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Research article

On the fixed space induced by a group action

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Abstract: This article studies connections between group actions and their corresponding vector spaces. Given an action of a group G on a non-empty set X, we examine the space $L(X)$ of scalar-valued functions on X and its fixed subspace: $L^G(X) = \{f \in L(X)\colon f(a \cdot x) = f(x) \text{ for all } a \in G, x \in X\}$. In particular, we show that $L^G(X)$ is an invariant of the action of G on X. In the case when the action is finite, we compute the dimension of $L^G(X)$ in terms of fixed points of X and prove several prominent results for $L^G(X)$, including Bessel's inequality and Frobenius reciprocity.

Keywords: Bessel's inequality; free action; Frobenius reciprocity; function space; group action
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1. Introduction

Let G be a finite group and let H be a subgroup of G. Denote by $C(G)$ and $C(H)$ the spaces of complex-valued class functions on G and on H, respectively. Frobenius reciprocity for class functions on G states that Res^G_H and Ind^G_H are Hermitian adjoint with respect to the Hermitian inner product defined by

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)} \quad \text{and} \quad \langle h, k\rangle_H = \frac{1}{|H|}\sum_{x\in H} h(x)\overline{k(x)}, \tag{1.1}$$

for all $f, g \in C(G)$, $h, k \in C(H)$. In other words, if f is a class function on H and if g is a class function on G, then

$$\langle \mathrm{Ind}^G_H f, g\rangle_H = \langle f, \mathrm{Res}^G_H g\rangle, \tag{1.2}$$

where Res^G_H is a linear transformation from $C(G)$ to $C(H)$ and Ind^G_H is a linear transformation from $C(H)$ to $C(G)$. This result is crucial and plays fundamental roles in proving well-known results in the representation theory of finite groups such as Mackey's irreducibility criterion; see, for instance, [9, Theorem 8.3.6].

The present article stems from the study of left regular representation of a finite gyrogroup in a series of articles [10–12]. It is well known that the conjugation relation in any group G may be viewed as a group action of G on itself by the formula $g \cdot x = gxg^{-1}$ for all $g, x \in G$. This suggests studying Frobenius reciprocity in the setting of group actions. In this article, we generalize Frobenius reciprocity to the family of functions that are invariant under a given group action. We remark that there are other versions of Frobenius reciprocity; see, for instance, [2, Theorem 10.8, p. 233] and [4, Theorem 2.3].

Let $\mathbb{F}$ be a field and let X be a non-empty set with an action of a group G (that is, X is a G-set). Define $L(X) = \{f \colon f \text{ is a function from } X \text{ to } \mathbb{F}\}$. Recall that $L(X)$ is a vector space. Furthermore, G acts linearly on $L(X)$ by the formula

$$(a \star f)(x) = f(a^{-1} \cdot x), \qquad x \in X, \tag{1.3}$$

for all $a \in G, f \in L(X)$, where $\star$ is the induced G-action on $L(X)$ and $\cdot$ is the given G-action on X. Therefore, we can speak of the fixed subspace of $L(X)$:

$$\mathrm{Fix}\,(L(X)) = \{f \in L(X) \colon a \cdot f = f \text{ for all } a \in G\}.$$

It is not difficult to check that $a \cdot f = f$ for all $a \in G$ if and only if $f(a \cdot x) = f(x)$ for all $a \in G$, $x \in X$. Therefore, the fixed subspace of $L(X)$ associated with the action given by (1.3) can be expressed as

$$L^G(X) = \{f \in L(X) \colon f(a \cdot x) = f(x) \text{ for all } a \in G,\ x \in X\}. \tag{1.4}$$

In the case when X is a finite-dimensional vector space, (1.3) induces an action of G on the space $\mathbb{F}[X]$ of polynomial functions on X. The study of this action along with the corresponding fixed subspace is a fundamental topic in invariant theory [3, 5–7]. The following example indicates that several familiar families of functions in the literature may be viewed as $L^G(X)$ with appropriate group actions.

Example 1.1. Let X be a non-empty set and let $\mathbb{F}$ be a field.

(a) If $G = \mathrm{Sym}\,(X)$ and let G acts on X by evaluation, then $L^G(X)$ is the family of constant functions.

(b) If G is a group and let G acts on itself by conjugation, then $L^G(G)$ is the usual family of class functions.

(c) Suppose that A is an abelian group. Fix $t \in A$ and set $G = \langle t \rangle = \{nt : n \in \mathbb{Z}\}$. Then, G acts on A by addition and $L^G(A)$ is the family of periodic functions defined on A with period t.

(d) Let $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Recall that a modular function $f \colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ must satisfy the condition that

$$f\left(\frac{az+b}{cz+d}\right) = f(z), \qquad z \in \mathbb{C}^\infty,$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ [1, p. 34]. Therefore, modular functions are elements in $L^G(\mathbb{C}^\infty)$, where G is the modular group and acts on $\mathbb{C}^\infty$ by the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \dfrac{az+b}{cz+d}$.

(e) Let $(K, \oplus)$ be a gyrogroup [13]. As in Section 4 of [10], the space

$$L^{\mathrm{gyr}}(K) = \{f \in L(K) \colon f(a \oplus \mathrm{gyr}[x, y]z) = f(a \oplus z) \text{ for all } a, x, y, z \in K\}$$

arises as a representation space of K associated with a gyrogroup version of left regular representation. Using the change of variable, $w = a \oplus z$, we obtain that

$$L^{\mathrm{gyr}}(K) = \{f \in L(K) \colon f(a \oplus \mathrm{gyr}[x, y](\ominus a \oplus z)) = f(z) \text{ for all } a, x, y, z \in K\}.$$

Let G be the subgroup of $\mathrm{Sym}\,(K)$ generated by the set

$$\{L_a \circ \mathrm{gyr}[x, y] \circ L_a^{-1} \colon a, x, y \in K\},$$

where L_a is the left gyrotranslation by a defined by $L_a(z) = a \oplus z$ for all $z \in K$ and $\mathrm{gyr}[x, y]$ is the gyroautomorphism generated by x and y. It is clear that G acts on K by evaluation. Furthermore, $L^{\mathrm{gyr}}(K) = L^G(K)$.

In Section 2, we study basic properties of arbitrary group actions related to their corresponding fixed subspaces. In Section 3, we reduce to the case of a finite action (that is, an action of a finite group on a finite set) and compute the dimension of the fixed subspace. This leads to some interesting properties of fixed-point-free actions. Once the usual Hermitian inner product on $L(X)$ is introduced, where X is a finite G-set, an orthogonal decomposition of $L(X)$ is obtained and several interesting results such as Fourier expansion, Bessel's inequality, and Frobenius reciprocity are established for the case of functions invariant under the action of G on X. In Section 4, we conclude our work.

2. Basic properties

Let G be a group and let X be a G-set. Recall that the action of G on X induces an equivalence relation $\sim$ given by

$$x \sim y \quad \text{if and only if} \quad y = a \cdot x \text{ for some } a \in G, \tag{2.1}$$

for all $x, y \in X$. Recall also that the orbit of $x \in X$ under the action of G is given by $\mathrm{orb}\, x = \{a{\cdot}x : a \in G\}$. Hence, the collection $\{\mathrm{orb}\, x \colon x \in X\}$ forms a partition of X. This partition leads to a characterization of elements in $L^G(X)$, as we will see shortly, and eventually to a standard basis for $L^G(X)$ if there are only finitely many orbits of G on X. Suppose that X is a G-set and let $\mathcal{P}$ be the partition of X determined by (2.1). Note that $f \in L^G(X)$ if and only if $f(a \cdot x) = f(x)$ for all $x \in X$, $a \in G$ if and only if f is constant on $\mathrm{orb}\, x$ for each $x \in X$. This leads to a natural question: Can a space of functions on a set endowed with a partition be viewed as $L^G(X)$ for a suitable group action? The answer to this question is affirmative. In fact, by [5, Corollary 1.1.7], if X is a non-empty set and if $\mathcal{P} = \{X_i \colon i \in I\}$ is a partition of X, then the permutation group $S_{\mathcal{P}} = \{\sigma \in \mathrm{Sym}\,(X) \colon \sigma(X_i) = X_i \text{ for all } i \in I\}$ acts on X by evaluation and induces its orbits on X as the cells of the partition.

Let X be a G-set and let $\mathcal{P}$ be the partition of X determined by the equivalence relation (2.1). For each $C \in \mathcal{P}$, the indicator function δ_C is defined by

$$\delta_C(x) = \begin{cases} 1 & \text{if } x \in C; \\ 0 & \text{if } x \in X \setminus C. \end{cases} \tag{2.2}$$

As noted earlier, δ_C belongs to $L^G(X)$ for all $C \in \mathcal{P}$. In fact, we obtain the following theorem.

Theorem 2.1. *Suppose that X is a G-set and let $\mathcal{P}$ be the partition of X determined by the equivalence relation* (2.1)*. Then, $\mathcal{B} = \{\delta_C : C \in \mathcal{P}\}$ is a linearly independent set in $L^G(X)$. Furthermore, $\mathcal{P}$ is finite if and only if $\mathcal{B}$ forms a basis for $L^G(X)$. In particular,* $\dim(L^G(X)) \geq |\mathcal{P}|$ *and equality holds if $\mathcal{P}$ is finite.*

Proof. By definition, $\mathcal{B}$ is linearly independent. Assume that $\mathcal{P}$ is finite, say $\mathcal{P} = \{C_1, C_2, \dots, C_n\}$. Fix $c_i \in C_i$ for all $i = 1, 2, \dots, n$. Then, $f = f(c_1)\delta_{C_1} + f(c_2)\delta_{C_2} + \cdots + f(c_n)\delta_{C_n}$ for all $f \in L^G(X)$ and so $\mathcal{B}$ spans $L^G(X)$. To prove the converse, suppose that $\mathcal{P}$ is infinite. Assume to the contrary that $\mathcal{B}$ forms a basis for $L^G(X)$. Define f by $f(x) = 1$ for all $x \in X$. Then, $f \in L^G(X)$ and so $f = a_1\delta_{C_1} + a_2\delta_{C_2} + \cdots + a_n\delta_{C_n}$ for some $C_1, C_2, \dots, C_n \in \mathcal{P}$. Since $\mathcal{P}$ is infinite, there is an orbit $C \in \mathcal{P} \setminus \{C_1, C_2, \dots, C_n\}$. Choose $c \in C$. Then, $f(c) = 1$, whereas

$$(a_1\delta_{C_1} + a_2\delta_{C_2} + \cdots + a_n\delta_{C_n})(c) = a_1\delta_{C_1}(c) + a_2\delta_{C_2}(c) + \cdots + a_n\delta_{C_n}(c) = 0.$$

Hence, $f \neq a_1\delta_{C_1} + a_2\delta_{C_2} + \cdots + a_n\delta_{C_n}$, a contradiction. This shows that $\mathcal{B}$ is not a basis for $L^G(X)$.

Since $\mathcal{B}$ is linearly independent, it follows that $\dim(L^G(X)) \geq |\mathcal{B}| = |\mathcal{P}|$. Moreover, if $\mathcal{P}$ is finite, then $\mathcal{B}$ is a basis for $L^G(X)$ and so $\dim(L^G(X)) = |\mathcal{P}|$. □

According to Theorem 2.1, $\{\delta_C : C \in \mathcal{P}\}$ does not form a basis for $L^G(X)$ in the case when $\mathcal{P}$ is infinite. It turns out that $\{\delta_C : C \in \mathcal{P}\}$ forms a basis for the following subspace of $L^G(X)$:

$$L^G_{\text{fs}}(X) = \{f \in L^G(X) : f \text{ is non-zero on finitely many orbits in } X\}, \tag{2.3}$$

so that the dimension of $L^G_{\text{fs}}(X)$ equals $|\mathcal{P}|$.

Next, let us state some properties between group actions and their corresponding spaces. Their proofs are straightforward and hence are omitted.

Theorem 2.2. *Let G be a group and let X be a G-set. Then, the following are equivalent:*

(1) $L(X) = L^G(X)$*;*

(2) $|\operatorname{orb} x| = 1$ *for all $x \in X$;*

(3) *G acts trivially on X.*

Theorem 2.3. *Let $\mathbb{F}$ be a field and let X be a G-set. Then, the following are equivalent:*

(1) *The action of G on X is transitive;*

(2) $\dim(L^G(X)) = 1$*;*

(3) $L^G(X) = \{f_\alpha : \alpha \in \mathbb{F}\} = \operatorname{span} f_1$*, where $f_\alpha(x) = \alpha$ for all $x \in X$, $\alpha \in \mathbb{F}$.*

We close this section with the following result, which indicates that $L^G(X)$ is an invariant of the action of G on X. Therefore, in certain circumstances, one can use the notion of $L^G(X)$ to distinguish inequivalent group actions.

Proposition 2.1. *Let X and Y be G-sets. If $\Phi \colon X \to Y$ is an equivalence, then the map τ defined by*

$$\tau(f) = f \circ \Phi^{-1}, \qquad f \in L(X), \tag{2.4}$$

is a linear isomorphism from $L(X)$ to $L(Y)$ that restricts to a linear isomorphism from $L^H(X)$ to $L^H(Y)$ for any subgroup H of G. Consequently, if X and Y are equivalent as G-sets, then $L(X) \cong L(Y)$ and $L^H(X) \cong L^H(Y)$ as vector spaces for any subgroup H of G.

Proof. The proof that τ is a linear isomorphism is straightforward. Let H be a subgroup of G and let $f \in L^H(X)$. We claim that $\tau(f) \in L^H(Y)$. Let $a \in H$ and let $y \in Y$. By surjectivity, there is an element $x \in X$ such that $y = \Phi(x)$. Thus, $\tau(f)(a \cdot y) = \tau(f)(a \cdot \Phi(x)) = \tau(f)(\Phi(a \cdot x)) = f(\Phi^{-1}(\Phi(a \cdot x))) = f(a \cdot x) = f(x) = f(\Phi^{-1}(y)) = (f \circ \Phi^{-1})(y) = \tau(f)(y)$. Hence, $\tau(f) \in L^H(Y)$ and so τ maps $L^H(X)$ to $L^H(Y)$.

Let $g \in L^H(Y)$ and set $f = g \circ \Phi$. Note that f is a map from X to $\mathbb{F}$ and that

$$f(a \cdot x) = g(\Phi(a \cdot x)) = g(a \cdot \Phi(x)) = g(\Phi(x)) = f(x)$$

for all $a \in H$ and $x \in X$. Hence, $f \in L^H(X)$. Furthermore, $\tau(f) = f \circ \Phi^{-1} = (g \circ \Phi) \circ \Phi^{-1} = g$. This proves that τ is surjective. Therefore, the restriction $\tau : L^H(X) \to L^H(Y)$ is a linear isomorphism. □

The converse to Proposition 2.1 is not, in general, true. That is, the condition that "$L^H(X) \cong L^H(Y)$ as vector spaces for some subgroup H of G" does not imply that "$X \cong Y$ as G-sets". In fact, let X be a set having at least two distinct elements, namely that $x, y \in X$ and $x \neq y$. Then, $H = \{\mathrm{id}_G\}$ acts transitively on $\{x\}$ and on $\{x, y\}$ by evaluation. By Theorem 2.3, $\dim(L^H(\{x\})) = 1 = \dim(L^H(\{x, y\}))$ and so $L^H(\{x\}) \cong L^H(\{x, y\})$. However, $\{x\}$ and $\{x, y\}$ are not equivalent G-sets.

3. The case of finite actions

If G is a finite group and if X is a finite G-set (that is, if the action is finite), we may use the Cauchy-Frobenius lemma (also called the Burnside lemma) to compute the dimension of $L^G(X)$. Moreover, the space $L(X)$ (and hence also $L^G(X)$) possesses a standard Hermitian inner product (the base field is assumed to be the field of complex numbers). This allows us to prove further related properties between group actions and their corresponding spaces, including Bessel's inequality and Frobenius reciprocity.

3.1. Dimensions and fixed points

Using results in the previous section, we obtain a formula for computing the dimension of $L^G(X)$, where G and X are finite, in terms of fixed points of X. As a consequence of this result, we obtain an interesting result of free (also called fixed-point-free) actions.

Lemma 3.1. *Let G be a finite group and let X be a finite G-set. For any subgroup H of G,*

$$\dim(L^H(X)) = \frac{1}{|H|} \sum_{a \in H} |\mathrm{Fix}\, a|, \tag{3.1}$$

where $\mathrm{Fix}\, a = \{x \in X \colon a \cdot x = x\}$.

Proof. Let $\mathrm{orb}_H\, x = \{a \cdot x : a \in H\}$ and let $\mathcal{P} = \{\mathrm{orb}_H\, x : x \in X\}$. As proved earlier, $\dim(L^H(X))$ equals $|\mathcal{P}|$, the number of orbits of H on X. By the famous Cauchy-Frobenius lemma, $|\mathcal{P}| = \dfrac{1}{|H|} \sum_{a \in H} |\mathrm{Fix}\, a|$. □

Lemma 3.2. *Let G be a finite group and let X be a finite G-set. For any subgroup H of G,*

$$|G| \dim(L^G(X)) - |H| \dim(L^H(X)) = \sum_{a \in G \setminus H} |\mathrm{Fix}\, a|. \tag{3.2}$$

Proof. Note that $\mathrm{Fix}_H\, a = \mathrm{Fix}_G\, a$ for all $a \in H$ because H acts on X by the action inherited from G. It follows from Lemma 3.1 that

$$|G| \dim(L^G(X)) - |H| \dim(L^H(X)) = \sum_{a\in G} |\mathrm{Fix}\, a| - \sum_{a\in H} |\mathrm{Fix}\, a| = \sum_{a\in G\setminus H} |\mathrm{Fix}\, a|. \qquad \square$$

Recall that an action of a group G on a set X is free if $\mathrm{stab}\, x = \{e\}$ for all $x \in X$; that is, if for all $a \in G, x \in X$, $a \cdot x = x$ implies $a = e$. It is clear that an action of G on X is free if and only if $\mathrm{Fix}\, a = \emptyset$ for all $a \in G \setminus \{e\}$. By Lemma 3.2, the ratio of $\dim(L^H(X))$ and $\dim(L^G(X))$ is simply the index of H in G when the action of G on X is free, as shown in the following theorem.

Theorem 3.1. *Let G be a finite group with a subgroup H and let X be a finite non-empty set. If G acts freely on X, then*

$$\frac{\dim(L^H(X))}{\dim(L^G(X))} = [G\colon H], \tag{3.3}$$

where $[G\colon H]$ denotes the index of H in G.

Proof. Since G acts freely on X, $\mathrm{Fix}\, a = \emptyset$ for all $a \in G \setminus H$. By Lemma 3.2,

$$|G| \dim(L^G(X)) - |H| \dim(L^H(X)) = \sum_{a\in G\setminus H} |\mathrm{Fix}\, a| = 0.$$

Hence, $\dfrac{\dim(L^H(X))}{\dim(L^G(X))} = \dfrac{|G|}{|H|} = [G\colon H]$. $\square$

3.2. Orthogonal decomposition and Frobenius reciprocity

In this section, let G be a (finite or infinite) group and let X be a finite G-set unless otherwise stated. We also suppose that $\mathbb{F} = \mathbb{C}$. Thus, $L(X)$ admits the Hermitian inner product defined by

$$\langle f, g\rangle = \frac{1}{|X|} \sum_{x\in X} f(x)\overline{g(x)}, \tag{3.4}$$

where $\bar{\cdot}$ denotes complex conjugation.

Proposition 3.1. *$L(X)$ forms a complex inner product space. If $\mathcal{P} = \{\mathrm{orb}\, x\colon x \in X\}$, then*

$$\mathcal{B} = \left\{ \sqrt{\frac{|X|}{|C|}}\delta_C \colon C \in \mathcal{P} \right\}$$

forms an orthonormal basis for $L^G(X)$.

Proof. As an application of results in Section 2, $\mathcal{B}$ forms a basis for $L^G(X)$. Next, we prove that $\mathcal{B}$ is orthonormal. Let $\sqrt{\frac{|X|}{|C|}}\delta_C, \sqrt{\frac{|X|}{|D|}}\delta_D \in \mathcal{B}$ with $C, D \in \mathcal{P}$. If $C \neq D$, then

$$\langle \sqrt{\frac{|X|}{|C|}}\delta_C, \sqrt{\frac{|X|}{|D|}}\delta_D\rangle = \frac{1}{\sqrt{|C||D|}} \sum_{x\in X} \delta_C(x)\overline{\delta_D(x)} = 0,$$

because for each $x \in X$, either $x \in C$ or $x \in D$. If $C = D$, then

$$\langle \sqrt{\frac{|X|}{|C|}}\delta_C, \sqrt{\frac{|X|}{|D|}}\delta_D\rangle = \frac{1}{\sqrt{|C||D|}} \sum_{x\in X} \delta_C(x)\overline{\delta_D(x)} = \frac{1}{|C|} \sum_{x\in X} |\delta_C(x)|^2 = 1,$$

because $\sum_{x\in X} |\delta_C(x)|^2 = |C|$. □

One advantage of the inner product defined by (3.4) is shown in the following theorem, which indicates that the action of G on X is preserved by this inner product.

Proposition 3.2. *The action given by* (1.3) *is unitary in the sense that* $\langle a \cdot f, a \cdot g\rangle = \langle f, g\rangle$ *for all* $f, g \in L(X)$, $a \in G$. *In particular, the map* $f \mapsto a \cdot f$, $f \in L(X)$, *is a unitary operator on* $L(X)$ *for all* $a \in G$.

Proof. The proposition follows from the fact that the map $x \mapsto a^{-1} \cdot x$ is a bijection from X to itself. □

To obtain an orthogonal decomposition of $L(X)$, we define a map σ by

$$\sigma(f) = \sum_{x\in X} f(x), \qquad f \in L(X). \tag{3.5}$$

Theorem 3.2. *Let* σ *be the map defined by* (3.5). *Then, the following assertions hold:*

(1) σ *is a linear functional from* $L(X)$ *to* $\mathbb{C}$.

(2) $\ker\sigma$ *is an invariant subspace of* $L(X)$ *under the action given by* (1.3).

(3) $\ker\sigma = (\operatorname{span} f_1)^{\perp}$, *where* $f_1(x) = 1$ *for all* $x \in X$.

(4) $\dim(\ker\sigma) = |X| - 1$.

(5) $\ker\left(\sigma\big|_{L^G(X)}\right)$ *is an invariant subspace of* $L^G(X)$ *and its dimension equals the number of orbits on* X *minus* 1. *Here,* $\sigma\big|_{L^G(X)}$ *is the restriction of* σ *to* $L^G(X)$.

(6) $L^G(X)^{\perp} \subseteq \ker\sigma$; *equality holds if and only if the action of* G *on* X *is transitive.*

Proof. The proofs of Parts (1), (3) and (5) are immediate. Part (2) holds because the map $x \mapsto a^{-1} \cdot x$ is a bijection from X to itself. To prove Part (4), note that $\operatorname{span} f_1$ is a finite-dimensional subspace of $L(X)$. By the projection theorem in linear algebra and Part (3), $L(X) = \operatorname{span} f_1 \oplus \ker\sigma$ and so $\dim(\ker\sigma) = |X| - 1$. That $L^G(X)^{\perp} \subseteq \ker\sigma$ is clear. By Theorem 2.3 and Part (3), $L^G(X)^{\perp} = \ker\sigma$ if and only if $L^G(X) = \operatorname{span} f_1$ (since $L^G(X)$ and $\operatorname{span} f_1$ are finite-dimensional subspaces of $L(X)$) if and only if the action of G on X is transitive. This proves Part (6). □

Corollary 3.1. *Let* G *be a group and let* X *be a finite* G*-set. Then,*

(1) $L(X) = L^G(X) \oplus L^G(X)^{\perp}$;

(2) $L(X) = \operatorname{span} f_1 \oplus \ker\sigma$;

(3) $L^G(X) = \operatorname{span} f_1 \oplus \ker\left(\sigma\big|_{L^G(X)}\right)$.

Here, $\oplus$ *denotes orthogonal direct sum decomposition.*

Proof. Part (1) follows from the projection theorem. Part (2) follows as in the proof of Part (4) of Theorem 3.2. Part (3) holds since span f_1 is a subspace of $L^G(X)$. $\square$

According to Proposition 3.1, we have an orthonormal basis for $L^G(X)$, which is an orthonormal set in $L(X)$. Thus, several prominent results in linear algebra can be deduced from this fact.

Theorem 3.3. *Let G be a group and let X be a finite G-set. Suppose that*

$$\mathcal{P} = \{\operatorname{orb} x\colon\ x \in X\} = \{C_1, C_2, \ldots, C_n\}.$$

Fix the ordered (orthonormal) basis of $L^G(X)$*:*

$$\mathcal{B} = \left(\sqrt{\frac{|X|}{|C_1|}}\delta_{C_1},\ \sqrt{\frac{|X|}{|C_2|}}\delta_{C_2}, \ldots,\ \sqrt{\frac{|X|}{|C_n|}}\delta_{C_n} \right).$$

Then, the following assertions hold:

(1) **(Fourier expansion)** *The Fourier expansion with respect to* $\mathcal{B}$ *of a function* $f \in L(X)$ *is*

$$\widehat{f} = \left(\frac{1}{|C_1|} \sum_{x\in C_1} f(x) \right) \delta_{C_1} + \left(\frac{1}{|C_2|} \sum_{x\in C_2} f(x) \right) \delta_{C_2} + \cdots + \left(\frac{1}{|C_n|} \sum_{x\in C_n} f(x) \right) \delta_{C_n}; \tag{3.6}$$

that is, the Fourier coefficients of f are given by

$$\langle f, \sqrt{\frac{|X|}{|C_i|}}\delta_{C_i} \rangle = \frac{1}{\sqrt{|X||C_i|}} \sum_{x\in C_i} f(x), \tag{3.7}$$

for all $i = 1, 2, \ldots, n$*.*

(2) **(Bessel's inequality)** *For all* $f \in L(X)$*,*

$$\sum_{i=1}^{n} \frac{1}{|C_i|} \left| \sum_{x\in C_i} f(x) \right|^2 \leq \sum_{x\in X} |f(x)|^2. \tag{3.8}$$

(3) *If G acts non-trivially on X, then there exists a function* $f \in L(X)$ *with* $\|\widehat{f}\| < \|f\|$*. That is, the equality in Bessel's identity is not attained.*

Proof. Recall that the Fourier coefficients of f are

$$\langle f, \sqrt{\frac{|X|}{|C_i|}}\delta_{C_i} \rangle = \frac{1}{|X|} \sum_{x\in C_i} f(x) \sqrt{\frac{|X|}{|C_i|}} = \frac{1}{\sqrt{|X||C_i|}} \sum_{x\in C_i} f(x)$$

for all $i = 1, 2, \ldots, n$. Hence, the Fourier expansion of f with respect to $\mathcal{B}$ is

$$\begin{aligned}
\widehat{f} &= \langle f, \sqrt{\frac{|X|}{|C_1|}}\delta_{C_1}\rangle \sqrt{\frac{|X|}{|C_1|}}\delta_{C_1} + \cdots + \langle f, \sqrt{\frac{|X|}{|C_n|}}\delta_{C_n}\rangle \sqrt{\frac{|X|}{|C_n|}}\delta_{C_n} \\
&= \left(\frac{1}{\sqrt{|X||C_1|}}\sum_{x\in C_1} f(x)\right)\sqrt{\frac{|X|}{|C_1|}}\delta_{C_1} + \cdots + \left(\frac{1}{\sqrt{|X||C_n|}}\sum_{x\in C_n} f(x)\right)\sqrt{\frac{|X|}{|C_n|}}\delta_{C_n} \\
&= \left(\frac{1}{|C_1|}\sum_{x\in C_1} f(x)\right)\delta_{C_1} + \cdots + \left(\frac{1}{|C_n|}\sum_{x\in C_n} f(x)\right)\delta_{C_n}.
\end{aligned}$$

This proves Part (1).

Recall that Bessel's inequality states that $\|\widehat{f}\| \leq \|f\|$. Hence,

$$\sum_{i=1}^{n}\left|\langle f, \sqrt{\frac{|X|}{|C_i|}}\delta_{C_i}\rangle\right|^2 \leq \langle f, f\rangle.$$

Direct computation shows that $\sum_{i=1}^{n}\left|\langle f, \sqrt{\frac{|X|}{|C_i|}}\delta_{C_i}\rangle\right|^2 = \frac{1}{|X|}\sum_{i=1}^{n}\frac{1}{|C_i|}\left|\sum_{x\in C_i} f(x)\right|^2$ and that $\langle f, f\rangle = \frac{1}{|X|}\sum_{x\in X}|f(x)|^2$. Hence, (3.8) follows. This proves Part (2).

Suppose that G acts non-trivially on X. By Theorem 2.2, $L^G(X) \subsetneq L(X)$. This implies that $\mathcal{B}$ is not an orthonormal basis for $L(X)$ and so there exists a function f in $L(X)$ with $\|\widehat{f}\| < \|f\|$ by [8, Theorem 9.17]. This proves Part (3). □

Next, we extend Frobenius reciprocity from the space of class functions to that of functions invariant under a given group action in a natural way. Let G be a group and let X be a G-set. Recall that a (non-empty) subset Y of X is invariant if $a \cdot y \in Y$ for all $a \in G, y \in Y$; that is, if $G \cdot Y = Y$. It is not difficult to check that the following are equivalent:

(1) Y is an invariant subset of X;

(2) For all $a \in G, x \in X$, $a \cdot x \in Y$ if and only if $x \in Y$.

Let X be a G-set and let Y be an invariant subset of X. Define a map Res^X_Y from $L(X)$ to $L(Y)$ by

$$\mathrm{Res}^X_Y f(y) = f(y), \qquad y \in Y, \tag{3.9}$$

for all $f \in L(X)$. Also, for each $f \in L(Y)$, define $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in Y; \\ 0, & \text{otherwise,} \end{cases} \tag{3.10}$$

for all $x \in X$. Then, $\tilde{f} \in L(X)$. In fact, we have the following lemma.

Lemma 3.3. *The map $\epsilon\colon L(Y) \to L(X)$ given by $\epsilon(f) = \tilde{f}$ is linear and maps $L^G(Y)$ to $L^G(X)$.*

Proof. The proof that ϵ is linear is straightforward. By the remark above, $\epsilon(f) \in L^G(X)$ for all $f \in L^G(Y)$. □

Theorem 3.4. *Let X be a G-set with an invariant subset Y. Then,* $\mathrm{Res}_Y^X \colon L(X) \to L(Y)$ *is linear and maps $L^G(X)$ surjectively onto $L^G(Y)$.*

Proof. The proof that Res_Y^X is linear is straightforward. Let $f \in L^G(Y)$. By Lemma 3.3, $\tilde{f} \in L^G(X)$ and $\mathrm{Res}_Y^X \tilde{f}(y) = \tilde{f}(y) = f(y)$ for all $y \in Y$. So $\mathrm{Res}_Y^X \tilde{f} = f$. This proves that Res_Y^X is surjective. □

Let G be a finite group, let X be a finite G-set, and let Y be an invariant subset of X. Define a map Ind_Y^X on $L(Y)$ by

$$\mathrm{Ind}_Y^X f(x) = \frac{|X|}{|G||Y|} \sum_{b \in G} \tilde{f}(b^{-1} \cdot x), \qquad x \in X, \tag{3.11}$$

for all $f \in L(Y)$. Then, Ind_Y^X is a linear transformation from $L(Y)$ to $L^G(X)$, as shown in the following theorem.

Theorem 3.5. *The map* Ind_Y^X *defined by* (3.11) *is a linear transformation from $L(Y)$ to $L^G(X)$.*

Proof. The proof that Ind_Y^X is linear is straightforward. Let $f \in L(Y)$. Given $a \in G$ and $x \in X$, we have by inspection that

$$\begin{aligned}
\mathrm{Ind}_Y^X f(a \cdot x) &= \frac{|X|}{|G||Y|} \sum_{b \in G} \tilde{f}(b^{-1} \cdot (a \cdot x)) \\
&= \frac{|X|}{|G||Y|} \sum_{b \in G} \tilde{f}((b^{-1} a) \cdot x) \\
&= \frac{|X|}{|G||Y|} \sum_{c \in G} \tilde{f}(c^{-1} \cdot x) \\
&= \mathrm{Ind}_Y^X f(x).
\end{aligned}$$

The third equality holds since if b runs over all of G, then so does $a^{-1}b$ (that is, the change of variable $c = a^{-1}b$ is permitted). Thus, $\mathrm{Ind}_Y^X f \in L^G(X)$. □

The following theorem asserts that the linear transformations Res_Y^X and Ind_Y^X are Hermitian adjoint with respect to the Hermitian inner product defined earlier. This is a group-action version of Frobenius reciprocity.

Theorem 3.6. *(Frobenius reciprocity) Let G be a finite group, let X be a finite G-set, and let Y be an invariant subset of X. Then,*

$$\langle \mathrm{Ind}_Y^X f, g \rangle = \langle f, \mathrm{Res}_Y^X g \rangle \tag{3.12}$$

for all $f \in L^G(Y), g \in L^G(X)$.

Proof. Direct computation shows that

$$
\begin{aligned}
\langle \mathrm{Ind}_Y^X f, g\rangle &= \frac{1}{|X|}\sum_{x\in X}\mathrm{Ind}_Y^X f(x)\overline{g(x)} = \frac{1}{|X|}\sum_{x\in X}\left(\frac{|X|}{|G||Y|}\sum_{b\in G}\tilde{f}(b^{-1}\cdot x)\right)\overline{g(x)} \\
&= \frac{1}{|G||Y|}\sum_{x\in X}\left(\sum_{b\in G}\tilde{f}(b^{-1}\cdot x)\right)\overline{g(x)} = \frac{1}{|G||Y|}\sum_{x\in Y}\left(\sum_{b\in G}f(b^{-1}\cdot x)\right)\overline{g(x)} \\
&= \frac{1}{|G||Y|}\sum_{x\in Y}|G|f(x)\overline{g(x)} = \frac{1}{|Y|}\sum_{x\in Y}f(x)\overline{g(x)} \\
&= \frac{1}{|Y|}\sum_{x\in Y}f(x)\overline{\mathrm{Res}_Y^X g(x)} = \langle f, \mathrm{Res}_Y^X g\rangle.
\end{aligned}
$$

The fifth equality holds since $f \in L^G(Y)$, which implies that $f(b^{-1}\cdot x) = f(x)$ for all $b \in G$, $x \in Y$. □

We remark that the inner product used on the right hand side of (3.12) is computed by the same formula as in (3.4) with Y in place of X. This makes sense because if Y is an invariant subset of X, then the G-action on X restricts to the G-action on Y. In other words, the restriction of the Hermitian inner product of $L(X)$ to $L(Y)$ does define an inner product on $L(Y)$.

4. Conclusions

We show that every group action is associated with a vector space over an arbitrary field, and in certain circumstances this notion can be used to distinguish non-equivalent group actions. We then study algebraic properties of group actions compared with properties of their corresponding spaces. We also prove several prominent results, including Bessel's inequality and Frobenius reciprocity.

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Conflict of interest

The author declares no conflict of interest.

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