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Research article

Nonlinear normal modes in a network with cubic couplings

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Abstract: We consider a network with cubic couplings. This is related to the well known Fermi-Pasta-Ulam-Tsingou model. We show that nonlinear periodic orbits extend from particular eigenvectors of the graph Laplacian, these are termed *nonlinear normal modes*. We present large classes of graphs where this occurs. These are the graphs whose Laplacian eigenvectors have components in $\{1, -1\}$ (bivalent), and $\{1, -1, 0\}$ with a condition (soft-regular trivalent), the bipartite complete graphs and their extensions obtained by adding an edge between vertices having the same component. Finally, we study the stability of these solutions for chains and cycles.

Keywords: networks; graph Laplacian; nonlinear normal mode (NNM); Fermi-Pasta-Ulam-Tsingou (FPUT) model; graph spectra and applications
Mathematics Subject Classification: 94C15, 70K75, 34A05

1. Introduction

Normal modes are a standard description of linear mechanical systems. For networks, these are the eigenvectors of the graph Laplacian matrix describing the couplings between the different coordinates of the system. These eigenvectors are orthogonal, correspond to real frequencies and allow to decouple the motion into each mode which then evolve separately from each other.

Nonlinearities will, in general, couple these normal modes. In special cases however, the linear normal modes give rise to periodic nonlinear orbits, commonly called *Nonlinear Normal Modes (NNM)*.

For the ϕ^4 on-vertex nonlinearity, such eigenvectors were studied by Aoki [1] for chains and cycles. The present authors extended this study to arbitrary networks [2] and showed that only eigenvectors composed of $\{-1, 0, 1\}$ (so-called bivalent and trivalent) extend to nonlinear periodic orbits. They also examined the stability of such solutions.

For on-edge nonlinearities such as the celebrated Fermi-Pasta-Ulam-Tsingou system of masses coupled by cubic response oscillators (see [3] for a review), such nonlinear periodic orbits play an active role in the distribution of energy among the Fourier modes. These NNM have been investigated for chain or cycle graphs by Chechin et al [4] using powerful group theoretical methods. A stability analysis has been performed explicitly for these periodic orbits by Bountis et al [5]. In another context, nonlinear periodic orbits continued from normal modes have been studied for vibrating mechanical systems, see the extensive review by [6].

In this article, we examine the Fermi-Pasta-Ulam-Tsingou model on arbitrary networks and find the conditions for the existence of such nonlinear normal modes. We show that bivalent and trivalent-soft-regular eigenvectors give rise to nonlinear normal modes. We also identify other eigenvectors giving nonlinear normal modes (complete bi-partite graphs). This generalizes what was done in the literature on chains and cycles. Finally we consider the stability of these nonlinear normal modes and derive the linearized equations around these modes. These decouple for the cycles and we give their explicit form. We conclude the article with a number of numerical results indicating stability or instability of these nonlinear normal modes.

The article is organized as follows. Section 2 presents the nonlinear normal modes, definition and properties. Graphs yielding nonlinear normal modes are given in section 3 and section 4 studies the stability of these solutions by linearization and gives some numerical results.

2. Nonlinear normal modes: definition and properties

We consider the nonlinear graph wave equation with a cubic intersite nonlinearity, known as the Fermi-Pasta-Ulam-Tsingou (FPUT) model [7] on a connected graph $\mathcal{G}$ with N nodes

$$\ddot{u}_i = -(\Delta u)_i - \sum_{k\sim i}(u_i - u_k)^3, \quad i \in \{1, \ldots, N\}, \tag{2.1}$$

where $u = (u_1, \ldots, u_N)^T$ is the field amplitude, $\ddot{u} \equiv \frac{d^2 u}{dt^2}$, $k \sim i$ indicates adjacency of vertices k and i, the sum on the right is taken over the neighbors k of i, and where Δ is the graph Laplacian [8]. This $N \times N$ matrix is $\Delta = D - A$, where A is the adjacency matrix such that $A_{ij} = 1$ if nodes i and j are connected ($i \neq j$) and $A_{ij} = 0$ otherwise, and D is the diagonal matrix where the entry $d_i = \sum_{j=1}^{N} A_{ij}$ is the degree of vertex i. Note that the matrix vector product Δu has the components

$$-(\Delta u)_i = -d_i u_i + \sum_{k\sim i} u_k.$$

This inhomogeneous Fermi-Pasta-Ulam-Tsingou (FPUT) model was derived by Panayotaros and Martinez-Farias [9] from a nonlinear elastic network model of protein vibrations, see also [10]. It is also an extension to a general graph of the Fermi-Pasta-Ulam-Tsingou lattice model [7]. In another context, equation (2.1) is a reduction of a model of coupled phase oscillators where the sine term is replaced by its Taylor expansion. Such systems were introduced by Kuramoto [11], they describe for example an electrical grid [12] or an array of Josephson junctions [13]. For the original Fermi-Pasta-Ulam-Tsingou model, studies concentrated on lattices where the graph Laplacian $(\Delta u)_i = u_{i+1} - 2u_i + u_{i-1}$ (for a one-dimensional lattice *i.e.* chain) is a finite difference discretization of the continuous Laplacian. We formulate the FPUT model using the graph Laplacian to describe general networks of arbitrary topology.

To illustrate the meaning of the system of differential equations (2.1), consider the graph shown in Figure 1 with four vertices and four edges.

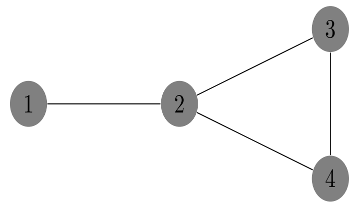

Figure 1. A paw graph.

The equations of motion corresponding to (2.1) are

$$\begin{aligned} \ddot{u}_1 &= -u_1 + u_2 - (u_1 - u_2)^3, \\ \ddot{u}_2 &= -3u_2 + u_1 + u_3 + u_4 - (u_2 - u_1)^3 - (u_2 - u_3)^3 - (u_2 - u_4)^3, \\ \ddot{u}_3 &= -2u_3 + u_2 + u_4 - (u_3 - u_2)^3 - (u_3 - u_4)^3, \\ \ddot{u}_4 &= -2u_4 + u_2 + u_3 - (u_4 - u_2)^3 - (u_4 - u_3)^3. \end{aligned} \tag{2.2}$$

Since the graph Laplacian Δ is a real symmetric positive-semi definite matrix, it is natural, following [14], to expand u using a basis of the eigenvectors v^j of Δ, such that

$$\Delta v^j = \lambda_j v^j. \tag{2.3}$$

The vectors v^j can be chosen orthogonal with respect to the scalar product in $\mathbb{R}^N$ and the eigenvalues are positive $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N$. The first eigenvalue $\lambda_1 = 0$ (the graph is connected) corresponds to the so-called Goldstone mode [2] whose components are equal on a network $v^1 = (1, 1, \ldots, 1)^T$.

Let us find the condition for the existence of a nonlinear periodic solution of (2.1), following [1], of the explicit form

$$u(t) = a_j(t) v^j, \tag{2.4}$$

the equations of motions (2.1) reduce to

$$\ddot{a}_j v_i^j = -\lambda_j a_j v_i^j - a_j^3 \sum_{k \sim i} \left(v_i^j - v_k^j\right)^3. \tag{2.5}$$

Two situations occur for a vertex i

(i) if $v_i^j = 0$ (soft nodes [14]), then we have

$$\sum_{k\sim i} \left(v_k^j\right)^3 = 0. \tag{2.6}$$

(ii) if $v_i^j \neq 0$ then dividing (2.5) by v_i^j we obtain

$$\ddot{a}_j = -\lambda_j a_j - \left[\frac{1}{v_i^j} \sum_{k\sim i} \left(v_i^j - v_k^j\right)^3\right] a_j^3. \tag{2.7}$$

These equations should be independent of the vertex i and this imposes

$$\frac{1}{v_i^j} \sum_{k\sim i} \left(v_i^j - v_k^j\right)^3 = \gamma_j, \quad \forall i \in \{1, \ldots, N\}, \tag{2.8}$$

where γ_j is a constant.

This gives rise to the following definition.

Definition 2.1. *The solution (2.4) associated to an eigenvector v^j of a graph is called a nonlinear normal mode (NNM) if $\sum_{k\sim i} \left(v_k^j\right)^3 = 0$ for $v_i^j = 0$ and $\frac{1}{v_i^j} \sum_{k\sim i} \left(v_i^j - v_k^j\right)^3 = \gamma_j$ constant for $v_i^j \neq 0$.*

For completeness, we recall the following definitions

Definition 2.2 (Bivalent graph [15])**.** *A graph is bivalent if there exists an eigenvector of the graph Laplacian composed from $-1, +1$. Such a vector is called bivalent.*

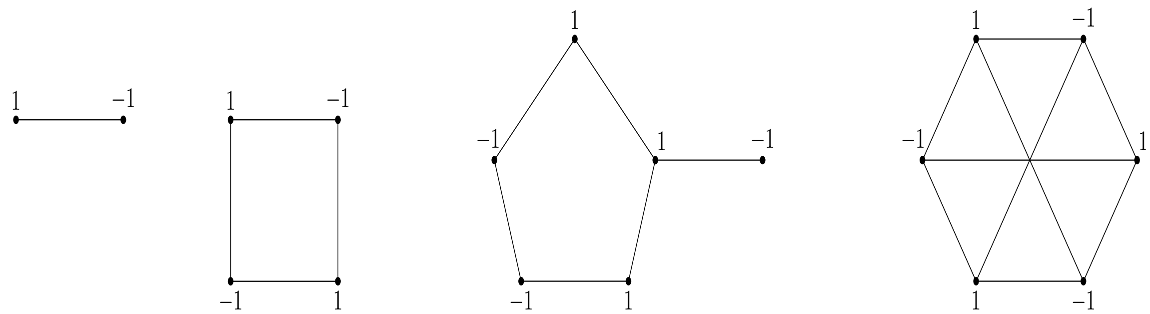

Figure 2. Examples of bivalent graphs.

Definition 2.3 (Trivalent graph [15])**.** *A graph is trivalent if there exists an eigenvector of the graph Laplacian composed from $-1, 0, +1$. Such a vector is called trivalent.*

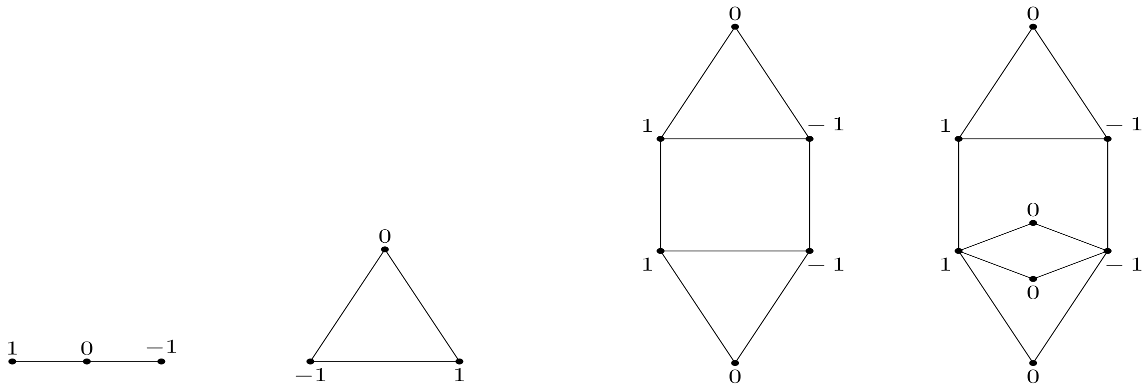

Figure 3. Examples of trivalent graphs.

Definition 2.4 (Trivalent-soft-regular graph [15])**.** *A graph is trivalent-soft-regular if there exists an eigenvector of the graph Laplacian composed from* $-1, 0, +1$ *such that the vertices with non zero component have same degree.*

The graph on the left of Figure 4 is 3-soft regular for the eigenvector $(0, 1, 1, 0, -1, -1)^T$ since all the non-zero vertices have the same degree 3. The graph on the right of Figure 4 is non-soft regular for the eigenvector $(0, 1, 1, 0, -1, -1, 0, 0)^T$ since the non-zero vertices have different degrees.

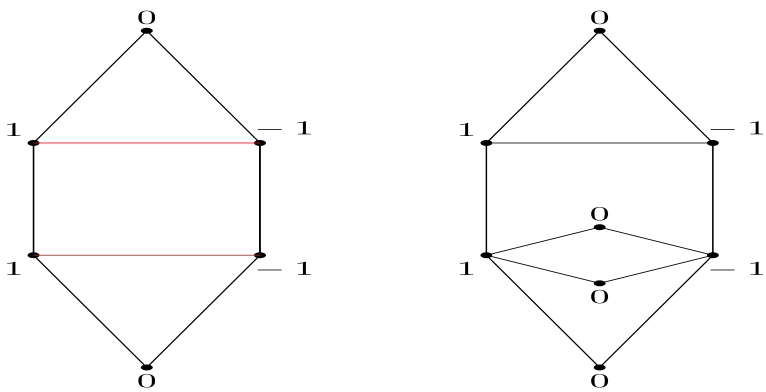

Figure 4. 3-soft regular graph for the Laplacian eigenvector $(0, 1, 1, 0, -1, -1)^T$ (left). Non-soft regular graph for the Laplacian eigenvector $(0, 1, 1, 0, -1, -1, 0, 0)^T$ (right).

Merris proved the following theorem,

Theorem 2.5 (**Link between two equal nodes** [16])**.** *Let* v *be an eigenvector of* $\Delta(\mathcal{G})$ *affording an eigenvalue* λ*. If* $v_i = v_j$*, then* v *is an eigenvector of* $\Delta(\mathcal{G}')$ *affording the eigenvalue* λ*, where* $\mathcal{G}'$ *is the graph obtained from* $\mathcal{G}$ *by deleting or adding the edge* e_{ij} *depending whether* e_{ij} *is an edge of* $\mathcal{G}$ *or not.*

Figure 5 shows how the transformation (Theorem 2.5) can be used to extend an eigenvector and its eigenvalue to the transformed graphs by adding edges (represented by red lines) between nodes having the same value.

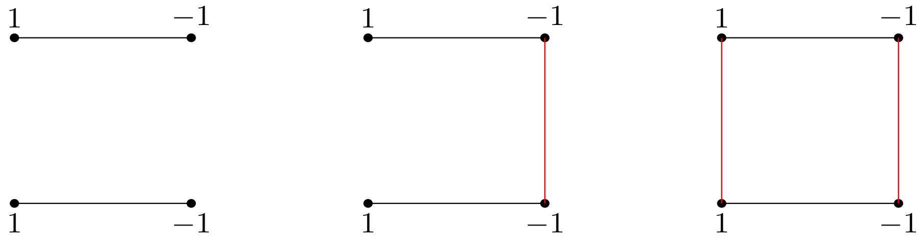

Figure 5. Three graphs obtained by adding or deleting edges between equal nodes, affording (the same eigenvalue) $\lambda = 2$.

Theorem 2.6. *The property (NNM) is preserved by the transformation (Theorem 2.5).*

The proof is elementary because adding a link between i and k gives $v_i^j - v_k^j = 0$.

Consider an eigenvector v and its normalization $v' = \frac{v}{\|v\|}$. We have the following property

$$\gamma_v = \| v \|^2 \gamma_{v'}. \tag{2.9}$$

The eigenvectors presented in the article are not normalized for clarity.

Finally, we introduce the definition of Alternate perfect matching and a theorem. These were derived in reference [15] and are presented here for completeness.

Definition 2.7 (Perfect matching)**.** *A perfect matching of a graph $\mathcal{G}$ is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.*

Definition 2.8 (Alternate perfect matching)**.** *An alternate perfect matching for a vector v on the nodes of a graph $\mathcal{G}$ is a perfect matching for the nonzero nodes such that edges e_{ij} of the matching satisfy $v_i = -v_j$ $(\neq 0)$.*

The left of Figure 4 shows the alternate perfect matching (represented by red lines) for the eigenvector $(0, -1, -1, 0, 1, 1)^T$ on the nodes of the 6-cycle.

Theorem 2.9 (**Add/Delete an alternate perfect matching** [16])**.** *Let v be an eigenvector of $\Delta(\mathcal{G})$ affording an eigenvalue λ. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by adding (resp. deleting) an alternate perfect matching for v. Then, v is an eigenvector of $\Delta(\mathcal{G}')$ affording the eigenvalue $\lambda + 2$ (resp. $\lambda - 2$).*

Adding an alternate perfect matching is illustrated in Figure 6. This transformation preserves the soft regularity of the graph and increases the eigenvalue by 2.

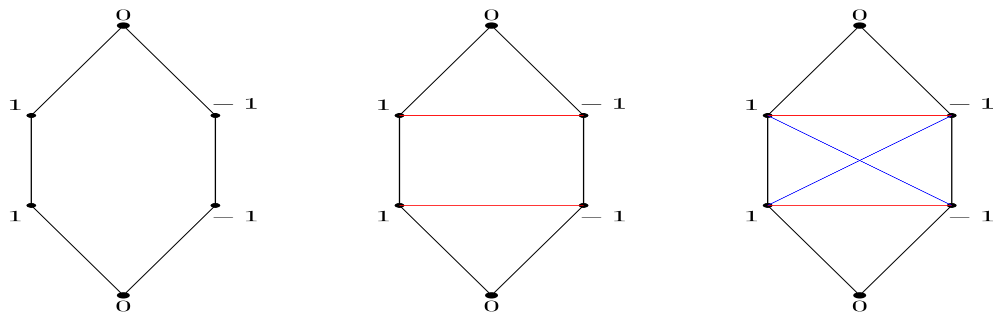

Figure 6. Graphs obtained by adding an alternate perfect matching for the eigenvector $(0, 1, 1, 0, -1, -1)^T$. The eigenvalues are $\lambda = 1$ (left), $\lambda = 3$ (middle) and $\lambda = 5$ (right).

3. Graphs yielding nonlinear normal modes

In this section, we present large classes of graphs that verify condition (2.8). These are the bivalent and trivalent-soft-regular and complete bipartite graphs and their extension by adding a link between two equal vertices (Theorem 2.5). The chains and cycles also posess NNM; their eigenvectors can be bivalent, trivalent or can form a complete bipartite graph.

3.1. Bivalent graphs

Bivalent eigenvectors which are identified for the discrete Φ^4 model in [2], satisfy the condition (2.8) and yield nonlinear normal modes also for the FPUT model.

Take a bivalent graph. It can be reduced using the property (Theorem 2.5) to a regular bipartite graph [15]. Then condition (NNM) is satisfied with

$$\gamma_j = 2^3 d_j, \tag{3.1}$$

where d_j is the degree which is independant of the vertex i.

3.2. Trivalent soft regular graphs

Trivalent soft regular graphs satisfy the condition (2.8), where

$$\gamma_j = 8d_j - 7s_j, \tag{3.2}$$

for nonsoft nodes.

Trivalent non-soft regular graphs do not satisfy the condition (2.8). For example consider the smallest trivalent non-soft regular graphs shown in Figure 7. The condition (2.8) is not satisfied for these graphs, since $\gamma_1 = 16, \gamma_3 = 10$ (left) and $\gamma_1 = 2^3, \gamma_3 = 2$ (right). General trivalent non-soft regular graphs have the same structure as the two graphs of Figure 7.

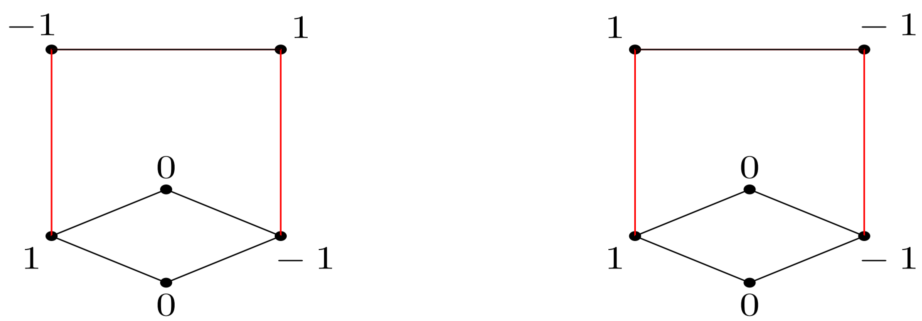

Figure 7. The smallest trivalent non-soft regular graphs.

These graphs are obtained from the elementary graphs of Figure 8 via the link transformation (Theorem 2.5) (right panel of Figure 7) and the alternate perfect matching (Theorem 2.9) (left panel of Figure 7). The eigenvalues are $\lambda = 4$ (left) and $\lambda = 2$ (right).

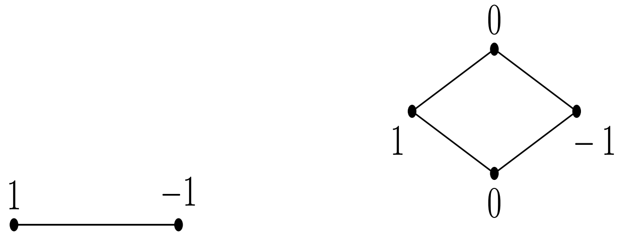

Figure 8. The smallest bivalent graph and the soft-regular trivalent graph for the same eigenvalue $\lambda = 2$.

3.3. Bipartite complete graphs

Definition 3.1 (Complete bipartite graph $K_{n,N-n}$)**.** *A complete bipartite graph $K_{n,N-n}$ is such that every vertex of the set $\{1, \dots n\}$ is connected to every vertex of the set $\{n+1, \dots N\}$.*

The eigenvalues with their multiplicities denoted as exponents are

$$0^1, \;\; n^{N-n-1}, \;\; (N-n)^{n-1}, \;\; N^1.$$

Eigenvectors for n can be chosen as $e^{n+1} - e^i$ $(i = n+2, \dots, N)$.
Eigenvectors for $N-n$ can be chosen as $e^1 - e^i$ $(i = 2, \dots, n)$.
The eigenvector for N is $(N-n, \dots, N-n, -n, \dots, -n)^T$.

The two first classes of eigenvectors are trivalent-soft-regular and therefore give rise to a NNM. The last eigenvector also gives rise to a NNM because it satisfies the condition (2.8) where

$$\gamma = N^3. \tag{3.3}$$

3.4. Examples

3.4.1. Chains

We have the following NNM for chains:

$$N \text{ even}, \quad v^{\frac{N}{2}+1} = (1, -1, -1, 1, \dots)^T, \quad \lambda_{\frac{N}{2}+1} = 2. \tag{3.4}$$

$$N \bmod 3 = 0, \quad v^{\frac{N}{3}+1} = (1, 0, -1, -1, 0, 1 \dots),^T \quad \lambda_{\frac{N}{3}+1} = 1. \tag{3.5}$$

$$N \bmod 3 = 0, \quad v^{\frac{2N}{3}+1} = (1, -2, 1, | \, 1, -2, 1, | \dots, | \, 1, -2, 1)^T, \quad \lambda_{\frac{2N}{3}+1} = 3, \;\; \gamma_{\frac{2N}{3}+1} = 3^3. \tag{3.6}$$

3.4.2. Cycles

For cycles, the following are NNM:

$$N \bmod 2 = 0, \quad v^N = (1,-1,1,-1,\dots)^T, \quad \lambda_N = 4. \tag{3.7}$$

$$N \bmod 4 = 0, \quad \lambda_{\frac{N}{2}} = \lambda_{\frac{N}{2}+1} = 2, \quad v^{\frac{N}{2}} = (1,0,-1,0,1,0,-1,0\dots)^T, \tag{3.8}$$

$$v^{\frac{N}{2}+1} = (0,1,0,-1,0,1,0,-1\dots)^T. \tag{3.9}$$

$$N \bmod 3 = 0, \quad v^{\frac{2N}{3}+1} = (0,1,-1,0,1,-1\dots)^T, \quad \lambda_{\frac{2N}{3}+1} = 3. \tag{3.10}$$

$$N \bmod 6 = 0, \quad v^{\frac{N}{3}+1} = (0,1,1,0,-1,-1,0,1,1,0,-1,-1\dots)^T, \quad \lambda_{\frac{N}{3}+1} = 1. \tag{3.11}$$

$$N \bmod 3 = 0, \quad v^{\frac{2N}{3}} = (2,-1,-1,|\,2,-1,-1\,|\dots,|\,2,-1,-1)^T, \quad \lambda_{\frac{2N}{3}} = 3, \;\; \gamma_{\frac{2N}{3}} = 3^3. \tag{3.12}$$

3.4.3. Complete graphs

An sub-class of trivalent regular graphs are the complete graphs.

We recall the definitions of a complete graph.

Definition 3.2 (Complete graph K_N)**.** *A clique or complete graph K_N is a graph where every pair of distinct vertices is connected by a unique edge.*

The clique K_N has eigenvalue N with multiplicity $N-1$ and eigenvalue 0. The eigenvectors for eigenvalue N can be chosen as $v^k = e^1 - e^k$, $k = 2,\dots,N$. These eigenvectors are trivalent soft so the results above apply and the $N-1$ eigenvectors give rise to nonlinear normal modes.

Note that these eigenvectors are not orthogonal. Due to this, it is likely that these solutions are unstable.

3.4.4. Star graphs

Star graphs S_{N-1} are the bipartite complete graphs $K_{1,N-1}$. All the eigenvectors $v^2,\dots,v^N$ extend to nonlinear normal modes. The normal modes and the corresponding constants γ_j are

$$\begin{aligned}
&\lambda_2 = 1, \;\; v^2 = (0,1,-1,0,\dots,0)^T, \;\; \gamma_2 = 1,\\
&\lambda_3 = 1, \;\; v^3 = (0,0,1,-1,0,\dots,0)^T, \;\; \gamma_3 = 1,\\
&\lambda_{N-1} = 1, \;\; v^{N-1} = (0,0,\dots,0,1,-1)^T, \;\; \gamma_{N-1} = 1,\\
&\lambda_N = N, \;\; v^N = (N-1,-1,\dots,-1)^T, \;\; \gamma_N = N^3.
\end{aligned} \tag{3.13}$$

Note that the eigenvectors associated to the eigenvalue 1 are not orthogonal. On the other hand, the eigenvector corresponding to the eigenvalue N is orthogonal to the other eigenvectors.

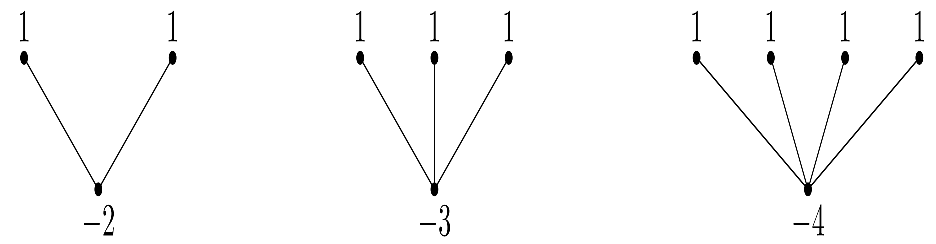

Figure 9. The Star graphs S_2, S_3 and S_4.

Figure 9 shows the star graphs S_2, S_3 and S_4 and the associated NNM.

4. Linearization around the nonlinear normal modes

To analyse the stability of (2.7), we perturb a nonlinear mode $w = a_j(t)v^j$ satisfying (2.7) and write

$$u = w + y,$$

where $||y|| \ll ||w||$. Plugging the above expression into (2.1), we get for each coordinate i

$$\ddot{y}_i = -(\Delta y)_i - \sum_{k\sim i}\left[3\,(w_i - w_k)^2(y_i - y_k) + 3\,(w_i - w_k)(y_i - y_k)^2 + (y_i - y_k)^3\right], \tag{4.1}$$

where we have used the fact that w is a solution of (2.1).

Equation (4.1) can be linearized to

$$\ddot{y}_i = -(\Delta y)_i - 3\sum_{k\sim i}(w_i - w_k)^2(y_i - y_k). \tag{4.2}$$

Using relation (3), we obtain the linearized equations

$$\ddot{y}_i = -(\Delta y)_i - 3a_j^2\sum_{k\sim i}(v_i^j - v_k^j)^2\,(y_i - y_k). \tag{4.3}$$

In general, these do not decouple. For cycles however, they do and we give some details in the next section.

4.1. Decoupled linearization equations for cycles

1. For the bivalent mode in cycles with N even

$$v^N = (1, -1, 1, -1, \ldots, 1, -1)^T,$$

we have

$$\forall i \in \{1, \ldots, N\},\ \ \forall k \sim i,\ \ (v_i^N - v_k^N)^2 = 4.$$

Then (4.3) becomes

$$\ddot{y}_i = -(\Delta y)_i - 12a_N^2 \sum_{k\sim i}(y_i - y_k). \tag{4.4}$$

$$\ddot{y} = -\left(1 + 12a_N^2\right)\Delta y. \tag{4.5}$$

Expanding y on the eigenvectors of the Laplacian, $y = \sum_{k=1}^{N} z_k(t)v^k$ we decouple (4.5) and obtain N one dimensional Hill-like equations for each amplitude z_k

$$\ddot{z}_k = -\left(1 + 12a_N^2\right)\lambda_k z_k, \quad k \in \{1, \ldots, N\}. \tag{4.6}$$

where a_N is solution of the Duffing equation

$$\ddot{a}_N = -4a_N - 16a_N^3 \tag{4.7}$$

2. For the trivalent-soft-regular modes in cycles with N multiple of 4

$$v^{\frac{N}{2}} = (1, 0, -1, 0, \ldots), \quad v^{\frac{N}{2}+1} = (0, 1, 0, -1 \ldots),$$

we have

$$\forall i \in \{1, \ldots, N\}, \;\; \forall k \sim i, \;\; (v_i^j - v_k^j)^2 = 2.$$

Then (4.3) becomes

$$\ddot{y} = -\left(1 + 6a_j^2\right)\Delta y. \tag{4.8}$$

Expanding y on the eigenvectors of the Laplacian, $y = \sum_{k=1}^{N} z_k(t)v^k$ we obtain N one dimensional Hill-like equations for each amplitude z_k

$$\ddot{z}_k = -\left(1 + 6a_j^2\right)\lambda_k z_k, \quad k \in \{1, \ldots, N\}. \tag{4.9}$$

where a_j is solution of the Duffing equation ($j = \frac{N}{2}, \frac{N}{2} + 1$)

$$\ddot{a}_j = -2a_j - 2a_j^3 \tag{4.10}$$

4.2. *Numerical simulations*

In this section, we use numerical simulations to test the stability of the nonlinear normal modes. We solve the ODEs with a Runge Kutta 4-5 variable step method. To test numerically if a nonlinear normal mode is stable, we choose it as initial condition, add a small perturbation to the other normal modes and see if the solution remains close to the nonlinear normal mode. We study the stability of NNM for chains and cycles; the results are shown in Tables 1 and 2 respectively.

Table 1. Nonlinear normal modes in chains and their associated behaviour.

N	Nonlinear normal modes for chains	λ	Behaviour
N mod 2 = 0	$v^{\frac{N}{2}+1} = (1, -1, -1, 1, \ldots)^T$	2	stable
N mod 3 = 0	$v^{\frac{N}{3}+1} = (1, 0, -1, -1, 0, 1 \ldots)^T$	1	unstable
N mod 3 = 0	$v^{\frac{2N}{3}+1} = (1, -2, 1, \mid 1, -2, 1, \mid \ldots)^T$	3	stable

Table 2. Nonlinear normal modes in cycles and their associated behaviour.

N	Nonlinear normal modes for cycles	λ	Behaviour
$N \bmod 2 = 0$	$v^N = (1, -1, 1, -1, \dots)^T$	4	unstable
$N \bmod 4 = 0$	$v^{\frac{N}{2}} = (1, 0, -1, 0, 1, 0, -1, 0 \dots)^T$	2	unstable
$N \bmod 4 = 0$	$v^{\frac{N}{2}+1} = (0, 1, 0, -1, 0, 1, 0, -1 \dots)^T$	2	unstable
$N \bmod 3 = 0$	$v^{\frac{2N}{3}+1} = (0, 1, -1, 0, 1, -1 \dots)^T$	3	stable
$N \bmod 3 = 0$	$v^{\frac{2N}{3}} = (2, -1, -1, \mid 2, -1, -1 \mid \dots)^T$	3	stable
$N \bmod 6 = 0$	$v^{\frac{N}{3}+1} = (0, 1, 1, 0, -1, -1, \dots)^T$	1	unstable

Surprisingly the bivalent solutions are stable for chains and unstable for cycles, see the first lines of Tables 1 and 2. In both cases, the $(2, -1, -1)$ solutions are stable.

5. Conclusions

We examined arbitrary networks of masses coupled by cubic response springs as in the Fermi-Pasta-Ulam-Tsingou model and found the conditions for the existence of nonlinear normal modes.

We show that bivalent and soft-regular trivalent graphs give rise to such nonlinear periodic orbits. This is more restrictive than the case of on site nonlinearity for which bivalent and all trivalent graphs yield nonlinear normal modes. We found another set of eigenvectors associated to complete bipartite graphs that support NNM.

Finally, we analyzed the stability of these NNM. The equations decouple for cycles and we presented numerical calculations of the stability for cycles and chains. We plan to compare these results to the analysis to see if we can obtain a better understanding.

Our results complement the now classical studies on the FPUT system based on the Poincaré theorem for the continuation of periodic orbits [17]. These hold for small or intermediate amplitudes. Our new exact solutions are valid for arbitrary amplitude.

From this study, one expects NNM to exist for other odd polynomial nonlinearities. It is not clear what happens for mixed even/odd or non polynomial nonlinearities.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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