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Research article

On almost set-Menger spaces in bitopological context

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Abstract: In this paper, we define the ij-almost-set Menger (ij-ASM) property in bitopological spaces. We put up some equivalences of ij-almost-set Menger bitopological spaces and investigate the behaviours of such spaces under some different types of mappings. We later take the preservation of these properties under union, subspaces, products into consideration and give some related examples. We finally introduce the concept of ij-almost P_γ-set in bitopological spaces.

Keywords: selection principles; almost Menger property; set Menger; bitopological spaces; covering properties
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1. Introduction

1.1. Selection principles

Covering properties of a topological space is one of the most active research fields and has a long history which appears in papers [1–4]. More recently, the theory known as infinite combinatorial topology or selection principles in mathematics was introduced by M. Scheepers [5, 6] applying selection principles to different open covers of a topological space and initiated a systematic study of selection principles. One of the most important features of the theory is to gather topology with the other fields of mathematics such as game theory, Ramsey theory, algebraic structures, etc. The theory has been widely studied and is still being studied. It also gives new viewpoints and frames for theoretical and applicable areas (see [7–9]). Several topological properties and concepts are defined and characterized by way of two classical selection principles stated in [6] as follows:

Let X be an infinite set, and $\mathcal{A}$ and $\mathcal{B}$ are the families of subsets of it.

$S_1(\mathcal{A}, \mathcal{B})$ is the selection principle: For every sequence $(A_n : n \in \mathbb{N})$ of members of $\mathcal{A}$, there exists a sequence $(b_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection principle: For every sequence $(A_n : n \in \mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that $B_n \subset A_n$ where B_n is a finite set for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In [3] Menger introduced the Menger basis property for metric spaces. In [1], Hurewicz introduced a property which is nowadays known as the Menger property (MP) and proved that the Menger basis property is equivalent to the MP. It is known that a topological space X having the MP is equivalent to $X \in S_{fin}(\mathcal{O}, \mathcal{O})$ in Scheepers' notation where $\mathcal{O}$ is the family of open covers of X and defined as the following:

X has the MP if for all the sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ where $\mathcal{D}_n$ is an open cover of X for each n, there exists a finite subset $C_n \subset \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \bigcup C_n = X$.

Hurewicz showed that the statement that a metrizable space is σ-compact if and only if it has MP is equivalent to Menger conjecture. In [2], he introduced a property which is stronger than the MP currently referred to as the Hurewicz Property that is defined as the following.

Let X be a topological space and $(\mathcal{D}_n : n \in \mathbb{N})$ be any sequence of open covers of X. If there is a sequence $(C_n : n \in \mathbb{N})$ such that each C_n is the finite subset of $\mathcal{D}_n$ for each n and all member x of X, $\mid \{n \in \mathbb{N} : x \notin \bigcup C_n\} \mid < \omega$ holds. It is well known that every σ-compact topological space has the Hurewicz property, and every topological space which has the Hurewicz property has the Menger property.

The Menger property has recently been studied extensively [10–12]. Also, general forms of the Menger property have been studied. Kočinac [13, 14] introduced the almost Menger property. A topological space X called almost Menger if every sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(C_n)_{n\in\mathbb{N}}$ where each C_n is finite subset of $\mathcal{D}_n$ for eacn $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} \overline{\bigcup C_n}$. Kocev [15] studied this notion systematically. Also, the weak Menger property has been introduced in [16] and studied in [17, 18]. These generalizations of the Menger property got their place in many papers [19–24].

1.2. Relatively topological spaces and set covering properties

In [25], a cardinal function sL was defined by Arhangel'skii. Let X be a topological space. Then the $sL(X)$ of X is the minimal cardinality κ such that for every subset $S \subset X$ and every open cover $\mathcal{D}$ of $\overline{S}$, there is a subfamily $\mathcal{D}^* \subset \mathcal{D}$ providing $\mid \mathcal{D}^* \mid \leq \kappa$ and $S \subset \overline{\bigcup \mathcal{D}^*}$. If $sL(X) = \omega$, then the space X is called *s-Lindelöf* space.

In any topological subject, it could be important to know "how a given subspace of a topological space is located in the space". Let X be a topological space, and denote with $\mathcal{P}(X)$ the power set of X. Let $Y \subset X$ and $P \subset \mathcal{P}(X)$. The properties of the subset Y or each member of P depend on how Y or members of P are placed in X. That is why it is so natural that two investigation areas arise. One of them is assigning a relative S of the subset Y of X for given a topological property S, and second one is assigning a property P-S showing how every member P located in X. With his collaborators, A.V. Arhangel'skii first applied these investigations [25–27]. In this sense, the same investigations have been applied to selection principles, too (see [21, 28–31]).

Motivated by the definition of an "s-Lindelöf cardinal function" of Arhangel'skii and this line of investigations, in [32] Kočinac and Konca defined set-Menger, set-Rothberger, set-Hurewicz spaces and their weaker forms considering the Menger, Rothberger and Hurewicz covering properties. Later, they defined in [33] the star versions of related spaces, and they initiated the investigation new sorts of selective covering properties known as set covering properties (see also [34]).

Definition 1.1. *[32] Let X be a topological space and $P \subset \mathcal{P}(X)$ such that $\emptyset \notin P$. X is said to be:*

(1) P-Menger if for all $S \in P$ *and each sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of open covers of* $\overline{S}$*, there is a sequence* $(\mathcal{D}^*_n)_{n\in\mathbb{N}})$ *where* $\mathcal{D}^*_n$ *is the finite subfamily of* $\mathcal{D}_n$ *for each n and* $S \subset \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{D}^*_n$*.*

(2) Almost P-Menger if $S \subset \bigcup_{n\in\mathbb{N}} \overline{\bigcup \mathcal{D}^*_n}$*.*

If $P = \mathcal{P} \setminus \{\emptyset\}$ *it is said that X is set-Menger (for short SM) or almost set-Menger (ASM), respectively.*

In this sense, we extend this investigation and introduce the ij-almost-set-Menger (ij-ASM) property and study it in bitopological spaces. Bitopological selection principles, which is the active research field, have been discussed in several papers [35–43]. The paper is generated as follows. In Section 2, we introduce the ij-almost-set-Menger property in bitopological context. After giving some examples for this kind of spaces, we investigate the equivalences of these spaces to other spaces like set-Menger and ij-almost Lindelöf. We also look at the behavior of ij-almost-set-Menger spaces under some different types of mappings defined in bitopological spaces. Later, in Section 3, We keep in sight the preservation of this property under union, subspaces and products. We introduce the class of ij-almost P_γ-set and investigate some properties in Section 4.

2. Definitions, equivalences and examples

We use usual notations and terminology for topological spaces as in [44]. Our notation and terminology will follow [45] for bitopological spaces. By $\mathbb{N}, \mathbb{P}$ and $\mathbb{R}$, we denote the sets of natural, irrational and real numbers, respectively. During the paper, by (X, σ) we denote a topological space while (X, σ_1, σ_2) (sometimes X) denotes a bitopological space (or shortly, bispace) which is a set X equipped with two topologies, in general unrelated, σ_1 and σ_2 (see [46]). For any $A \subset X$, σ_i-$cl(A)$ denotes closure of A, and σ_i-$int(A)$ denotes the interior of A with respect to the topology σ_i ($i = 1, 2$).

Definition 2.1. *A bitopological space* (X, σ_1, σ_2) *is said to be* ij-almost-set-Menger *(ij-ASM, for short)* $(i, j = 1, 2)$ *if, for all nonempty* $A \subset X$ *and for each* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *sequence of* σ_i*-open covers of* σ_i*-*$cl(A)$*, there exists a sequence* $(C_n)_{n\in\mathbb{N}}$ *of finite families such that* $C_n \subset \mathcal{D}_n$ *for each* $n \in \mathbb{N}$ *and* $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{V\in C_n} \sigma_j$*-*$cl(V)$

In the light of this definition, we can give the following proposition.

Proposition 2.1. *Let* (X, σ_1, σ_2) *be a bitopological space,*
(1) If (X, σ_1) *is a Menger space (or SM), then* (X, σ_1, σ_2) *is* 12*-ASM.*
(2) If (X, σ_1) *is ASM and* $\sigma_2 \leq \sigma_1$ *(*σ_2 *is coarser than* σ_1*), then* (X, σ_1, σ_2) *is* 12*-ASM.*

Proof. (1) Obvious from the corresponding definitions.

(2) Let $A \subset X$, and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be any sequence of σ_1-open covers of σ_1-$cl(A)$. Since (X, σ_1) is ASM, then there exists a sequence $(C_n)_{n\in\mathbb{N}}$ where $C_n \subset \mathcal{D}_n$ is finite for each $n \in \mathbb{N}$ and $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{V\in C_n} \sigma_1$-$cl(V)$ holds. Since $\sigma_2 \leq \sigma_1$, then σ_1-$cl(V) \subset \sigma_2$-$cl(V)$ for all $V \in C_n$ and $n \in \mathbb{N}$. This shows that (X, σ_1, σ_2) is 12-ASM.

□

Example 2.1. *Assume* $\mathbb{R}$ *equipped with cocountable topology* σ_1 *and the Sorgenfrey topology* σ_2*. Since* $(\mathbb{R}, \sigma_1)$ *is SM,* $(\mathbb{R}, \sigma_1, \sigma_2)$ *is* 12*-ASM.*

Example 2.2. *Assume the bispace* $(\mathbb{R}, \sigma_1, \sigma_2)$ *where* σ_1 *and* σ_2 *are the Smirnov's deleted topology and usual topology, respectively (see [47]). The topological space* $(\mathbb{R}, \sigma_1)$ *is Menger (and SM) and* $\sigma_2 \leq \sigma_1$*. So the bispace* $(\mathbb{R}, \sigma_1, \sigma_2)$ *is* 12*-ASM.*

On the other hand, the assertion converse in Proposition 2.1(1) does not hold in general as the following example illustrates.

Example 2.3. *Let* $\mathbb{P}$ *be the irrational numbers set and* $a \in \mathbb{P}$ *fixed point. Assume particular point topology* $\sigma_1 = \{U \subseteq \mathbb{R} : a \in U\} \cup \{\emptyset\}$ *and* $\sigma_2 = \{U \cap \mathbb{P} : U \in \sigma_1\}$*. Then, the followings are obtained.*

(1) Since $a \in U$ *for all* $U \in \sigma_1$ *it is obvious that* $\sigma_2 \leq \sigma_1$*.*

(2) $\mathcal{D} = \{\{x, a\} : x \in \mathbb{R}\}$ *is an open cover for* $(\mathbb{R}, \sigma_1)$*. Choose* $\mathcal{D}_n = \mathcal{D}$ *for all* $n \in \mathbb{N}$*, then* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *is the sequence of open covers of* $(\mathbb{R}, \sigma_1)$*. This sequence assures that* $(\mathbb{R}, \sigma_1)$ *is not Menger (so not SM).*

(3) Since each nonempty open subset is dense in $(\mathbb{R}, \sigma_1)$*, then* $(\mathbb{R}, \sigma_1)$ *is ASM.*

(4) $(\mathbb{R}, \sigma_1, \sigma_2)$ *is* 12*-ASM with the statement* 1.

That is why it is so natural to discuss under what conditions the converse assertion in Proposition 2.1(1) holds. In this sense, we firstly give the following definition.

Definition 2.2. *[48] A bispace* (X, σ_1, σ_2) *is said to be an* ij*-regular space* $(i, j = 1, 2,\ i \neq j)$ *if, for each element* $x \in X$ *and each* σ_i*-closed set* F *with* $x \notin F$*, there are a* σ_i*-open set* U *and* σ_j*-open set* V *such that* $x \in U$*,* $V \supset F$ *and* $U \cap V = \emptyset$*.*

Theorem 2.1. *If* (X, σ_1, σ_2) *is* ij*-ASM and an* ij*-regular bispace, then* (X, σ_i) *is SM.*

Proof. Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of σ_i-open covers of σ_i-$cl(A)$. Then, for each $n \in \mathbb{N}$ and $x \in \sigma_i$-$cl(A)$, we can choose $U_x^n \in \mathcal{D}_n$ such that U_x^n contains x. Since (X, σ_1, σ_2) is an ij-regular bispace, there exist $V_x^n \in \sigma_i$ for each $x \in \sigma_i$-$cl(A)$ and U_x^n such that $x \in V_x^n \subset \sigma_j$-$cl(V_x^n) \subset U_x^n$ (see [48]). Now let $\mathcal{G}_n = \{V_x^n : x \in \sigma_i\text{-}cl(A)\}$ for all $n \in \mathbb{N}$. $(\mathcal{G}_n)$ is a σ_i-open cover of σ_i-$cl(A)$ for all $n \in \mathbb{N}$ and

$$\sigma_j\text{-}cl(\mathcal{G}_n) = \{\sigma_j\text{-}cl(V_x^n) : V_x^n \in \mathcal{G}_n\}$$

is a refinement of $\mathcal{D}_n$. On the other side, since (X, σ_1, σ_2) is ij-ASM, there is a sequence $(C_n)_{n\in\mathbb{N}}$ such that C_n is the finite subset of $\mathcal{G}_n$ and $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{V\in C_n} \sigma_j\text{-}cl(V)$. For each $n \in \mathbb{N}$ and $V \in C_n$ we can choose $U_V \in \mathcal{D}_n$ such that σ_j-$cl(V) \subset U_V$ due to that σ_j-$cl(\mathcal{G}_n)$ refines $\mathcal{D}_n$. Let $\mathcal{G}_n^* = \{U_V : V \in C_n\}$. Then each $\mathcal{G}_n^* \subset \mathcal{D}_n$ is finite and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{G}_n^* \supset A$ which completes the proof. □

The class of ij-ASM bispaces can be characterized in terms of ij-regular open sets. We now give the characterization such this spaces with ij-regular open sets.

Definition 2.3. *[45, 49] Let* (X, σ_1, σ_2) *be a bispace and* $A \subset X$*. A subset* A *of* X *is* ij*-regular open set (respectively* ij*-regular closed set) if* $A = \sigma_i\text{-}int(\sigma_j\text{-}cl(A))$(respectively$A = \sigma_i\text{-}cl(\sigma_j\text{-}int(A))$).

It can easily be seen that every ij-regular open set in (X, σ_1, σ_2) is σ_i-open.

Theorem 2.2. *A bispace* (X, σ_1, σ_2) *is* ij*-ASM if and only if for each* $A \subset X$ *and every sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of covers of* σ_i*-*$cl(A)$ *by* ij*-regular open sets in* X*, there is a sequence* $(C_n)_{n\in\mathbb{N}}$ *where each* C_n *is finite subset of* $\mathcal{D}_n$ *for all* n*, and* $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{V\in C_n} \sigma_j\text{-}cl(V)$*.*

Proof. (⇒) Obvious.
(⇐) Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be the sequence of σ_i-open covers of σ_i-$cl(A)$. If we put $C_n = \{\sigma_i\text{-}int(\sigma_j\text{-}cl(U)) : U \in \mathcal{D}_n\}$ for all n, we obtain a sequence $(C_n)_{n\in\mathbb{N}}$ of covers of σ_i-$cl(A)$ by ij-regular open sets in X. Then, by assumption, there is $C_n^* \subset C_n$ for all n such that C_n^* is the finite subset and $A \subset \bigcup_{n\in\mathbb{N}} \bigcup \sigma_j\text{-}cl(C_n^*)$ where $\sigma_j\text{-}cl(C_n^*) = \{\sigma_j\text{-}cl(V) : V \in C_n^*\}$. For every $n \in \mathbb{N}$ and every $V \in C_n$, there is $U_V \in \mathcal{D}_n$ such that $V = \sigma_i\text{-}int(\sigma_j\text{-}cl(U_V))$. Then, the sequence $(\mathcal{D}_n^*)_{n\in\mathbb{N}}$ is the desired one where $\mathcal{D}_n^* = \{U_V : V \in C_n^*\}$. For seeing this, one can observe that $\sigma_j\text{-}cl(U_V)$ is a ji-regular closed subset of X, and

$$\sigma_j\text{-}cl(V) = \sigma_j\text{-}cl(\sigma_i\text{-}int(\sigma_j\text{-}cl(U_V))) = \sigma_j\text{-}cl(U_V).$$

Thus, $A \subset \bigcup_{n\in\mathbb{N}} \bigcup \sigma_j\text{-}cl(\mathcal{D}_n^*)$. □

We now give the relations between ij-ASM and ij-Almost Lindelöf bispaces.

Definition 2.4. *[50] A bipsace (X, σ_1, σ_2) is said to be ij-almost Lindelöf (for short ij-AL) if for every σ_i-open cover $\mathcal{D}$ of X, there is a countable subset $\{V_n : n \in \mathbb{N}\}$ of $\mathcal{D}$ such that $X = \bigcup_{n\in\mathbb{N}} \sigma_j\text{-}cl(V_n)$.*

In the following theorem, we see that every ij-ASM bispace is ij-AL.

Theorem 2.3. *Every ij-ASM bispace is ij-AL.*

Proof. Let (X, σ_1, σ_2) be a bispace and $\mathcal{D}$ be any σ_i-open cover of X. Let $\mathcal{D}_n = \mathcal{D}$ for each $n \in \mathbb{N}$ and $A \subset X$. Put $B = X \setminus A$. Since $\mathcal{D}$ is a σ_i-open cover of X, then $\mathcal{D}_n$ is a σ_i-open cover of both σ_i-$cl(A)$ and σ_i-$cl(B)$. Then, we clearly obtain a sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of σ_i-open covers of σ_i-$cl(A)$ and σ_i-$cl(B)$. Since (X, σ_1, σ_2) is ij-ASM, there exist $C_n^A, C_n^B \subset \mathcal{D}_n$ where C_n^A and C_n^B are finite for all $n \in \mathbb{N}$ with $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{U\in C_n^A} \sigma_j\text{-}cl(V)$ and $B \subset \bigcup_{n\in\mathbb{N}} \bigcup_{U\in C_n^B} \sigma_j\text{-}cl(U)$. Since C_n^A and C_n^B are finite, then the family $\mathcal{W}_n = C_n^A \cup C_n^B$ is a finite family for each $n \in \mathbb{N}$. Then $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{D}$, which is clearly providing that $X = A \cup B \subset \bigcup_{n\in\mathbb{N}} \bigcup_{W\in\mathcal{W}_n} \sigma_j\text{-}cl(W)$. So, X is ij-AL. □

The following example shows the inverse implication, in general not true.

Example 2.4. *Consider $\mathbb{R}$ endowed with the two topologies; σ_1 is the Sorgenfrey topology, and σ_2 is the family of sets $U \setminus C$, where $U \in \sigma_1$ and $C \subset \mathbb{R}$ and $|C| \leq \omega$. The bispace $(\mathbb{R}, \sigma_1, \sigma_2)$ is 12-AL, since (R, σ_1) is Lindelöf. But it fails to be 12-ASM since $(\mathbb{R}, \sigma_1)$ is not almost Menger so not ASM (see [17]) and $\sigma_1\text{-}cl(U) = \sigma_2\text{-}cl(U)$ for every σ_1-open set U.*

It is a quite natural question under what conditions these properties are equivalent. Let us give the definition of P-space.

Definition 2.5. *[51] A space X is called P-space if every intersection of countably many open sets is open.*

Theorem 2.4. *Let (X, σ_1, σ_2) be ij-AL. If (X, σ_i) is a P-space, then (X, σ_1, σ_2) is ij-ASM.*

Proof. Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of σ_i-open covers of σ_i-$cl(A)$. We may suppose that every $\mathcal{D}_n$ is closed under finite unions without loss of generality. Now, if we put $\mathcal{G} = \{\bigcap_{n\in\mathbb{N}} U_n : U_n \in \mathcal{D}_n\}$, then since (X, σ_i) is P-space, we obtain an σ_i-open over of σ_i-$cl(A)$. On the other hand, since (X, σ_1, σ_2) is ij-AL, σ_i-$cl(A)$ is ij-AL. Then there is a countable subset $\mathcal{G}^* = \{G_n : n \in \mathbb{N}\}$ of $\mathcal{G}$ providing $\sigma_i\text{-}cl(A) \subset \bigcup_{n\in\mathbb{N}} \sigma_j\text{-}cl(G_n)$. Let $G_n = \bigcap_{m\in\mathbb{N}} U_m^n$ where $U_m^n \in \mathcal{D}_m$. Since $G_n \subset U_n^n$ for each $n \in \mathbb{N}$, we clearly obtain $A \subset \bigcup_{n\in\mathbb{N}} \sigma_j\text{-}cl(U_n^n)$. So (X, σ_1, σ_2) is ij-ASM. □

Corollary 2.1. *Let (X, σ_1, σ_2) be an ij-regular bispace, and (X, σ_1) is P-space. Then, the following expressions are equivalent:*

(1) (X, σ_1) is Menger,

(2) (X, σ_1) is ASM,

(3) (X, σ_1, σ_2) is ij-ASM,

(4) (X, σ_1, σ_2) is ij-AL,

(5) (X, σ_1) is Lindelöf.

In what follows, we study some behaviors of ij-ASM bispaces under some types of mappings.

Definition 2.6. *[45] Let (X, σ_1, σ_2) and (Y, ρ_1, ρ_2) be bispaces and $f : X \to Y$ be a mapping. f is said to be d-continuous (pairwise continuous) if the mappings $f_i : (X, \sigma_i) \to (Y, \rho_i)$ are continuous (i-continuous) for $i = 1, 2$.*

Theorem 2.5. *Let (X, σ_1, σ_2) be ij-ASM bispace, and let (Y, ρ_1, ρ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then (Y, ρ_1, ρ_2) is ij-ASM*

Proof. Let $B \subset Y$, $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of open sets of Y providing $\rho_i\text{-}cl(B) \subset \bigcup \mathcal{D}_n$ for all $n \in \mathbb{N}$ and $A = f^{-1}(B)$. Since f is d-continuous, $f^{-1}(U) \in \sigma_i$ for all $n \in \mathbb{N}$ and $U \in \mathcal{D}_n$. Moreover, by the d-continuity of f, we obtain $\sigma_i\text{-}cl(A) \subset f^{-1}(\rho_i\text{-}cl(B)) \subset f^{-1}(\bigcup \mathcal{D}_n)$ for all $n \in \mathbb{N}$. Then, $(\mathcal{D}^A_n)_{n\in\mathbb{N}}$ is the sequence of σ_i-open covers of $\sigma_i\text{-}cl(A)$ where $\mathcal{D}^A_n = \{f^{-1}(U) : U \in \mathcal{D}_n\}$ for each $n \in \mathbb{N}$. Since (X, σ_1, σ_2) is an ij-ASM bispace, then there is a $C^A_n \subset \mathcal{D}^A_n$ such that C^A_n is finite for all $n \in \mathbb{N}$, and $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{V' \in C^A_n} \sigma_j\text{-}cl(V')$ holds. We can choose a $U_{V'} \in \mathcal{D}_n$ such that $V' = f^{-1}(U_{V'})$ for all $V' \in C^A_n$ and $n \in \mathbb{N}$. Let $C_n = \{U_{V'} : V' \in C^A_n\}$. Then each C_n is the finite subset of $\mathcal{D}_n$ for all n, and

$$\begin{aligned}
B = f(A) &\subset f(\bigcup_{n\in\mathbb{N}} \bigcup_{V' \in C^A_n} \sigma_j\text{-}cl(V')) \\
&\subset \bigcup_{n\in\mathbb{N}} \bigcup_{V' \in C^A_n} \rho_j\text{-}cl(f(V')) \\
&= \bigcup_{n\in\mathbb{N}} \bigcup_{V \in C_n} \rho_j\text{-}cl(V)
\end{aligned}$$

which concludes that (Y, ρ_1, ρ_2) is ij-ASM. □

Definition 2.7. *[52] A mapping $f : (X, \sigma_1, \sigma_2) \to (Y, \rho_1, \rho_2)$ is 12-continuous if $f^* : (X, \sigma_1) \to (Y, \rho_2)$ is continuous.*

Proposition 2.2. *Let (X, σ_1, σ_2) be 12-ASM bispace and $f : (X, \sigma_1, \sigma_2) \to (Y, \rho_1, \rho_2)$ is 21-continuous. If $\sigma_2 \leq \sigma_1$, then (Y, ρ_1, ρ_2) is 12-ASM.*

Definition 2.8. *[53] Let (X, σ_1, σ_2) and (Y, ρ_1, ρ_2) be bispaces and $f : (X, \sigma_1, \sigma_2) \to (Y, \rho_1, \rho_2)$ be a mapping. f is ij-strongly-θ-continuous if each $x \in X$ and every $U \in \rho_i$ such that $f(x) \in U$, there exists an open set $V \in \sigma_i$ such that $x \in V$ and $f(\sigma_j\text{-}cl(V)) \subset U$.*

Clearly, if f is an ij-strongly-θ-continuous mapping, then f is i-continuous.

Theorem 2.6. *Let (X,σ_1,σ_2) and (Y,ρ_1,ρ_2) be bispaces, and $f:(X,\sigma_1,\sigma_2)\to(Y,\rho_1,\rho_2)$ is ij-strongly-θ-continuous and surjective. Then, (Y,ρ_i) is SM.*

Proof. Let $B\subset Y$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of ρ_i-open covers of ρ_i-$cl(B)$. Let $A=f^{-1}(B)$ and $x\in\sigma_i$-$cl(A)$. We obtain $f(x)\in f(\sigma_i\text{-}cl(f^{-1}(B)))=\rho_i\text{-}cl(B)$ for all $x\in\sigma_i$-$cl(A)$, since f is i-continuous. Then, we can choose a $U^n_x\in\mathcal{D}_n$ such that $f(x)\in U^n_x$ for each $n\in\mathbb{N}$. Since f is ij-strongly-θ-continuous, there is a σ_i-open V^n_x such that $x\in V^n_x$ and $f(\sigma_j\text{-}cl(V^n_x))\subset U^n_x$. Then, $(\mathcal{D}^A_n)_{n\in\mathbb{N}}$ is a sequence of σ_i-open covers of σ_i-$cl(A)$ where $\mathcal{D}^A_n=\{V^n_x : x\in\sigma_i\text{-}cl(A)\}$. Since (X,σ_1,σ_2) is ij-ASM, there is $\mathcal{C}^A_n\subset\mathcal{D}^A_n$ such that $\mathcal{C}^A_n$ is a finite subset for each $n\in\mathbb{N}$ providing that $A\subset\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{C}^A_n$. Let F_n be a finite subset of σ_i-$cl(A)$ for each $n\in\mathbb{N}$ and let $\mathcal{C}^A_n=\{V^n_x : x\in F_n\}$. Then, $\mathcal{C}_n=\{U^n_x : x\in F_n\}$ is the finite subset of $\mathcal{D}_n$ for each $n\in\mathbb{N}$. Indeed, we have

$$\begin{aligned} f(f^{-1}(A)) = B &\subset f(\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n}\sigma_j\text{-}cl(V^n_x)) \\ &\subset \bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} f(\sigma_j\text{-}cl(V^n_x)) \\ &\subset \bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} U^n_x \\ &= \bigcup_{n\in\mathbb{N}}\bigcup \mathcal{C}_n. \end{aligned}$$

So, (Y,ρ_i) is SM. □

Since every ij-strogly-θ-continuous mapping is i-continuous, we can give the following result.

Corollary 2.2. *If $f:(X,\sigma_1,\sigma_2)\to(Y,\rho_1,\rho_2)$ is an i-continuous mapping, and (X,σ_1,σ_2) is ij-regular and an ij-ASM bispace, then (Y,ρ_i) is SM.*

What about the pre-images of ij-ASM bispaces? We need some definitions for looking at the behavior.

Definition 2.9. *[45] Let (X,σ_1,σ_2) and (Y,ρ_1,ρ_2) be bitopological spaces. A mapping $f:X\to Y$ is called d-closed if induced mappings $f_i:(X,\sigma_i)\to(Y,\rho_i)$ are closed for $i=1,2$.*

Definition 2.10. *A bispace (X,σ_1,σ_2) is called d-compact if the spaces (X,σ_i) are compact for $i=1,2$.*

Definition 2.11. *[54] Let (X,σ_1,σ_2) and (Y,ρ_1,ρ_2) be bispaces and $f:X\to Y$ is d-closed and d-continuous mapping. f is called perfect if for all $y\in Y$, the set $f^{-1}(y)$ is d-compact in X.*

Definition 2.12. *[55] Let (X,σ_1,σ_2) and (Y,ρ_1,ρ_2) be bispaces and $f:X\to Y$ be a mapping. f is called ij-preopen if $f(V)\subset\rho_i-int(\rho_j-cl(f(V)))$ for all $V\in\sigma_i$.*

Proposition 2.3. *[56] $f:(X,\sigma_1,\sigma_2)\to(Y,\rho_1,\rho_2)$ is an ij-preopen mapping if and only if $f^{-1}(\rho_i-cl(U))\subset\sigma_i-cl(f^{-1}(U))$ for all $U\in\rho_i$.*

Theorem 2.7. *Let (Y,ρ_1,ρ_2) be ij-ASM, and $f:(X,\sigma_1,\sigma_2)\to(Y,\rho_1,\rho_2)$ is a perfect ji-preopen mapping. Then, (X,σ_1,σ_2) is ij-ASM.*

Proof. Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be the sequence of open covers of σ_i-$cl(A)$. Let $B = f(A) \subset Y$ and $y \in \rho_i$-$cl(f(A))$. Then, there is a finite subset C_y^n of $\mathcal{D}_n$ for each $n \in \mathbb{N}$ such that $f^{-1}(y) \subset \bigcup C_y^n$. Let $\bigcup C_y^n = V_y^n$. Since f is i-closed, $U_y^n = Y \setminus f(X \setminus V_y^n)$ is a ρ_i-open neighbourhood of y. For every $n \in \mathbb{N}$, let $\mathcal{H}_n = \{U_y^n : y \in \rho_i\text{-}cl(f(A))\}$. Then, $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of ρ_i-open covers of ρ_i-$cl(f(A))$. Since (Y, ρ_i, ρ_j) is ij-ASM, there is finite $\mathcal{H}_n^* \subset \mathcal{H}_n$ for all n such that $f(A) \subset \bigcup_{n\in\mathbb{N}} \bigcup_{H\in\mathcal{H}_n^*} \rho_j$-$cl(H)$. Let $F_n \subset f(A)$ be finite for all $n \in \mathbb{N}$ and $\mathcal{H}_n^* = \{U_{y_i}^n : i \in F_n\}$. Then, $\mathcal{D}_n^* = \bigcup_{i\in F_n} C_{y_i}^n \subset \mathcal{D}_n$ is finite for all $n \in \mathbb{N}$. Then, since f is ji-preopen, we have the following:

$$\begin{aligned} A \subset f^{-1}(f(A)) &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{i\in F_n} f^{-1}(\rho_i\text{-}cl(U_{y_i}^n)) \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{i\in F_n} \sigma_j\text{-}cl(f^{-1}(U_{y_i}^n)) \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{i\in F_n} \sigma_j\text{-}cl(V_{y_i}^n) \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{i\in F_n} \sigma_j\text{-}cl(\cup C_{y_i}^n) \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{U^*\in\mathcal{D}_n^*} \sigma_j\text{-}cl(U^*). \end{aligned}$$

Hence, (X, σ_1, σ_2) is ij-ASM. □

Definition 2.13. *A mapping $f : (X, \sigma_1, \sigma_2) \to (Y, \rho_1, \rho_2)$ is called k-continuous if the inverse image of every ρ_i-open set is ij-regular open.*

Theorem 2.8. *A k-continuous surjection image of an ij-ASM bispace is ij-ASM.*

3. Union, subspaces and products

We consider the preservation of ij-ASM property under union, subspaces and products in this section.

Theorem 3.1. *Being ij-ASM bispace is closed under countable union.*

Proof. Let $\{(X_n, \sigma_{1n}, \rho_{1n}) : n \in \mathbb{N}\}$ be countable family of ij-ASM bispaces and $X = \bigcup_{n\in\mathbb{N}} X_n$. Suppose that τ and σ are the first and second topologies on X, respectively. Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be the sequence of τ-open covers of τ-$cl(A)$. Without loss of generality, we may assume that $A_n \subset X_n$ for each $n \in \mathbb{N}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$. Let N_n be an infinite subset of $\mathbb{N}$, $N_n \cap N_m = \emptyset$ for each $n, m \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{n\in\mathbb{N}} N_n$. Since σ_{in}-$cl(A_n) \subset \tau$-$cl(A)$ for each $n \in \mathbb{N}$, $\mathcal{S}_n = (\mathcal{D}_k : k \in N_n)$ is the sequence of σ_{in}-open covers of σ_{in}-$cl(A_n)$. Since $(X_n, \sigma_{1n}, \rho_{2n})$ is ij-ASM, there is finite $C_k \subset \mathcal{D}_k$ for each $k \in N_n$ and $n \in \mathbb{N}$ such that $A_n \subset \bigcup_{k\in N_n} \bigcup_{V\in C_k} \rho_{j_n}$-$cl(V)$ holds. Then $\mathcal{D} = \{\bigcup_{k\in N_n} \sigma\text{-}cl(C_k) : n \in \mathbb{N}\}$ is the desired cover of A. □

Theorem 3.2. *Every σ_i-closed and σ_j-open subspace of an ij-ASM bispace is ij-ASM.*

Proof. Let $(A, \sigma_{1A}, \sigma_{2A})$ be a σ_i-closed and σ_j-open subspace of ij-ASM (X, σ_1, σ_2). Let B any subset of A and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be sequence of σ_{iA}-open covers of σ_{iA}-$cl(B)$. Then we can choose V_U for each $U \in \mathcal{D}_n$ and $n \in \mathbb{N}$ such that $U = A \cap V_U$. Let $\mathcal{D}_n^* = \{V_U : U \in \mathcal{D}_n\}$. Since A is σ_i-closed, we have

$$\sigma_{iA}\text{-}cl(B) = A \cap \sigma_i\text{-}cl(B) = \sigma_i\text{-}cl(A \cap B) = \sigma_i\text{-}cl(B).$$

Then, $(\mathcal{D}_n^*)_{n\in\mathbb{N}}$ is sequence of σ_i-open covers of σ_i-$cl(B)$. Since (X, σ_1, σ_2) is ij-ASM, there is a finite subset C_n^* of $\mathcal{D}_n^*$ for each $n \in \mathbb{N}$ such that $\bigcup_{V_U\in C_n^*} \sigma_j\text{-}cl(V_U)$ is a cover of B. Let $C_n = \{U = A \cap V_U : V_U \in C_n^*\}$ for each $n \in \mathbb{N}$. As $A \in \sigma_j$ and for all $n \in \mathbb{N}$ and $U \in C_n$, we have

$$\sigma_{j_A}(U) = \sigma_{j_A}(A \cap V_U) = A \cap \sigma_j\text{-}cl(V_U)$$

holds and thus $B \subset \bigcup_{n\in\mathbb{N}} \bigcup_{U\in C_n} \sigma_{j_A}\text{-}cl(U)$. Hence $(A, \sigma_{1A}, \sigma_{2A})$ is ij-ASM. □

In this manner, being ij-ASM bispace is not hereditary property as the following example illustrates.

Example 3.1. *An ij-ASM bispace whose a subspace is not ij-ASM.*

Consider the set $X = [0, \Omega]$ is the set of ordinals such that $\alpha \leq \Omega$ for all $\alpha \in X$ where Ω denotes the first uncountable ordinal together with the order topology σ_1 and σ_2 is the discrete topology on X. Then the bispace (X, σ_1, σ_2) is 12-ASM, since (X, σ_1) is compact so it is ASM (see [47]). If we consider the subset $Y = X \setminus \{\Omega\}$ with its corresponding topologies σ_{1Y} and σ_{2Y}, the bispace $(Y, \sigma_{1Y}, \sigma_{2Y})$ is not 12-AL (see [57]), so by the Theorem 2.3, it is not 12-ASM.

Theorem 3.3. *Let (X, σ_1, σ_2) be ij-ASM bispace and (Y, ρ_1, ρ_2) be a d-compact bispace. Then $(X \times Y, \sigma_1 \times \rho_1, \sigma_2 \times \rho_2)$ is ij-ASM.*

Proof. Let A and B be any subsets of X and Y, respectively, and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be any sequence of $\sigma_i \times \rho_i$-open covers of $\sigma_i\text{-}cl(A) \times \rho_i\text{-}cl(B) = \sigma_i \times \rho_i\text{-}cl(A \times B)$. Without loss of generality, we can assume that $\mathcal{D}_n = \mathcal{A}_n \times \mathcal{B}_n$ where $\mathcal{D}_n$ is a σ_i-open cover of σ_i-$cl(A)$, and $\mathcal{B}_n$ is a ρ_i-open cover of ρ_i-$cl(B)$ for each $n \in \mathbb{N}$. Let $x \in \sigma_i$-$cl(A)$. Since Y is ρ_i-compact, ρ_i-$cl(B)$ is ρ_i-compact. Then, we can choose a finite subset C_n of $\mathcal{D}_n$ for each $n \in \mathbb{N}$ such that $\{x\} \times \rho_i\text{-}cl(B) \subset \bigcup C_n$. Say $C_n = \mathcal{A}_x^{(n)} \times \mathcal{B}_x^{(n)}$ for all n. If $U_x^{(n)} = \bigcap \mathcal{A}_x^{(n)}$, one can observe that

$$\{x\} \times \rho_i\text{-}cl(B) \subset \bigcup \left((\bigcap \mathcal{A}_x^{(n)}) \times \mathcal{B}_x^{(n)} \right) \subset \bigcup \left(\mathcal{A}_x^{(n)} \times \mathcal{B}_x^{(n)} \right)$$

for each $n \in \mathbb{N}$. Let $\mathcal{G}_n = \{U_x^{(n)} : x \in \sigma_i\text{-}cl(A)\}$ for each $n \in \mathbb{N}$. Then, $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a sequence of σ_i-open covers of σ_i-$cl(A)$. Since (X, σ_1, σ_2) is ij-ASM, there is a finite subset $\mathcal{H}_n$ of $\mathcal{G}_n$ such that $\mathcal{H}_n = \{U_{x_k^n}^{(n)} : k \in F_n\}$ where F_n is the finite subset of σ_i-$cl(A)$ for each $n \in \mathbb{N}$, and

$$A \subset \bigcup_{n\in\mathbb{N}} \bigcup \sigma_j\text{-}cl(\mathcal{H}_n)$$

holds. If we choose $\mathcal{D}_n^* = \bigcup \left(\mathcal{A}_{x_k^n}^{(n)} \times \mathcal{B}_{x_k^n}^{(n)} \right)$, then $\mathcal{D}_n^*$ is the finite subset of $\mathcal{D}_n$ and we have

$$\begin{aligned} A \times B \subset \sigma_i\text{-}cl(A) \times B &\subset \Big(\bigcup_{n\in\mathbb{N}} \bigcup \sigma_j\text{-}cl(\mathcal{H}_n) \Big) \times B \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{U\in\mathcal{D}_n^*} \Big(\sigma_j \times \rho_j\text{-}cl(U) \Big). \end{aligned}$$

So, $(X \times Y, \sigma_1 \times \rho_1, \sigma_2 \times \rho_2)$ is ij-ASM. □

Definition 3.1. *[6] An open cover $\mathcal{D}$ of a topological space (X, σ) is an ω-cover if $X \notin \mathcal{D}$ and each finite subset of X is contained in some element of $\mathcal{D}$.*

Theorem 3.4. *Let (X, σ_1, σ_2) be a bispace. The power bitopological space $(X^n, \sigma_1^n, \sigma_2^n)$ (see [58]) is ij-ASM if and only if for every $A \subset X$ and for every sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of σ_i-ω-covers of σ_i-$cl(A)$, there is a sequence $(C_n)_{n\in\mathbb{N}}$ where $C_n \subset \mathcal{D}_n$ is finite for each $n \in \mathbb{N}$ and for all finite subset F of A, there is at least $n \in \mathbb{N}$ and $V \in C_n$ such that $F \subset \sigma_j$-$cl(V)$.*

Proof. ($\Rightarrow$) Let $A \subset X$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of σ_i-ω-covers of σ_i-$cl(A)$. Let K_t be infinite subset of $\mathbb{N}$ with $K_t \cap K_n = \emptyset$ for all $t, n \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{t\in\mathbb{N}} K_t$. For every $n \in \mathbb{N}$ and $k \in K_t$, let $\mathcal{D}_k^t = \{U^t : U \in \mathcal{D}_k\}$. Then $(\mathcal{D}_k^t)_{k\in K_t}$ is a sequence of σ_i^t-open covers of $(\sigma_i\text{-}cl(A))^t = \sigma_i^t\text{-}cl(A^t)$. Since $(X^t, \sigma_1^t, \sigma_2^t)$ is ij-ASM, there is a finite subset $C_k^t \subset \mathcal{D}_k^t$ for each $k \in K_t$ and $A^t \subset \bigcup_{k\in K_t} \bigcup_{V\in C_k^t} \sigma_j^t$-$cl(V)$ holds. For every $k \in K_t$ and $V \in C_k^t$, we can choose $U_V \in \mathcal{D}_k$ such that $V = U_V^t$. Now say $C_k = \{U_V : V \in C_k^t\}$ for each $k \in K_t$. Then, the sequence $(C_k)_{k\in K_t}$ is the desired sequence. It obviously is that each C_k is finite subset of $\mathcal{D}_k$ and if $F = \{x_1, x_2, ..., x_p\} \subset A$, then there is an at least $k \in K_p$ and $V \in C_k^p$ such that $(x_1, x_2, ..., x_p) \in \sigma_j^p$-$cl(V)$. On the other hand, $V = U_V^p$ for an $U_V \in \mathcal{D}_k$. Then, we have

$$\sigma_j\text{-}cl(V) = \sigma_j^p\text{-}cl(U_V^p) = \left(\sigma_j\text{-}cl(U_V)\right)^p$$

and hence $F \subset \sigma_j$-$cl(U_V)$.

($\Leftarrow$) Let $A \subset X^t$ and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of σ_i^t-open covers of σ_i^t-$cl(A)$. Let $\mathcal{D}_n = \{U_k^{(n)} : k \in S_n\}$ for each $n \in \mathbb{N}$ and $A = A_1 \times A_2 \times ... \times A_t$. Let $F_p \subset \sigma_i$-$cl(A_p)$ be finite subset for each $p \in \{1, 2, ..., t\}$. Then, $F_1 \times F_2 \times ... \times F_t$ is a finite subset of X^t. Then, there is a finite subset $S_n^{F_1} \subset S_n$ such that $F_1 \times F_2 \times ... \times F_t \subset \bigcup_{k\in S_n^{F_1}} U_k^{(n)}$. On the other hand, there is a σ_i-open set V_{F_p} for each $p \in \{1, 2, ..., t\}$ such that $F_p \subset V_{F_p}$ and $V_{F_1} \times V_{F_2} \times ... \times V_{F_p} \subset \bigcup_{k\in S_n^{F_1}} U_k^{(n)}$ (see [47]). Then, for all finite subsets F_{A_p} of σ_i-$cl(A_p)$ for each $p \in \{1, 2, ..., t\}$, $C_n^{(p)} = \{V_{F_{A_p}} : F_{A_p} \subset \sigma_i\text{-}cl(A_p)$ is finite $\}$ is a σ_i-ω-cover of σ_i-$cl(A_p)$ for each $n \in \mathbb{N}$. By assumption, there is finite subset $\mathcal{G}_n^{(p)} \subset C_n^{(p)}$ for each $n \in \mathbb{N}$ and $p \in \{1, 2, ..., t\}$, and for every finite subset P of A_p, one can find a $n \in \mathbb{N}$ and $G \in \mathcal{G}_n^{(p)}$ such that $P \subset \sigma_j$-$cl(G)$. Let $R_n^{(p)}$ be a finite index set for each $n \in \mathbb{N}$ and $p \in \{1, 2, ..., t\}$. Assume that, $\mathcal{G}_n^{(p)} = \{V_{F_{A_p}^r} : r \in R_n^{(p)}\}$. In this sense, if $K_n = \{k \in S_n^{F_{A_p}^r} : p \in \{1, 2, ...t\}$ and $r \in R_n^{(p)}\}$, then

$$\bigcup_{n\in\mathbb{N}} \bigcup_{k\in K_n} \sigma_j^t\text{-}cl(U_k^{(n)}) \supset A$$

holds. To see this, let $x = (x_1, x_2, ..., x_t) \in A$. Then, $\{x_p\} \subset A_p$ for each $p \in \{1, 2, ..., t\}$. Thus, there is $n_{x_p} \in \mathbb{N}$ and $G_{x_p} \in \mathcal{G}_{n_{x_p}}^{(p)}$ such that $\{x_p\} \subset \sigma_j$-$cl(G_{x_p})$. Let $G_{x_p} = V_{F_{A_p}^{r_{x_p}}}$ for some $r_{x_p} \in R_{n_{x_p}}^{(p)}$. Then, we have

$$\begin{aligned}\{(x_1, x_2, ..., x_p)\} &\subset \sigma_j\text{-}cl\left(V_{F_{A_1}^{r_{x_1}}}\right) \times ... \times \sigma_j\text{-}cl\left(V_{F_{A_t}^{r_{x_t}}}\right)\\ &\subset \sigma_j^t\text{-}cl\left(V_{F_{A_1}^{r_{x_1}}} \times ... \times V_{F_{A_t}^{r_{x_t}}}\right)\\ &\subset \bigcup_{k\in S_n^{F_{A_p}^{r_{x_1}}}} \sigma_j^t\text{-}cl(U_k^{(n)}).\end{aligned}$$

Hence, there is $k \in K_n$ such that $x \in \sigma_j^t$-$cl(U_k^{(n)})$. So, $(X^t, \sigma_1^t, \sigma_j^t)$ is ij-ASM. □

Question 3.1. *An ij-ASM bispace (X, σ_1, σ_2) such that $(X^2, \sigma_1^2, \sigma_2^2)$ is not ij-ASM?*

4. ij-Almost P_γ-sets

The concept of a γ-set was introduced by Gerlits and Nagy in [59]. Later, in [15] Kocev introduced the concept of an almost γ-set and studied. In this section, we will give the definition of ij-almost P_γ-set based upon the definitions of γ-set and almost γ-set. We will investigate the characterization of this class of ij-almost P_γ-sets with ij-regular open sets and their preservation under d-continuous surjection.

Definition 4.1. *Let (X, σ_1, σ_2) be a bitopological space and $A \subset X$ and let $\mathcal{D}$ be an infinite σ_i-open cover of A. If the set $\{U \in \mathcal{D} : x \notin \sigma_j\text{-}cl(U)\}$ is finite for all $x \in A$, then we say that $\mathcal{D}$ is an ij-almost P_γ-cover of A.*

Definition 4.2. *A bispace (X, σ_1, σ_2) is called ij-almost P_γ-set (shortly ij-$AP_\gamma S$ if for all $A \subset X$ and for any sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of σ_i-ω-covers of σ_i-$cl(A)$, there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{D}_n$ for each $n \in \mathbb{N}$ and the set $\{U_n : n \in \mathbb{N}\}$ is an ij-almost P_γ-cover of A.*

Based upon this definiton, We can give following proposition,

Proposition 4.1. *Let (X, σ_1, σ_2) be a bispace. If (X, σ_1) is γ-set (see [59]), then (X, σ_1, σ_2) is ij-$AP_\gamma S$.*

Remark 4.1. *Statement converse in Proposition* 4 *is not true in general.*

Example 4.1. *Endow the real line by the two topologies: σ_1 is the particular point topology (see Example 2.3), and σ_2 is the indiscrete topology. Then, the bispace $(\mathbb{R}, \sigma_1, \sigma_2)$ is clearly 12-$AP_\gamma S$, while (X, σ_1) is not a γ-set.*

Theorem 4.1. *A bispace (X, σ_1, σ_2) is ij-$AP_\gamma S$ if and only if for every $A \subset X$ and every sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of σ_i-ω-open covers of σ_i-$cl(A)$ by ij-regular open subsets of X, there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{D}_n$ for each $n \in \mathbb{N}$ and the set $\{U_n : n \in \mathbb{N}\}$ is an ij-almost P_γ-cover of A.*

Proof. ($\Rightarrow$) It is an obvious consequence from the fact that every ij-regular open set is σ_i-open.
($\Leftarrow$) Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of σ_i-ω covers of σ_i-$cl(A)$. Then, $(\mathcal{D}^*_n)_{n\in\mathbb{N}}$ is the sequence of σ_i-ω covers of σ_i-$cl(A)$ by ij-regular open sets of X where $\mathcal{D}^*_n = \{\sigma_i\text{-}int(\sigma_j\text{-}cl(U) : U \in \mathcal{D}_n\}$ for eacn $n \in \mathbb{N}$. There exists a sequence $(U^*_n)_{n\in\mathbb{N}}$ with $U^*_n \in \mathcal{D}^*_n$ for every $n \in \mathbb{N}$ and $\mathcal{D}^* = \{U^*_n : n \in \mathbb{N}\}$ is P_γ-cover of the set A. On the other hand, we can choose an $U_n \in \mathcal{D}_n$ such that $U^*_n = \sigma_i\text{-}int(\sigma_j\text{-}cl(U_n)$. Then, one can easily see that $\mathcal{D} = \{U_n : n \in \mathbb{N}\}$ is the desired cover of A. Hence, (X, σ_1, σ_2) is ij-A$P_\gamma S$. □

Theorem 4.2. *d-continuous surjection of an ij-$AP_\gamma S$ bispace is ij-$AP_\gamma S$.*

Proof. Let (X, σ_1, σ_2) be ij-A$P_\gamma S$ and $f : (X, \sigma_1, \sigma_2) \to (Y, \rho_1, \rho_2)$ be a d-continuous surjection. Let $B \subset Y$, $f^{-1}(B) = A$, and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of ρ_i-ω-open covers of ρ_i-$cl(B)$ by ij-regular open subsets of Y. Since f is i-continuous, $\sigma_i\text{-}cl(A) \subset f^{-1}(\rho_i\text{-}cl(B))$ holds, and $(C_n)_{n\in\mathbb{N}}$ is the sequence of σ_i-ω covers of σ_i-$cl(A)$ where $C_n = \{f^{-1}(U) : U \in \mathcal{D}_n\}$ for each $n \in \mathbb{N}$. Since (X, σ_1, σ_2) is ij-A$P_\gamma S$, there is a sequence $(V_n)_{n\in\mathbb{N}}$ providing that $V_n \in C_n$ for each $n \in \mathbb{N}$ and $C = \{V_n : n \in \mathbb{N}\}$ is ij-almost P_γ-cover of A. On the other side, there is a $U_n \in \mathcal{D}_n$ such that $V_n = f^{-1}(U_n)$ for each $n \in \mathbb{N}$. Then, $\mathcal{D}^* = \{U_n : n \in \mathbb{N}\}$ is an ij-almost P_γ-cover of B. To see this, let $y \in B$ and $f(x) = y$ for some $x \in A$. Since C is an ij-almost P_γ-cover of A, the set $F_x = \{V_n \in C : x \notin \sigma_j\text{-}cl(V_n)\}$ is finite. Then, there is $n_0 \in \mathbb{N}$ such that $x \in \sigma_j\text{-}cl(V_n)$ for all $n > n_0$. Then, $f(x) = y \in f(\sigma_j\text{-}cl(V_n))$, and thus $y \in \rho_j\text{-}cl(U_n)$ for all $n > n_0$. Therefore, we conclude that the $F^y = \{U_n \in \mathcal{D}^* : y \notin \rho_j\text{-}cl(U_n)\}$ is finite, and since C is infinite, $\mathcal{D}^*$ is infinite. So, (Y, ρ_1, ρ_2) is ij-A$P_\gamma S$. □

5. Conclusions

In this paper, we dealed with the almost-set-Mengerness in bitopological spaces. Further investigations may be the similar properties of almost-set-Hurewicz and almost-set-Rothberger property (we began to investigate) in bitopological spaces. we now give the related definitions as the followings.

Definition 5.1. *A bitopological space* (X, σ_1, σ_2) *is called:*

(1) ij-almost-set-Hurewicz (for short, ij-ASH) *if for all* $A \subset X$ *and for every sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of* σ_i*-open covers of* σ_i*-cl*(A) *there is a sequence* $(C_n)_{n\in\mathbb{N}}$ *such that* C_n *is a finite subset of* $\mathcal{D}_n$ *and each* $x \in A$ *belongs to all but finitely many sets* σ_j*-cl*$(\cup C_n)$*. (in other words, the set* $\{\cup C_n : n \in \mathbb{N}\}$ *is* ij*-almost* P_γ*-cover of A.)*

(2) ij-almost-set-Rothberger (ij-ASR) *if for all* $A \subset X$ *and for every sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of* σ_i*-open covers of* σ_i*-cl*(A) *there is a sequence* $(U_n)_{n\in\mathbb{N}}$ *such that* $U_n \in \mathcal{D}_n$ *and* $A \subset \bigcup_{n\in\mathbb{N}} \sigma_j$*-cl*$(U_n)$*.*

Also, we give the definition of an ij-weakly-set-Menger bispace as follows:

Definition 5.2. (X, σ_1, σ_2) *is said to be* ij*-weakly-set-Menger (for short,* ij*-WSM) if for all* $A \subset X$ *and for every sequence* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *of* σ_i*-open covers of* σ_i*-cl*(A)*, there is a sequence* $(C_n)_{n\in\mathbb{N}}$ *such that* $C_n \subset \mathcal{D}_n$ *is a finite subset for each* $n \in \mathbb{N}$ *and* $A \subset \sigma_j$*-cl*$\Big(\bigcup_{n\in\mathbb{N}} \bigcup C_n\Big)$

It would be interesting to study and investigate the properties of ij-WSM bispaces in bitopological context as well as the relations between those kind of bispaces and ij-ASM bispaces. Furthermore, if these properties have game-theoretic characterization can be scrutinized in bitopological context and those ones can open a way to applicable area. The possible applications of statistical convergence to the open covers of topological spaces and selection properties were given in [60, 61]. This notions also can be studied and extended under the convergence in binary metric spaces and their induced topologies [62]. Also, the results obtained may be generalized to fuzzy bitopological spaces and associated with the fixed point theory [63, 64].

We also define ij-weakly-set-Hurewicz (ij-WSH) and ij-weakly-set-Rothberger (ij-WSR) bitopological spaces in a similar way.

Conflict of interest

The authors declare that they have no competing interests.

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