\in[k_j,\xi_{j+1}]} d_\infty(\eta(\xi-\delta),\gamma(\xi-\delta)) \\
&\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_{k_j}^{\xi} |(\xi-\vartheta)^{\varsigma-1}| (M_{\mathcal{H}_1} \sup_{\xi\in[k_j,\xi_{j+1}]} d_\infty(\eta(\vartheta),\gamma(\vartheta)) \\
&\qquad\qquad + M_{\mathcal{H}_2} \sup_{\xi\in[k_j,\xi_{j+1}]} d_\infty(\eta(\vartheta-\delta),\gamma(\vartheta-\delta)))d\vartheta \\
&\le M_{h_j} H_1(\eta,\gamma) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_1} H_1(\eta,\gamma) + \frac{1-\varsigma}{B(\varsigma)} M_{\mathcal{H}_2} H_1(\eta,\gamma) \\
&\quad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) H_1(\eta,\gamma) \int_{k_j}^{\xi} |(\xi-\vartheta)^{\varsigma-1}| d\vartheta \\
&\le M_{h_j} H_1(\eta,\gamma) + \frac{1-\varsigma}{B(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) H_1(\eta,\gamma) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) H_1(\eta,\gamma) T^{\varsigma} \\
&\le \left(M_{h_j} + \frac{1-\varsigma}{B(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} (M_{\mathcal{H}_1} + M_{\mathcal{H}_2}) T^{\varsigma} \right) H_1(\eta,\gamma) \\
&\le L_{2_j} H_1(\eta,\gamma).
\end{aligned}
$$

From the above four cases, we conclude that

$$
H_1((\Lambda\eta),(\Lambda\gamma)) = \sup_{\xi\in J} d_\infty[\Lambda\eta(\xi),\Lambda\gamma(\xi)] \le L H_1(\eta,\gamma). \tag{3.1}
$$

Thus, for sufficiently small T, Λ is a strict contraction mapping and hence by Banach fixed point theorem Λ has a unique fixed point which is the solution of system (1.1). The Theorem 3.1 is existence of local solution because our mapping $\Lambda : PC(J : \mathbb{K}^n) \to PC(J : \mathbb{K}^n)$ is not strict contraction for all values of T. In Eq (3.1), we can see that a constant $L < 1$ if T is sufficiently small. Thus, we can say that our solution exists locally.

4. Existence of global solution

To show the existence of global solution, we need the Gronwall's inequality:

Lemma 4.1. *[37] (**Gronwall's inequality**) Let $\mathfrak{F}(\xi,k) \ge 0$ be a continuous function on $0 \le k < \xi \le T$. If, there are positive constant a, b, ς such that*

$$
\mathfrak{F}(\xi,k) \le a + b \int_k^{\xi} (\xi-\kappa)^{\varsigma-1} \mathfrak{F}(\kappa,k) d\kappa, \text{ for } 0 \le k < \xi \le T,
$$

then there is a constant C such that $\mathfrak{F}(\xi,k) \le C$ for $0 \le k < \xi \le T$.

For the convenience, we set the following notations
$C_1 = \max_{1\le j\le n}\{a_1, c_{1_j}, a_{2_j}\}$, $C_2 = \max_{1\le j\le n}\{b_1, b_{2_j}\}$,
$C_3 = \max_{-\delta\le\xi\le T} \left\{ C_1 \exp\left(C_2 \int_{-\delta}^{T} |(\xi-\vartheta)^{\varsigma-1}| d\vartheta \right) \right\}$,

$a_1 = \left(\frac{1}{L_3}A_1 + \frac{1}{L_3}\frac{1-\varsigma}{B(\varsigma)}K(T) + \frac{1}{L_3}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)T^{\varsigma}\right)$, $L_3 = \left(1 - \frac{1-\varsigma}{B(\varsigma)}2K(T)\right)$, $A_1 = d_\infty(\Psi(0), 0)$,
$c_{1_j} = \frac{1}{1-M_{h_j}}$, $a_{2_j} = \left(\frac{1}{L_{4_j}}\frac{1-\varsigma}{B(\varsigma)}K(T) + \frac{1}{L_{4_j}}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)(T)^{\varsigma}\right)$, $b_1 = \left(\frac{1}{L_3}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\right)$,
$L_{4_j} = \left(1 - M_{h_j} - \frac{1-\varsigma}{B(\varsigma)}2K(T)\right)$, $b_{2_j} = \left(\frac{1}{L_{4_j}}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\right)$, $j = 1, 2, \cdots, n$.

Theorem 4.1. *Let the function $\mathcal{H} : (k_j, \xi_{j+1}] \times \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$ satisfies the assumptions $(B1)$ and $(B2)$ and there exists a real valued function $K(\xi)$ which is continuous and non decreasing such that*

$$d_{\mathcal{H}}([\mathcal{H}(\xi, \eta_1, \eta_2)]^\iota, [0]^\iota) \leq K(\xi)(1 + d_{\mathcal{H}}([\eta_1]^\iota, [0]^\iota) + d_{\mathcal{H}}([\eta_2]^\iota, [0]^\iota)),\ \forall\ \eta_1,\ \eta_2 \in \mathbb{K}^n.$$

Then, the Eq (1.1) has a unique solution $\mathcal{P}$ which exists for all $\xi \in [-\delta, T]$.

Proof. By Theorem 3.1, we can continue the solution of system (1.1) as long as $\|\mathcal{P}\|$ stays bounded. Therefore, we need to show that if $\mathcal{P}$ exists on $[-\delta, T)$, then it is bounded as $\xi \uparrow T$. Also, the solution of system (1.1) is given by

$$\mathcal{P}(\xi) = \begin{cases} \Psi(\xi), \quad \xi \in [-\delta, 0), \\ \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta-\delta))d\vartheta, \\ \qquad\qquad\qquad\qquad \forall\ \xi \in [0, \xi_1], \\ h_j(\xi, \mathcal{P}(\xi_j^-)),\ \xi \in (\xi_j, k_j], \qquad j = 1, 2, \cdots, n, \\ h_j(k_j, \mathcal{P}(\xi_j^-)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta-\delta))d\vartheta \\ \qquad + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta)),\ \forall\ \xi \in (k_j, \xi_{j+1}],\ j = 1, 2, \cdots, n. \end{cases}$$

The proof of this theorem is divided into following four cases:
Case 1: For $\xi \in [-\delta, 0)$, we have $\mathcal{P}(\xi) = \Psi(\xi)$.
In this case, we get

$$H_1(\mathcal{P}, 0) \leq 0.$$

Case 2: For $\xi \in [0, \xi_1]$,

$$\mathcal{P}(\xi) = \Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta-\delta))d\vartheta.$$

Now, we have

$$\begin{aligned} &d_{\mathcal{H}}([\mathcal{P}(\xi)]^\iota, [0]^\iota) \\ &= d_{\mathcal{H}}([\Psi(0) + \frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta)) + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta-\delta))d\vartheta]^\iota, [0]^\iota) \\ &= d_{\mathcal{H}}([\Psi(0)]^\iota + [\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta))]^\iota \\ &\qquad + [\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^\xi (\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta-\delta))d\vartheta]^\iota, [0]^\iota) \\ &= d_{\mathcal{H}}([\Psi(0)]^\iota, [0]^\iota) + d_{\mathcal{H}}([\frac{1-\varsigma}{B(\varsigma)}\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi-\delta))]^\iota, [0]^\iota) \end{aligned}$$

$$
\begin{aligned}
&+ d_{\mathcal{H}}([\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}(\xi-\vartheta)^{\varsigma-1}\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))d\vartheta]^{\iota},[0]^{\iota})\\
&\leq d_{\mathcal{H}}([\Psi(0)]^{\iota},[0]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}d_{\mathcal{H}}([\mathcal{H}(\xi,\mathcal{P}(\xi),\mathcal{P}(\xi-\delta))]^{\iota},[0]^{\iota})\\
&+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|d_{\mathcal{H}}([\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))]^{\iota},[0]^{\iota})d\vartheta\\
&\leq d_{\mathcal{H}}([\Psi(0)]^{\iota},[0]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+d_{\mathcal{H}}([\mathcal{P}(\xi)]^{\iota},[0]^{\iota})+d_{\mathcal{H}}([\mathcal{P}(\xi-\delta)]^{\iota},[0]^{\iota}))\\
&+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+d_{\mathcal{H}}([\mathcal{P}(\vartheta)]^{\iota},[0]^{\iota})+d_{\mathcal{H}}([\mathcal{P}(\vartheta-\delta)]^{\iota},[0]^{\iota}))d\vartheta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d_{\infty}[\mathcal{P}(\xi),0] &= \sup_{\iota\in(0,1]} d_{\mathcal{H}}([\mathcal{P}(\xi)]^{\iota},[0]^{\iota})\\
&\leq A_1+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+d_{\infty}(\mathcal{P}(\xi),0)+d_{\infty}(\mathcal{P}(\xi-\delta),0))\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+d_{\infty}(\mathcal{P}(\vartheta),0)+d_{\infty}(\mathcal{P}(\vartheta-\delta),0))d\vartheta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
H_1(\mathcal{P},0) &= \sup_{\xi\in[0,\xi_1]} d_{\infty}(\mathcal{P}(\xi),0)\\
&\leq A_1+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+2H_1(\mathcal{P},0))\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+2H_1(\mathcal{P},0))d\vartheta\\
&\leq A_1+\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{1-\varsigma}{B(\varsigma)}2K(\xi)H_1(\mathcal{P},0)\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|d\vartheta\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta\\
\left(1-\frac{1-\varsigma}{B(\varsigma)}2K(\xi)\right)H_1(\mathcal{P},0) &\leq A_1+\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)T^{\varsigma}\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta\\
H_1(\mathcal{P},0) &\leq \frac{1}{L_3}A_1+\frac{1}{L_3}\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{1}{L_3}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)T^{\varsigma}\\
&\quad+\frac{1}{L_3}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta\\
&\leq a_1+b_1\int_0^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta.
\end{aligned}
$$

Case 3: Similarly, for $\xi \in (\xi_j, k_j],\ j = 1, 2, \cdots, n$ and $\mathcal{P}(\xi) = h_j(\xi, \mathcal{P}(\xi_j^-))$,

$$\begin{aligned} d_{\mathcal{H}}([\mathcal{P}(\xi)]^\iota, [0]^\iota) &= d_{\mathcal{H}}([h_j(\xi, \mathcal{P}(\xi_j^-))]^\iota, [0]^\iota) \\ &\leq M_{h_j} d_{\mathcal{H}}([\mathcal{P}(\xi_j^-)]^\iota, [0]^\iota). \end{aligned}$$

Therefore,

$$\begin{aligned} d_\infty[\mathcal{P}(\xi), 0] &= \sup_{\iota \in (0,1]} d_{\mathcal{H}}([\mathcal{P}(\xi)]^\iota, [0]^\iota) \\ &\leq M_{h_j} d_\infty(\mathcal{P}(\xi_j^-), 0). \end{aligned}$$

Thus,

$$\begin{aligned} H_1(\mathcal{P}, 0) &= \sup_{[\xi_j, k_j]} d_\infty[\mathcal{P}(\xi), 0] \\ H_1(\mathcal{P}, 0) &\leq M_{h_j} H_1(\mathcal{P}, 0) \\ H_1(\mathcal{P}, 0) &\leq \frac{1}{1 - M_{h_j}}, \\ H_1(\mathcal{P}, 0) &\leq c_{1_j}. \end{aligned}$$

Case 4: For $\xi \in (k_j, \xi_{j+1}],\ j = 1, 2, \cdots, n$, we have

$$\begin{aligned} \mathcal{P}(\xi) = h_j(k_j, \mathcal{P}(\xi_j^-)) &+ \frac{1 - \varsigma}{B(\varsigma)} \mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta)) \\ &+ \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_{k_j}^{\xi} (\xi - \vartheta)^{\varsigma - 1} \mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta - \delta)) d\vartheta. \end{aligned}$$

Now, we have

$$\begin{aligned} & d_{\mathcal{H}}([\mathcal{P}(\xi)]^\iota, [0]^\iota) \\ &= d_{\mathcal{H}}([h_j(k_j, \mathcal{P}(\xi_j^-)) + \frac{1 - \varsigma}{B(\varsigma)} \mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta)) \\ &\qquad + \frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_{k_j}^{\xi} (\xi - \vartheta)^{\varsigma - 1} \mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta - \delta)) d\vartheta]^\iota, [0]^\iota) \\ &= d_{\mathcal{H}}([h_j(k_j, \mathcal{P}(\xi_j^-))]^\iota + [\frac{1 - \varsigma}{B(\varsigma)} \mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta))]^\iota \\ &\qquad + [\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_{k_j}^{\xi} (\xi - \vartheta)^{\varsigma - 1} \mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta - \delta)) d\vartheta]^\iota, [0]^\iota) \\ &\leq d_{\mathcal{H}}([h_j(k_j, \mathcal{P}(\xi_j^-))]^\iota, [0]^\iota) + d_{\mathcal{H}}([\frac{1 - \varsigma}{B(\varsigma)} \mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta))]^\iota, [0]^\iota) \\ &\qquad + d_{\mathcal{H}}([\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)} \int_{k_j}^{\xi} (\xi - \vartheta)^{\varsigma - 1} \mathcal{H}(\vartheta, \mathcal{P}(\vartheta), \mathcal{P}(\vartheta - \delta)) d\vartheta]^\iota, [0]^\iota) \\ &\leq d_{\mathcal{H}}([h_j(k_j, \mathcal{P}(\xi_j^-))]^\iota, [0]^\iota) + \frac{1 - \varsigma}{B(\varsigma)} d_{\mathcal{H}}([\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta))]^\iota, [0]^\iota) \end{aligned}$$

$$+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|d_{\mathcal{H}}([\mathcal{H}(\vartheta,\mathcal{P}(\vartheta),\mathcal{P}(\vartheta-\delta))]^{\iota},[0]^{\iota})d\vartheta$$

$$\leq M_{h_j}d_{\mathcal{H}}([\mathcal{P}(\xi_j^-)]^{\iota},[0]^{\iota})+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+d_{\mathcal{H}}([\mathcal{P}(\xi)]^{\iota},[0]^{\iota})+d_{\mathcal{H}}([\mathcal{P}(\xi-\delta)]^{\iota},[0]^{\iota}))$$

$$+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+d_{\mathcal{H}}([\mathcal{P}(\vartheta)]^{\iota},[0]^{\iota})+d_{\mathcal{H}}([\mathcal{P}(\vartheta-\delta)]^{\iota},[0]^{\iota}))d\vartheta.$$

Therefore,

$$\begin{aligned}
d_{\infty}[\mathcal{P}(\xi),0]&=\sup_{\iota\in(0,1]}d_{\mathcal{H}}([\mathcal{P}(\xi)]^{\iota},[0]^{\iota})\\
&\leq M_{h_j}d_{\infty}(\mathcal{P}(\xi_j^-),0)+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+d_{\infty}(\mathcal{P}(\xi),0)+d_{\infty}(\mathcal{P}(\xi-\delta),0))\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+d_{\infty}(\mathcal{P}(\vartheta),0)+d_{\infty}(\mathcal{P}(\vartheta-\delta),0))d\vartheta.
\end{aligned}$$

Thus,

$$\begin{aligned}
H_1(\mathcal{P},0)&=\sup_{\xi\in[k_j,\xi_{j+1}]}d_{\infty}[\mathcal{P}(\xi),0]\\
&\leq M_{h_j}H_1(\mathcal{P},0)+\frac{1-\varsigma}{B(\varsigma)}K(T)(1+2H_1(\mathcal{P},0))\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|K(T)(1+2H_1(\mathcal{P},0))d\vartheta\\
&\leq M_{h_j}H_1(\mathcal{P},0)+\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{1-\varsigma}{B(\varsigma)}2K(\xi)H_1(\mathcal{P},0))\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|d\vartheta\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0))d\vartheta\\
L_{4_j}H_1(\mathcal{P},0)&\leq\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)(T)^{\varsigma}\\
&\quad+\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0))d\vartheta\\
H_1(\mathcal{P},0)&\leq\frac{1}{L_{4_j}}\frac{1-\varsigma}{B(\varsigma)}K(T)+\frac{1}{L_{4_j}}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}K(T)(T)^{\varsigma}\\
&\quad+\frac{1}{L_{4_j}}\frac{\varsigma}{B(\varsigma)\Gamma(\varsigma)}2K(T)\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0))d\vartheta\\
&\leq a_{2_j}+b_{2_j}\int_{k_j}^{\xi}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta.
\end{aligned}$$

From the above four cases, we conclude that, for $\xi\in[-\delta,T]$

$$H_1(\mathcal{P},0)\leq C_1+C_2\int_{-\delta}^{T}|(\xi-\vartheta)^{\varsigma-1}|H_1(\mathcal{P},0)d\vartheta.$$

So that,

$$H_1(\mathcal{P},0) \leq C_1 \exp\left\{C_2 \int_{-\delta}^{T} |(\xi-\vartheta)^{\varsigma-1}| d\vartheta\right\}$$
$$\leq C_3.$$

Thus, $H_1(\mathcal{P},0) = \|\mathcal{P}\| \leq C_3$. Hence, from Lemma 4.1, $\mathcal{P}$ is bounded. Therefore, we can extend our solution to the whole interval $[-\delta, T]$. Thus, our solution is global. For more details, please see [37]). □

Remark 5. *By using the above argument, we can prove that the system (1.1) has atleast one solution under the following weak assumptions*
(A1) *Function $\mathcal{H} : (k_j, \xi_{j+1}] \times \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n,\ j = 0, 1, \cdots, n$, is continuous and there exists a positive constant $M_H > 0$ such that*

$$|\mathcal{H}(\xi, \eta_1, \eta_2)| \leq M_H(1 + |\eta_1| + |\eta_2|), \quad \forall\, \xi \in (k_j, \xi_{j+1}],\ \eta_1,\ \eta_2 \in \mathbb{K}^n.$$

(A2) *Functions $h_j : (\xi_j, k_j] \times \mathbb{K}^n \to \mathbb{K}^n,\ j = 1, 2, \cdots, n$ are continuous and there exists a positive constants $M_j > 0$ such that*

$$|h_j(\xi, \eta_1)| \leq M_j(1 + |\eta_1|),\ \forall\, \xi \in (\xi_j, k_j],\ \eta_1 \in \mathbb{K}^n.$$

6. Example

We consider the following retarded fractional differential system with non-instantaneous impulsive condition

$$\begin{aligned}
{}^{ABC}{}_{0}D_{\xi}^{\frac{1}{2}}\mathcal{P}(\xi) &= \bar{2}\xi\mathcal{P}^2(\xi) + \bar{2}\xi^2\mathcal{P}^2(\xi - \frac{1}{2}), && \xi \in (0,1] \cup (1.5, 2],\\
\mathcal{P}(\xi) &= \frac{\sin(j\xi)}{e^{j\xi}}\mathcal{P}(\xi_j^-), && \xi \in (1, 1.5], j = 1, \qquad (6.1)\\
\mathcal{P}(\xi) &= \Psi(\xi) = \xi + 1, && \xi \in [-\frac{1}{2}, 0].
\end{aligned}$$

Here, we have $\varsigma = \frac{1}{2},\ \xi \in [-\frac{1}{2}, 2],\ 0 = k_0 < \xi_1 = 1 < k_1 = 1.5 < \xi_2 = 2 = T,\ \mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta)) = \bar{2}\xi\mathcal{P}^2(\xi) + \bar{2}\xi^2\mathcal{P}^2(\xi - \frac{1}{2})$ and impulsive function $h_j(\xi, \mathcal{P}(\xi_j^-)) = \frac{\sin(j\xi)}{e^{j\xi}}\mathcal{P}(\xi_j^-),\ j = 1$.
The ι-level of fuzzy number $\bar{2}$ is $[2]^\iota = [\iota+1, 3-\iota],\ \forall\, \iota \in [0,1]$. Then, the ι-level set of $\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta))$ is

$$\begin{aligned}
[\mathcal{H}(\xi, \mathcal{P}(\xi), \mathcal{P}(\xi - \delta))]^\iota &= [\bar{2}\xi\mathcal{P}^2(\xi) + \bar{2}\mathcal{P}^2(\xi - \frac{1}{2})]^\iota\\
&= \xi[(\iota+1)(\mathcal{P}_q^\iota(\xi))^2, (3-\iota)(\mathcal{P}_r^\iota(\xi))^2] + \xi^2[(\iota+1)(\mathcal{P}_q^\iota(\xi - \frac{1}{2}))^2, (3-\iota)(\mathcal{P}_r^\iota(\xi - \frac{1}{2}))^2].
\end{aligned}$$

Now, we have

$$d_{\mathcal{H}}([\mathcal{H}(\xi, \eta(\xi), \eta(\xi - \delta))]^\iota, [\mathcal{H}(\xi, \gamma(\xi), \gamma(\xi - \delta))]^\iota)$$

$$
\begin{aligned}
&= d_{\mathcal{H}}\{((\iota+1)\xi(\eta_q^\iota(\xi))^2, (3-\iota)\xi(\eta_r^\iota(\xi))^2 + \xi^2(\iota+1)(\eta_q^\iota(\xi-\frac{1}{2}))^2, \xi^2(3-\iota)(\eta_r^\iota(\xi-\frac{1}{2}))^2),\\
&\quad ((\iota+1)\xi(\gamma_q^\iota(\xi))^2, (3-\iota)\xi(\gamma_r^\iota(\xi))^2 + \xi^2(\iota+1)(\gamma_q^\iota)(\xi-\frac{1}{2}))^2, \xi^2(3-\iota)(\gamma_r^\iota(\xi-\frac{1}{2}))^2)\}\\
&= d_{\mathcal{H}}(\xi[(\iota+1)(\eta_q^\iota(\xi))^2, (3-\iota)(\eta_r^\iota(\xi))^2], \xi[(\iota+1)(\gamma_q^\iota(\xi))^2, (3-\iota)(\gamma_r^\iota(\xi))^2])\\
&\quad + d_{\mathcal{H}}([\xi^2(\iota+1)(\eta_q^\iota(\xi-\frac{1}{2}))^2, \xi^2(3-\iota)(\eta_r^\iota(\xi-\frac{1}{2}))^2],\\
&\qquad [\xi^2(\iota+1)(\gamma_q^\iota(\xi-\frac{1}{2}))^2, \xi^2(3-\iota)(\gamma_r^\iota(\xi-\frac{1}{2}))^2])\\
&\le \max\{|(\iota+1)\xi(\eta_q^\iota(\xi))^2 - (\iota+1)\xi(\gamma_q^\iota(\xi))^2|, |(3-\iota)\xi(\eta_r^\iota(\xi))^2 - (3-\iota)\xi(\gamma_r^\iota(\xi))^2|\}\\
&\quad + \max\{|(\iota+1)\xi^2(\eta_q^\iota(\xi-\frac{1}{2}))^2 - \xi^2(\iota+1)(\gamma_q^\iota(\xi-\frac{1}{2}))^2|,\\
&\qquad |(3-\iota)\xi^2(\eta_r^\iota(\xi-\frac{1}{2}))^2 - \xi^2(3-\iota)(\gamma_r^\iota(\xi-\frac{1}{2}))^2|\}\\
&\le \max\{(\iota+1)\xi|(\eta_q^\iota(\xi))^2 - (\gamma_q^\iota(\xi))^2|, (3-\iota)\xi|(\eta_r^\iota(\xi))^2 - (\gamma_r^\iota(\xi))^2|\}\\
&\quad + \max\{(\iota+1)\xi^2|(\eta_q^\iota(\xi-\frac{1}{2}))^2 - (\gamma_q^\iota(\xi-\frac{1}{2}))^2|, (3-\iota)\xi^2|(\eta_r^\iota(\xi-\frac{1}{2}))^2 - (\gamma_r^\iota(\xi-\frac{1}{2}))^2|\}\\
&\le T(3-\iota)\max\{|\eta_q^\iota(\xi)-\gamma_q^\iota(\xi)||\eta_q^\iota(\xi)+\gamma_q^\iota(\xi)|, |\eta_r^\iota(\xi)-\gamma_r^\iota(\xi)||\eta_r^\iota(\xi)+\gamma_r^\iota(\xi)|\}\\
&\quad + T^2(3-\iota)\max\{|\eta_q^\iota(\xi-\frac{1}{2})-\gamma_q^\iota(\xi-\frac{1}{2})||\eta_q^\iota(\xi-\frac{1}{2})+\gamma_q^\iota(\xi-\frac{1}{2})|,\\
&\qquad |\eta_r^\iota(\xi-\frac{1}{2})-\gamma_r^\iota(\xi-\frac{1}{2})||\eta_r^\iota(\xi-\frac{1}{2})+\gamma_r^\iota(\xi-\frac{1}{2})|\}\\
&\le (3-\iota)T\max_{\frac{-1}{2}\le\xi\le 2}\{|\eta_q^\iota(\xi)+\gamma_q^\iota(\xi)|\}\, d_{\mathcal{H}}([\eta(\xi)]^\iota, [\gamma(\xi)]^\iota)\\
&\quad + T^2(3-\iota)\max_{\frac{-1}{2}\le\xi\le 2}\{|\eta_r^\iota(\xi-\frac{1}{2})+\gamma_r^\iota(\xi-\frac{1}{2})|\} d_{\mathcal{H}}([\eta(\xi-\frac{1}{2})]^\iota, [\gamma(\xi-\frac{1}{2})]^\iota)\\
&\le c_1\, d_{\mathcal{H}}([\eta(\xi)]^\iota, [\gamma(\xi)]^\iota) + c_2 d_{\mathcal{H}}([\eta(\xi-\frac{1}{2})]^\iota, [\gamma(\xi-\frac{1}{2})]^\iota),
\end{aligned}
$$

where $c_1 = (3-\iota)T\max_{\frac{-1}{2}\le\xi\le 2}\{|\eta_q^\iota(\xi)+\gamma_q^\iota(\xi)|\}$, $c_2 = (3-\iota)T^2\max_{\frac{-1}{2}\le\xi\le 2}\{|\eta_r^\iota(\xi-\frac{1}{2})+\gamma_r^\iota(\xi-\frac{1}{2})|\}$ satisfies the condition $(B1)$.
Now, the ι-level set of fuzzy number $\bar{1}$ is $[\bar{1}]^\iota = [\iota, 2-\iota]$, $\forall\ \iota \in [0,1]$ and ι-level set of impulsive function $h_j(\xi, \mathcal{P}(\xi_j^-))$ is

$$
\begin{aligned}
[h_j(\xi, \mathcal{P}(\xi_j^-))]^\iota &= [\frac{\sin(j\xi)}{e^{j\xi}}\mathcal{P}(\xi_j^-)]^\iota\\
&= \frac{\sin(j\xi)}{e^{j\xi}}[(\iota, 2-\iota)[\mathcal{P}(\xi_j^-)]^\iota]\\
&= \frac{\sin(j\xi)}{e^{j\xi}}[\iota\mathcal{P}_q^\iota(\xi_j^-), (2-\iota)\mathcal{P}_r^\iota(\xi_j^-)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&d_{\mathcal{H}}([h_j(\xi, \eta(\xi_j^-))]^\iota, [h_j(\xi, \gamma(\xi_j^-))]^\iota)\\
&\qquad = d_{\mathcal{H}}(\frac{\sin(j\xi)}{e^{j\xi}}[\iota\eta_q^\iota(\xi_j^-), (2-\iota)\eta_r^\iota(\xi_j^-)], \frac{\sin(j\xi)}{e^{j\xi}}[\iota\gamma_q^\iota(\xi_j^-), (2-\iota)\gamma_r^\iota(\xi_j^-)])
\end{aligned}
$$

$$
\begin{aligned}
&= d_{\mathcal{H}}(\frac{\sin(j\xi)}{e^{j\xi}}[\iota\eta_q^{\iota}(\xi_j^-),(2-\iota)\eta_r^{\iota}(\xi_j^-)], \frac{\sin(j\xi)}{e^{j\xi}}[\iota\gamma_q^{\iota}(\xi_j^-),(2-\iota)\gamma_r^{\iota}(\xi_j^-)])\\
&\le \max\{\iota\frac{\sin(j\xi)}{e^{j\xi}}|\eta_q^{\iota}(\xi_j^-)-\gamma_q^{\iota}(\xi_j^-)|,(2-\iota)\frac{\sin(j\xi)}{e^{j\xi}}|\eta_r^{\iota}(\xi_j^-)-\gamma_r^{\iota}(\xi_j^-)|\}\\
&\le (2-\iota)\frac{\sin(jT)}{e^{jT}}\max\{|\eta_q^{\iota}(\xi_j^-)-\gamma_q^{\iota}(\xi_j^-)|,|\eta_r^{\iota}(\xi_j^-)-\gamma_r^{\iota}(\xi_j^-)|\}\\
&\le (2-\iota)\frac{\sin(jT)}{e^{jT}}d_{\mathcal{H}}([\eta(\xi_j^-)]^{\iota},[\gamma(\xi_j^-)]^{\iota})\\
&\le c_3 d_{\mathcal{H}}([\eta(\xi_j^-)]^{\iota},[\gamma(\xi_j^-)]^{\iota}),
\end{aligned}
$$

where $c_3 = (2-\iota)\frac{\sin(jT)}{e^{jT}}$, $j = 1$, satisfies the condition $(B2)$.

Thus, all the conditions of Theorem 3.1 are fulfilled. Hence, system (6.1) has a unique fuzzy solution.

7. Conclusions

In this work, we have considered the fractional order fuzzy delay differential system with non-instantaneous impulses. The main aim of this work is to establish the existence of local and global solutions to the considered system. In Section 3, we have studied the existence of a local solution and in Section 4, we have extended the local solution of Section 3 to a global solution. Fuzzy set theory, Banach fixed point theorem and non-linear function analysis are the major tools to establish these results. In Section 5, an example is given to validate obtained outcomes.

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Conflict of interest

The authors declare no conflict of interest.

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