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Research article

Fractional view analysis of delay differential equations via numerical method

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Abstract: In this article, we solved pantograph delay differential equations by utilizing an efficient numerical technique known as Chebyshev pseudospectral method. In Caputo manner fractional derivatives are taken. These types of problems are reduced to linear or nonlinear algebraic equations using the suggested approach. The proposed method's convergence is being studied with particular care. The suggested technique is effective, simple, and easy to implement as compared to other numerical approaches. To prove the validity and accuracy of the presented approach, we take two examples. The solutions we obtained show greater accuracy as compared to other methods. Furthermore, the current approach can be implemented for solving other linear and nonlinear fractional delay differential equations, owing to its innovation and scientific significance.

Keywords: Chebyshev pseudospectral method; Caputo operator; fractional pantograph delay differential equations
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

1. Introduction

Fractional calculus (FC) is considered to be the generalization of integer order derivative and integral to arbitrary order and was originated at the end of the seventeenth century from the letter among the mathematicians Leibniz and L'Hospital in 1695, see [1, 2] for more details. Fractional calculus can be viewed as one of the extensions of ordinary classical calculus. For a long time, scientists and engineers have been interested in fractional calculus, resulting in many physical and engineering processes. FC has played a significant role in different areas. It becomes a vital tool for finding the solutions to many problems relating to fluid mechanics [3], electromagnetism [4, 5], visco-elastic materials [6], propagation of spherical flames [7], continuum and statistical mechanics [8], dynamics of viscoelastic materials [9], earthquakes [10], signal processing [11], control [12], etc.

FC has been utilized in real-world modeling applications and is proved to be well explained by fractional differential equations (FDEs). FDEs in mathematical models have become increasingly popular in recent years. Most of the phenomena in nature are described by nonlinear differential equations [13–16]. The nonlinear equations represent the world's most significant occurring phenomena. Manipulating nonlinear phenomenons is of great importance in physics, applied mathematics, and problems associated with engineering [17, 18]. As a result, FDEs must be solved analytically or numerically to solve these problems. Since most FDEs analytical solution does not exist, mathematicians have tried to develop successful numerical approaches for solving them [19, 20]. Among the FDEs, fractional delay differential equations (FDDEs) are a type of differential equation that includes time delay. The manner of the variable determines the behavior of the unknown variable in these types of equations at any given time at past states, and there is a kind of time delay in the system. Nowadays, researchers gained more attention from FDDEs than simple FDEs because a little delay has a significant effect. The concept of PDDEs arises from work [21] for an electric locomotive. The pantograph term was derived due to the accident of pantograph devices for duplicating, drawing, and writing [22] and many applications [23–26].

Many researchers have looked into the solution of fractional pantograph delay differential equations (FPDDEs) numerically due to their emergence and numerous applications. The well-known among these methods are mentioned: We want to mention some well-known techniques. In [27], generalized fractional Bernoulli wavelet functions (GFBWFs) constructed on the Bernoulli wavelets are used to solve FPDDEs numerically. Changqing Yang et al. [28] solved FPDDEs using the Jacobi collocation method. For handling FPDDEs, the authors proposed fractional hybrid Bessel functions (FHBFs) built by the combination of fractional Bessel functions and block-pulse functions [29]. To achieve the numerical result for FPDDEs, fractional Boubaker polynomials were used [30]. M. S. Hashemia et al. [31] implemented the Generalized squared remainder minimization method [GSRM] for solving FPDDEs. In [32], a new fractional integration operational matrices method was implemented to solve a class of neutral pantograph delay differential equations with fractional order. To solve the FPDDEs numerically, fractional-order generalized Taylor wavelets (FOGTW) are proposed in [33]. Furthermore, Schaefer's and Banach fixed point theorems [34] prove Implicit FPDDEs existence and uniqueness. The problem's Ulam-Hyers and generalized Ulam-Hyers stability are also defined.

In this article, we solve a class of FPDDEs by using Chebyshev pseudospectral method. The suggested technique can be applied all FDDEs. We focus on FPDDE of the form

$$D_{\mu}^{\gamma}\zeta(\mu) = f(\mu, \zeta(\mu), \zeta(g(\mu))), \quad 0 < \mu \leq 1, \quad m < \gamma \leq m+1, \quad m = 1, 2, 3, \ldots \tag{1.1}$$

having initial conditions

$$\zeta(0) = \alpha_0, \quad \zeta''(0) = \alpha_1, \tag{1.2}$$

where α_0, α_1 are real constants, $D^{\gamma}_{\mu}\zeta(\mu)$ represents the fractional Caputo derivative of $\zeta(\mu)$, f and g are well-defined functions.

2. Preliminaries

In this section, we present FC basic definitions used in our present study.

Definition 2.1. A function $\zeta(\mu), \mu > 0$ is assumed to be in $\mathbb{C}_{\nu}, \nu \in \mathbb{R}$ space for a real number $p > \nu$ with $\zeta(\mu) = \mu^p \zeta_1(\mu)$, where $\zeta_1(\mu) \in [0, \infty)$ and will be in $\mathbb{C}^{\kappa}_{\nu}$ space if and only if $\zeta^{(\kappa)} \in \mathbb{C}_{\nu}, \kappa \in \mathbb{N}$.

Definition 2.2. The Caputo fractional derivative having order γ is stated as [35]

$$D^{\gamma}\zeta(\mu) = \frac{1}{\Gamma(\kappa - \gamma)} \int_0^{\mu} (\mu - \vartheta)^{\kappa-\gamma-1} \zeta^{(\kappa)}(\vartheta) d\vartheta \tag{2.1}$$

for $\kappa - 1 < \gamma \leq \kappa, \kappa \in \mathbb{N}, \mu > 0$ and $\zeta \in \mathbb{C}^m_{-1}$.

We have the Caputo derivative [35]

$$D^{\gamma}C = 0, \qquad \text{where } C \text{ is a constant;} \tag{2.2}$$

$$D^{\gamma}u^{\wp} = \begin{cases} 0, & \text{for } \wp \in \mathbb{N}_0 \text{ and } \wp < \lceil \gamma \rceil; \\ \frac{\Gamma(\wp+1)}{\Gamma(\wp+1-\gamma)} u^{\wp-\gamma}, & \text{for } \wp \in \mathbb{N}_0 \text{ and } \wp \geq \lceil \gamma \rceil, \end{cases} \tag{2.3}$$

where the ceiling function $\lceil \gamma \rceil$ denotes the lowest integer equal to or greater than γ and $\mathbb{N}_0 = 1, 2, \ldots$.

Definition 2.3. The fractional Riemann-Liouville integral operator is stated as [35]

$$I^{\gamma}\zeta(\mu) = \frac{1}{\Gamma(\gamma)} \int_0^{\mu} (\mu - \vartheta)^{\gamma-1} \zeta(\vartheta) d\vartheta \tag{2.4}$$

having the following properties:

$$D^{\gamma} I^{\gamma} \zeta(\mu) = \zeta(\mu),$$

$$I^{\gamma} D^{\gamma} \zeta(\mu) = \zeta(\mu) - \sum_{k=0}^{\kappa-1} \frac{\zeta^{(k)}(0^+)}{k!} \mu^k, \quad \mu \geq 0, \kappa - 1 < \gamma < \kappa.$$

3. Implementation of Chebyshev series expansion to derive a fractional derivatives

The renowned Chebyshev polynomials are well-define over the interval $[-1, 1]$ and can be defined by recurrence formulae as [36, 37]

$$W_{j+1}(\mu) = 2\mu W_j(\mu) - W_{j-1}(\mu), \quad j = 1, 2, \ldots, \tag{3.1}$$

where $W_0(\mu) = 1$ and $W_1(\mu) = \mu$. The analytical form of Chebyshev polynomial having degree j is as [37]

$$W_j(\mu) = \frac{j}{2} \sum_{r=0}^{\lfloor j/2 \rfloor} (-1)^r \frac{(j - r - 1)!}{r!(j - 2r)!} (2\mu)^{j-2r}. \tag{3.2}$$

Now, by using the above polynomials in the $[0,1]$ interval, we describe the Chebyshev shifted polynomials $\hat{W}_j(\mu)$. The Chebyshev polynomials $W_j(\mu)$ are determined as [37]

$$\hat{W}_j(\mu) = W_j(2\mu - 1) \tag{3.3}$$

with recurrence formula are as follow

$$\hat{W}_{j+1}(\mu) = 2(2\mu - 1)\hat{W}_j(\mu) - \hat{W}_{j-1}(\mu), \quad j = 1, 2, \ldots \tag{3.4}$$

with $\hat{W}_0(\mu) = 1$ and $\hat{W}_1(\mu) = 2\mu - 1$. The orthogonality condition is [38]

$$\int_0^1 \frac{\hat{W}_j(\mu)\hat{W}_k(\mu)}{\sqrt{\mu - \mu^2}} d\mu = \begin{cases} 0, & \text{for } m \neq j; \\ \frac{\pi}{2}, & \text{for } m = j \neq 0; \\ \pi, & \text{for } m = j = 0. \end{cases} \tag{3.5}$$

Thus, by utilizing the renowned relation

$$\hat{W}_j(\mu) = W_{2n}\left(\sqrt{\mu}\right). \tag{3.6}$$

Using (3.2) to obtain shifted Chebyshev polynomials analytical form considering order j:

$$\hat{W}_j(\mu) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r} \frac{j(2j - r - 1)!}{r!(2j - 2r)!} (2x)^{j-2r}. \tag{3.7}$$

A function $\zeta(\mu) \in L_2[0, 1]$ may be explained in manner of Chebyshev shifted polynomials as

$$\zeta(\mu) = \sum_{j=1}^{\infty} c_j \hat{W}_j(\mu) \tag{3.8}$$

and the coefficients c_j, $j = 1, 2, \ldots$ are described as

$$c_0 = \frac{1}{\pi} \int_0^1 \frac{f(\mu)\hat{W}_0(\mu)}{\sqrt{\mu - \mu^2}} d\mu \quad \text{and} \quad c_n = \frac{2}{\pi} \int_0^1 \frac{f(\mu)\hat{W}_j(\mu)}{\sqrt{\mu - \mu^2}} d\mu. \tag{3.9}$$

Hence, simply the first $(m + 1)$-terms are considered. Thus

$$\zeta_m(\mu) = \sum_{j=0}^{m} c_j \hat{W}_j(\mu). \tag{3.10}$$

4. Theorems

The error in determining $\zeta(\mu)$ by the summation of the first m terms is restricted by the summation of the absolute values of whole ignored coefficients.

Theorem 4.1. *If*

$$\zeta_m(\mu) = \sum_{k=0}^{m} c_k W_k(\mu), \tag{4.1}$$

then for all $\zeta(\mu)$*, all* m *and all* $\mu \in [-1, 1]$*, we get*

$$E_W(m) = |\zeta(\mu) - \zeta_m(\mu)| \leq \sum_{j=m+1}^{\infty} |c_n|. \tag{4.2}$$

Proof. For any $\mu \in [-1, 1]$ and all k, the Chebyshev polynomials are bounded by 1, $|W_k(\mu)| \leq 1$. The kth term is therefore constrained by $|c_k|$. By deducting the reduced series from the infinite series, bounding every term in the difference and adding the bounds, the theorem can be obtained. □

The fundamental approximation formula for the fractional derivative of $\zeta(\mu)$ is provided by the given theorem.

Theorem 4.2. *Let the Chebyshev polynomials estimate* $\zeta(\mu)$ *as in* (3.10) *with* $\gamma > 0$*. Then, we have*

$$D^\gamma(\zeta_m(\mu)) = \sum_{j=\lceil\gamma\rceil}^{m} \sum_{r=0}^{n-\lceil\gamma\rceil} c_j b_{j,r}^{\gamma} \mu^{j-r-\gamma}, \tag{4.3}$$

where $b_{j,r}^{\gamma}$ *is given by*

$$b_{j,r}^{\gamma} = (-1)^r 2^{2j-2r} \frac{j(2j-r-1)!(j-r)!}{r!(2j-2r)!\Gamma(j-r+1-\gamma)}. \tag{4.4}$$

Proof. As fractional Caputo differentiation is a linear process, thus

$$D^\gamma(\zeta_m(\mu)) = \sum_{j=0}^{m} c_n D^\gamma(\hat{W}_j(\mu)). \tag{4.5}$$

Now, to calculate $D^\gamma(\hat{W}_j(\mu))$, employing (2.2) and (2.3) to (3.7):

$$D^\gamma(\hat{W}_j(\mu)) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r} \frac{j(2j-r-1)!}{r!(2j-2r)!} D^\gamma(\mu)^{j-r}, \tag{4.6}$$

where $j = \lceil\gamma\rceil, \lceil\gamma+1\rceil, \ldots, m$. Given that $\hat{W}_j(\mu)$ is a polynomial of degree j, we have

$$D^\gamma(\hat{W}_j(\mu)) = 0 \tag{4.7}$$

for all $j = 0, 1, 2, \ldots, \lceil\gamma\rceil - 1$ and $\gamma > 0$. The outcome of combining (4.5)–(4.7) is as follows:

$$\begin{aligned} D^\gamma(\zeta_m(\mu)) &= \sum_{j=\lceil\gamma\rceil}^{m} \sum_{r=0}^{n-\lceil\gamma\rceil} \frac{c_j (-1)^r 2^{2j-2r} j(2j-r-1)!(j-r)!}{r!(2j-2r)!\Gamma(j-r+1-\gamma)} \mu^{j-r-\gamma} \\ &= \sum_{j=\lceil\gamma\rceil}^{m} \sum_{r=0}^{n-\lceil\gamma\rceil} c_j b_{j,r}^{\gamma} \mu^{j-r-\gamma}, \end{aligned} \tag{4.8}$$

which is the desired result. □

5. Chebyshev collocation method

In this section, we discuss the solution of FPDDE (1.1) having boundary conditions (1.2) by means of Chebyshev collocation method. To attain this goal, we estimated $\zeta(\mu)$ as

$$\zeta_m(\mu) = \sum_{j=0}^{m} c_j \hat{W}_j(\mu) \tag{5.1}$$

By means of Theorem 4.2 and (1.1), (5.1) we get

$$\sum_{j=\lceil\gamma\rceil}^{m} \sum_{r=0}^{n-\lceil\gamma\rceil} c_j b_{j,r}^{\gamma} \mu^{j-2r-\gamma} = F\left(\mu, \sum_{j=0}^{m} c_j \hat{W}_j(\mu), \sum_{j=0}^{m} c_j \hat{W}_j(g(\mu))\right) \tag{5.2}$$

for $0 < \mu < 1$ and $m + 1 < \gamma < m$.

Thus, we collocate (5.2) at μ_p points as $p = 0, 1, 2, \ldots, m - \lceil\gamma\rceil$:

$$\sum_{j=\lceil\gamma\rceil}^{m} \sum_{r=0}^{n-\lceil\gamma\rceil} c_j b_{j,r}^{\gamma} \mu_p^{j-2r-\gamma} = F\left(\mu_p, \sum_{j=0}^{m} c_j \hat{W}_j(\mu_p), \sum_{j=0}^{m} c_j \hat{W}_j(g(\mu_p))\right) \tag{5.3}$$

for $p = 0, 1, \cdots m - \lceil\gamma\rceil$ and $m + 1 < \gamma < m$.

Now, by substituting (5.1) in (1.2), we get the following $\lceil\gamma\rceil$ equations:

$$\sum_{i=0}^{m} (-1)^i c_i = \alpha_0, \quad \sum_{i=0}^{m} c_i = \alpha_1. \tag{5.4}$$

Thus, we get $(m + 1 - \lceil\gamma\rceil)$ algebraic equations from (5.3) and $(\lceil\gamma\rceil)$ algebraic equations from (5.4) for the unknowns c_j, $j = 0, 1, 2, \ldots, m$ which can be solved to calculate $\zeta_m(\mu)$.

6. Numerical implementation

In this section, we implemented the suggested approach for solving two problems. The proposed problems results are compared with other techniques. All the computational work is done through maple.

Example 1. Consider the following PDDE of the form [39]

$$D_\mu^\gamma \zeta(\mu) = \frac{1}{2}\zeta(q\mu) - \zeta(\mu) - \frac{1}{2}\exp(-q\mu), \quad 0 \le \mu \le R, \ \ 0 < \gamma \le 1, \ \ 0 < q \le 1 \tag{6.1}$$

with initial condition $\zeta(0) = 1$.

The problem accurate solution for $\gamma = 1$ is

$$\zeta(\mu) = \exp(-\mu).$$

The exact and proposed method solution for $q = 0.5$ and $m = 8$ are shown in Figure 1. Similarly, Figure 2 shows the solution graph of the suggested approach at different fractional-orders. Table 1 illustrates the behavior of exact, CPM solution with the aid of absolute error for Example 1. In Table 2, we give the absolute error comparison with the technique in [39, 40] at $q = 0.5$ and $q = 0.99$. Also, at $q = 1$, we compared our solution with the Runge-Kutta method, Taylor polynomial approach and Boubaker matrix method in Table 3. From both the tables it is clear that the error of the presented method is less than that of others which confirms that CPM approaches fast as compared to other methods.

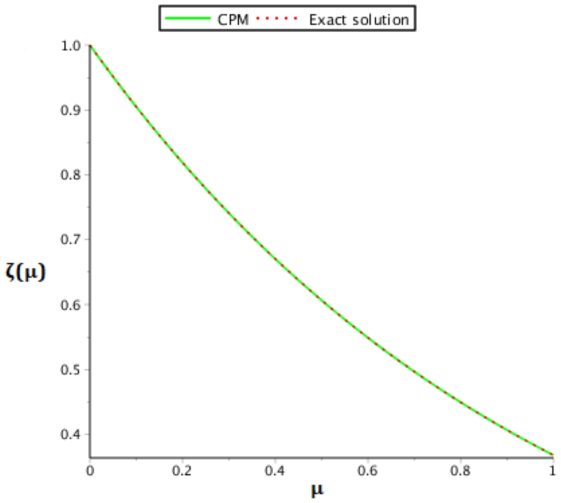

Figure 1. The exact and CPM solutions at $m = 8$ for Example 1.

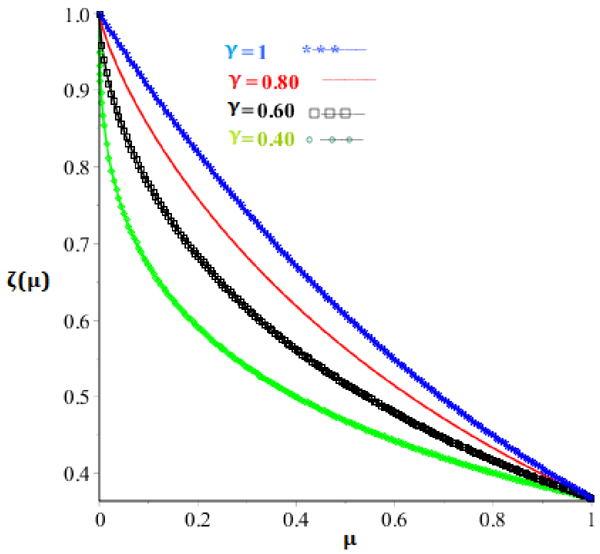

Figure 2. The Example 1 solution graph at different fractional orders.

Table 1. Behavior of exact, CPM solution along with absolute error for Example 1.

μ	Exact	CPM	CPM Error
0	1.00000000000000	1.00000000000000	0.0000000000E+00
0.1	0.90483741803482	0.90483741803596	1.1326881600E-12
0.2	0.81873075302927	0.81873075307798	4.8707551100E-11
0.3	0.74081822032894	0.74081822068171	3.5277246821E-10
0.4	0.67032004486301	0.67032004603563	1.1726267785E-09
0.5	0.60653065730985	0.60653065971263	2.4027784946E-09
0.6	0.54881163270937	0.54881163609402	3.3846474234E-09
0.7	0.49658530034770	0.49658530379141	3.4437015128E-09
0.8	0.44932896119523	0.44932896411722	2.9219862717E-09
0.9	0.40656965663434	0.40656965974059	3.1062560826E-09
1.0	0.36787943966839	0.36787944117144	1.5030479540E-09

Table 2. The absolute error comparison for Example 1.

m	Technique in [42] for $q = 0.5$	Technique in [42] for $q = 0.99$	Technique in [43] $q = 0.5$	Technique in [43] $q = 0.99$	CPM for $q = 0.5$	CPM for $q = 0.99$
6	1.351E-4	7.362E-7	3.5007E-8	3.7109E-8	1.6458E-10	1.6598E-10
8	1.102E-6	1.891E-9	5.0088E-10	8.8818E-16	1.1327E-12	1.1518E-12
10	5.662E-9	3.598E-12	3.2419E-14	4.6384E-8	3.6180E-15	3.6878E-15
14	1.854E-13	4.442E-16	0.00	3.8858E-16	3.00E-20	2.00E-20
16	5.552E-16	4.442E-16	5.8842E-15	3.9413E-15	0.00	0.00

Table 3. The absolute error comparison for Example 1 at $q = 1$.

μ	Runge-Kutta method ($q = 1$) [44]	Taylor polynomial approach [45]	Boubaker matrix method [40]	CPM
2^{-1}	0.5000E-5	0.1000E-9	1.2000E-9	1.5600E-12
2^{-2}	0.1870E-6	0.2000E-9	9.1000E-10	2.9150E-13
2^{-3}	0.6430E-8	0.2000E-9	1.0000E-9	1.1830E-14
2^{-4}	0.2100E-9	0.2000E-9	1.0000E-9	2.8200E-16
2^{-5}	0.6700E-11	0.1000E-9	9.0000E-10	5.3700E-18
2^{-6}	0.210E-12	0.000000	1.0000E-9	8.0000E-20

Example 2. Consider the following PDDE of the form [41]

$$D^{\gamma}_{\mu}\zeta(\mu) = -\zeta(\mu) + 5\zeta^2\left(\frac{\mu}{2}\right), \quad \mu \geq 0, 1 < \gamma \leq 2 \tag{6.2}$$

having initial conditions $\zeta(0) = 1$ and $\zeta'(0) = -2$.

The problem accurate solution for $\gamma = 2$ is

$$\zeta(\mu) = \exp(-2\mu).$$

The accurate and proposed method solution at $m = 12$ are illustrated in Figure 3. Table 4 illustrates accurate solution, CPM solution and CPM absolute error at $m = 12$. Similarly, Figure 4 illustrates the solution graph of the presented approach at different fractional-orders. From the Figures, it is observed that as the value of m increases, the absolute error tends to zero. It has been noted that the results of the suggested approach are more favourable for this example.

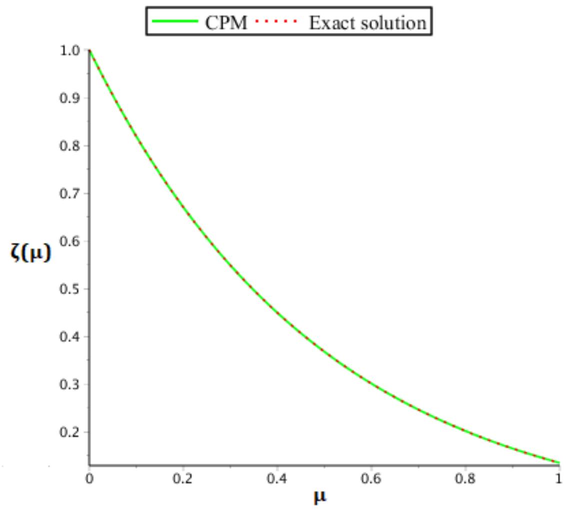

Figure 3. The exact and CPM solutions at $m = 8$ for Example 2.

Table 4. Behavior of Exact, CPM solution along with absolute error for Example 2.

μ	Exact	CPM	CPM Error
0	1.00000000000000	1.00000000000000	0.0000000000E+00
0.1	0.81873075307800	0.81873075307798	1.9310670000E-14
0.2	0.67032004603683	0.67032004603563	1.1963473900E-12
0.3	0.54881163610331	0.54881163609402	9.2906713200E-12
0.4	0.44932896414715	0.44932896411722	2.9930749730E-11
0.5	0.36787944123109	0.36787944117144	5.9648713620E-11
0.6	0.30119421200338	0.30119421191220	9.1182147850E-11
0.7	0.24659696406387	0.24659696394160	1.2226468655E-10
0.8	0.20189651814768	0.20189651799465	1.5302873585E-10
0.9	0.16529888840530	0.16529888822158	1.8372224028E-10
1.0	0.13533528345066	0.13533528323661	2.1405679253E-10

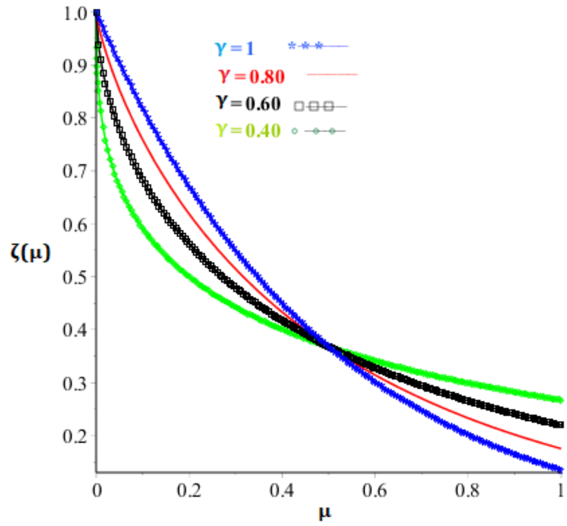

Figure 4. The Example 2 solution graph at various fractional orders.

7. Conclusions

In this article, we implemented Chebyshev pseudospectral method to solve FPDDEs. The Caputo derivative was approximated by a Chebyshev series expansion formula. Chebyshev polynomials features are used to convert DFDEs into easily solvable linear or nonlinear algebraic equations. The methodology is simple to use and has a higher rate of convergence than previous approaches. Some problems are solved to demonstrate the efficacy of the current approach. The obtained solutions are compared to other approaches such as the Runge-Kutta method, Taylor polynomial approach, and Boubaker matrix method were used to compare our results. CPM has greater accuracy than any of these approaches, as illustrated by comparison. CPM, on the other hand, can be easily implemented to various fractional delay or non-delay physics and real-life scientific models.

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Conflict of interest

The authors declare that they have no competing interests.

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