Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(12): 20490–20509.
DOI: 10.3934/math.20221122
Received: 15 July 2022
Revised: 26 August 2022
Accepted: 02 September 2022
Published: 19 September 2022

Research article

Uniqueness of meromorphic functions concerning fixed points

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Abstract: In this paper, we study a uniqueness question of meromorphic functions concerning fixed points and mainly prove the following theorem: Let f and g be two nonconstant meromorphic functions, let n, k be two positive integers with $n > 3k + 10.5 - \Theta_{\min}(k + 6.5)$, if $\Theta_{\min} \geq \frac{2.5}{k+6.5}$, otherwise $n > 3k + 8$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$. Our results extend and improve some results due to [8, 9, 19, 24].

Keywords: meromorphic function; entire function; unicity; fixed point
Mathematics Subject Classification: 30D35

1. Introduction and main results

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory, see [11, 14, 21, 23]: $T(r, f), N(r, f), m(r, f), \cdots$.

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set $E \in (0, +\infty)$ with finite logarithmic measure $\int_E dr/r < \infty$.

Let f and g be two nonconstant meromorphic functions. Define

$$\Theta(\infty, f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)}.$$

Similarly, we have the notation $\Theta(\infty, g)$. Let $\Theta_{\min} = \min\{\Theta(\infty, f), \Theta(\infty, g)\}$.

A meromorphic function α is said to be a small function of f if it satisfies $T(r, \alpha) = S(r, f)$. Let α be a small function of both f and g. If $f - \alpha$ and $g - \alpha$ have the same zeros counting multiplicities (ignoring multiplicities), then we call that f and g share α CM (IM). Let $N_0(r, \alpha, f, g)$ be counting function of common zeros of both $f-\alpha$ and $g-\alpha$ with counting multiplicities. If $N\left(r, \frac{1}{f-\alpha}\right)+N\left(r, \frac{1}{g-\alpha}\right)-2N_0(r, \alpha, f, g) \le S(r, f) + S(r, g)$, then we call that f and g share α CM almost.

Let f and g be two nonconstant meromorphic functions, and let f and g share 1 IM almost. We denote by $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function for 1-points of both f and g about which f has larger multiplicity than g, with multiplicity being not counted. Similarly, we have the notation $\overline{N}_L\left(r, \frac{1}{g-1}\right)$. Especially, if f and g share 1 CM, then $\overline{N}_L\left(r, \frac{1}{f-1}\right) = \overline{N}_L\left(r, \frac{1}{g-1}\right) = 0$.

We denote by $N_{(k}(r, f)$ the counting function for poles of f with multiplicity$\ge k$, and by $\overline{N}_{(k}(r, f)$ the corresponding one for which multiplicity is not counted. Set $N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f)$.

Hayman [10], Clunie [3], Mues [16], Bergweiler and Eremenko [1], Chen and Fang [4] proved the following theorem.

Theorem A. [1, 4] Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n(z)f'(z) = 1$ has infinitely many solutions.

Fang and Hua [7], Yang and Hua [22] obtained a unicity theorem corresponding to Theorem A.

Theorem B. [22] Let f and g be two nonconstant meromorphic (entire) functions, and let $n \ge 11$ ($n \ge 6$) be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Fang and Qiu [8] extended Theorem B and proved the following theorem.

Theorem C. Let f and g be two nonconstant meromorphic (entire) functions, and let $n \ge 11$ ($n \ge 6$) be a positive integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Hennekemper [12], Hennekemper et al. [13], Chen [5], Wang [17], Wang and Fang [18] extended Theorem A by proving the following theorem.

Theorem D. [18] Let f be a transcendental entire function, and let n, k be two positive integers with $n \ge k + 1$. Then $(f^n(z))^{(k)} = 1$ has infinitely many solutions.

Naturally, we pose the following problem.

Problem 1. Does there exist a corresponding unicity theorem to Theorem D?

Fang [9] studied this problem and proved the following result.

Theorem E. Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

For meromorphic functions, Bhoosnurmath and Dyavanal [2] proved the following theorem.

Theorem F. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

But there exists a gap in the proof of [2], which can be found in the appendix at the end of this paper. Up to now, we don't know whether Theorem F is valid or not.

In this paper, we extend Theorem E and prove the following results.

Theorem 1. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8 - \Theta_{\min}(k + 4)$, if $\Theta_{\min} \ge \frac{2}{k+4}$, otherwise $n > 3k + 6$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

Corollary 2. Let n, k be two positive integers with $n > 2k + 4$, and let f and g be two nonconstant meromorphic functions such that $\Theta_{\min} > \frac{k+3}{k+4}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

Theorem 3. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 6 - \Theta_{\min}(k + 2)$, if $\Theta_{\min} \geq \frac{1}{k+2}$, otherwise $n > 3k + 5$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

Corollary 4. Let n, k be two positive integers with $n > 2k + 4$, and let f and g be two nonconstant meromorphic functions such that $\Theta_{\min} > \frac{k+1}{k+2}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

From Corollary 2 or Corollary 4, we get Theorem E.

In 2009, Zhang [24] studied the case of entire functions sharing fixed points and proved the following theorem.

Theorem G. Let f and g be two nonconstant entire functions, let n, k be two positive integers with $n > 2k + 4$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$.

Xu et al. [19] studied the case of meromorphic functions.

Theorem H. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k+10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$.

Naturally, we ask the following problem.

Problem 2. In Theorem H, is $n > 3k + 10$ the best possibility?

In this paper, we study Problem 2 and prove the following results.

Theorem 5. Let f and g be two nonconstant meromorphic functions, let n, k be two positive integers with $n > 3k + 10.5 - \Theta_{\min}(k + 6.5)$, if $\Theta_{\min} \geq \frac{2.5}{k+6.5}$, otherwise $n > 3k + 8$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$.

Corollary 6. Let n, k be two positive integers with $n > 2k + 4$, let f and g be two nonconstant meromorphic functions such that $\Theta_{\min} > \frac{k+5.5}{k+6.5}$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$.

Theorem 7. Let f and g be two nonconstant meromorphic functions, let n, k be two positive integers with $n > 3k + 9 - \Theta_{\min}(k + 5)$, if $\Theta_{\min} \geq \frac{2}{k+5}$, otherwise $n > 3k + 7$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ CM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or

$f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$.

Corollary 8. Let n, k be two positive integers with $n > 2k + 4$, let f and g be two nonconstant meromorphic functions such that $\Theta_{\min} > \frac{k+4}{k+5}$, and let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ CM, then one of the following two cases holds: If $k = 1$, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$, or $f = tg$ for a constant t such that $t^n = 1$; if $k \geq 2$, then $f = tg$ for a constant t such that $t^n = 1$.

From Corollary 6 or Corollary 8, we get Theorem G. From Theorem 5, we get Theorem H.

2. Some lemmas

For the proof of our results, we need the following lemmas.

Lemma 1. [11, 14, 21, 23] Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then

$$\begin{aligned} T(r, f) \leq& \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\ \leq& \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)}(z) = 0$, but $f(z)(f^{(k)}(z) - c) \neq 0$.

Lemma 2. [11, 14, 21, 23] Let f be a nonconstant meromorphic function, and let k be a positive integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 3. [11, 14, 21, 23] Let f be a nonconstant meromorphic function, and let α_i $(i = 1, 2, 3)$ (one may be ∞) be three distinct small functions of f. Then

$$T(r, f) \leq \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f - \alpha_i}\right) + S(r, f).$$

Lemma 4. [23] Let f be a meromorphic function such that $f^{(k)} \not\equiv 0$, and let k be a positive integer. Then

$$\begin{aligned} T(r, f^{(k)}) \leq& T(r, f) + k\overline{N}(r, f) + S(r, f), \\ N\left(r, \frac{1}{f^{(k)}}\right) \leq& N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \end{aligned}$$

Lemma 5. [20, 23] Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n (\neq 0)$, $a_{n-1}, \cdots, a_0$ are constants. If f is a nonconstant meromorphic function, then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 6. [23] Let f_i $(i = 1, 2, 3)$ be nonconstant meromorphic functions such that $\sum_{i=1}^{3} f_i \equiv 1$. If f_i $(i = 1, 2, 3)$ are linearly independent, then

$$T(r, f_i) \leq \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r, f_i) + o(T(r)),$$

where $T(r) = \max_{1\leq i\leq 3}\{T(r, f_i)\}$ and $r \notin E$.

Lemma 7. [11, 21, 23] Let f be a nonconstant meromorphic function, let $n \geq 2$ be a positive integer, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be distinct small functions of f. Then

$$m\left(r, \frac{1}{f-\alpha_1}\right) + \cdots + m\left(r, \frac{1}{f-\alpha_n}\right) \leq m\left(r, \frac{1}{f-\alpha_1} + \cdots + \frac{1}{f-\alpha_n}\right) + S(r, f).$$

Lemma 8. Let n, k be two positive integers with $n > k + 5$, and let f and g be two meromorphic functions such that $(f^n)^{(k+2)} \not\equiv 0$ and $(g^n)^{(k+2)} \not\equiv 0$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then

$$T(r, f) = O(T(r, g)),\ T(r, g) = O(T(r, f)). \tag{2.1}$$

Proof. Since f and g are two nonconstant meromorphic functions, then

$$T(r, f) \geq \log r + O(1),\ T(r, g) \geq \log r + O(1). \tag{2.2}$$

By Lemma 2, Lemma 4, Lemma 7 and Nevanlinna's first fundamental theorem, we get

$$\begin{aligned}
&m\left(r, \frac{1}{f^n}\right) + m\left(r, \frac{1}{(f^n)^{(k)} - z}\right) \leq m\left(r, \frac{1}{(f^n)^{(k)}}\right) + m\left(r, \frac{1}{(f^n)^{(k)} - z}\right) + S(r, f)\\
\leq& m\left(r, \frac{1}{(f^n)^{(k+1)}}\right) + m\left(r, \frac{1}{(f^n)^{(k+1)} - 1}\right) + S(r, f) \leq m\left(r, \frac{1}{(f^n)^{(k+1)}} + \frac{1}{(f^n)^{(k+1)} - 1}\right) + S(r, f)\\
\leq& m\left(r, \frac{1}{(f^n)^{(k+2)}}\right) + S(r, f) = T(r, (f^n)^{(k+2)}) - N\left(r, \frac{1}{(f^n)^{(k+2)}}\right) + S(r, f)\\
\leq& T(r, (f^n)^{(k)}) + 2\overline{N}(r, f) - N\left(r, \frac{1}{(f^n)^{(k+2)}}\right) + S(r, f).
\end{aligned}$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, it follows from Nevanlinna's first fundamental theorem that

$$\begin{aligned}
&nT(r, f) + T(r, (f^n)^{(k)} - z)\\
\leq& N\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{(f^n)^{(k)} - z}\right) - N\left(r, \frac{1}{(f^n)^{(k+2)}}\right) + T(r, (f^n)^{(k)}) + 2\overline{N}(r, f) + S(r, f)\\
\leq& (k+2)\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(g^n)^{(k)} - z}\right) + T(r, (f^n)^{(k)}) + 2\overline{N}(r, f) + S(r, f)\\
\leq& (k+4)T(r, f) + N\left(r, \frac{1}{(g^n)^{(k)} - z}\right) + T(r, (f^n)^{(k)}) + S(r, f).
\end{aligned} \tag{2.3}$$

By (2.2), we get

$$T(r, (f^n)^{(k)} - z) \geq T(r, (f^n)^{(k)}) - \log r \geq T(r, (f^n)^{(k)}) - T(r, f). \tag{2.4}$$

Hence, it follows from (2.3) and (2.4) that

$$(n-k-5)T(r,f) \leq N\left(r, \frac{1}{(g^n)^{(k)}-z}\right) + S(r,f). \tag{2.5}$$

By Lemma 4, we have

$$\begin{aligned} N\left(r, \frac{1}{(g^n)^{(k)}-z}\right) \leq & T(r,(g^n)^{(k)}) + \log r + O(1) \\ \leq & T(r,g^n) + k\overline{N}(r,g^n) + \log r + S(r,g) \\ \leq & (n+k+1)T(r,g) + S(r,g). \end{aligned} \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$[n-k-5]T(r,f) \leq (n+k+1)T(r,g) + S(r,f) + S(r,g). \tag{2.7}$$

Similarly, we obtain

$$[n-k-5]T(r,g) \leq (n+k+1)T(r,f) + S(r,f) + S(r,g). \tag{2.8}$$

By (2.7), (2.8) and $n > k+5$, we get (2.1).

By using the same argument as used in proof of Lemma 8, we obtain the following lemma.

Lemma 9. Let n, k be two positive integers with $n > k+2$, and let f and g be two nonconstant meromorphic functions such that $(f^n)^{(k+1)} \not\equiv 0$ and $(g^n)^{(k+1)} \not\equiv 0$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then

$$T(r,f) = O(T(r,g)),\ T(r,g) = O(T(r,f)).$$

Lemma 10. [6] Let f be a nonconstant entire function, and let $k(\geq 2)$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0)$, b are two constants.

Lemma 11. [22] Let f and g be two nonconstant entire functions, and let $n(\geq 1)$ be a positive integer. If $f^n f' g^n g' \equiv 1$, then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

Lemma 12. [8] Let f and g be two nonconstant entire functions, and let $n(\geq 2)$ be a positive integer. If $f^n f' g^n g' \equiv z^2$, then $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$.

By Theorem 2.2 [15], we get the following result.

Lemma 13. Let f be a meromorphic function, and let k be a positive integer with $k \geq 2$. If $f(z)$ and $f^{(k)}(z)$ have finitely many zeros, then $f(z) = R(z)e^{P(z)}$, where $R(z)$ is a rational function and $P(z)$ is a polynomial.

3. Proof of theorems

3.1. Proof of Theorem 1

Set

$$F = (f^n)^{(k)},\ G = (g^n)^{(k)}.$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then F and G share 1 CM.

By Lemma 4, we obtain

$$T(r,F)=T(r,(f^n)^{(k)})\le T(r,f^n)+k\overline{N}(r,f)+S(r,f)\le(n+k)T(r,f)+S(r,f).$$

It follows $S(r,F)=S(r,f)$. Similarly, we get $S(r,G)=S(r,g)$.

Set

$$\phi=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}. \tag{3.1}$$

Next, we consider two cases.

Case 1. $\phi\equiv 0$. Then by (3.1), we have

$$\frac{F-1}{F}\equiv C\frac{G-1}{G}, \tag{3.2}$$

where C is a nonzero finite complex number. In the following, we consider two subcases.

Case 1.1. $C=1$. It follows from (3.2) that $F\equiv G$, that is $(f^n)^{(k)}\equiv(g^n)^{(k)}$. It follows $f^n=g^n+P$, where P is a polynomial with $\deg P\le k-1$.

If $P\not\equiv 0$, then we have

$$\frac{f^n}{P}-\frac{g^n}{P}=1. \tag{3.3}$$

Since f and g are two nonconstant meromorphic functions, then

$$T(r,f)\ge\log r+O(1),\ T(r,g)\ge\log r+O(1). \tag{3.4}$$

By Nevanlinna's first fundamental theorem and (3.4), we obtain

$$\begin{aligned}T\left(r,\frac{f^n}{P}\right)&\le T(r,f^n)+T(r,P)+O(1)\\&\le nT(r,f)+(k-1)\log r+O(1)\\&\le(n+k-1)T(r,f)+O(1).\end{aligned}$$

Hence, we get

$$S\left(r,\frac{f^n}{P}\right)=S(r,f). \tag{3.5}$$

By $n>2k+4$, Nevanlinna's second fundamental theorem and (3.3)–(3.5), we have

$$\begin{aligned}&nT(r,f)=T(r,f^n)\le T\left(r,\frac{f^n}{P}\right)+T(r,P)\\&\le\overline{N}\left(r,\frac{f^n}{P}\right)+\overline{N}\left(r,\frac{P}{f^n}\right)+\overline{N}\left(r,\frac{1}{\frac{f^n}{P}-1}\right)+(k-1)\log r+S(r,f)\\&\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+2(k-1)\log r+S(r,f)\\&\le 2kT(r,f)+\overline{N}\left(r,\frac{1}{g}\right)+S(r,f).\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$(n-2k)T(r,f) \le \overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \tag{3.7}$$

Similarly, we have

$$(n-2k)T(r,g) \le \overline{N}\left(r,\frac{1}{f}\right) + S(r,g). \tag{3.8}$$

By either $n > 3k+6$ or $n > 3k+8-\Theta_{\min}(k+4) \ge 2k+4$, we get

$$T(r,f) + T(r,g) \le S(r,f) + S(r,g),$$

a contradiction.

Hence, $P \equiv 0$. It follows $f \equiv tg$, where t is a constant such that $t^n = 1$.

Case 1.2. $C \neq 1$. Then by (3.2), we obtain

$$\frac{1}{F} - \frac{C}{G} = 1 - C. \tag{3.9}$$

Since f and g share ∞ IM, it follows from (3.9) that $F \neq \infty$ and $G \neq \infty$. Hence $\frac{1}{F} \neq 0$, then by (3.9) we deduce that $G \neq \frac{C}{C-1}$.

By Lemma 1, we obtain

$$\begin{aligned}
&nT(r,g) = T(r,g^n)\\
\le&\overline{N}(r,g^n) + N\left(r,\frac{1}{g^n}\right) + N\left(r,\frac{1}{(g^n)^{(k)} - \frac{C}{C-1}}\right) - N\left(r,\frac{1}{(g^n)^{(k+1)}}\right) + S(r,g)\\
\le&(k+1)\overline{N}\left(r,\frac{1}{g}\right) + S(r,g).
\end{aligned}$$

It follows from either $n > 3k+6$ or $n > 3k+8-\Theta_{\min}(k+4) \ge 2k+4$ that $T(r,g) \le S(r,g)$, a contradiction.

Case 2. $\phi \not\equiv 0$.

Let z_0 be a pole of f with multiplicity l_1. Then by f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2. Set $l = \min\{l_1, l_2\}$, by (3.1), we deduce that z_0 is a zero of ϕ with multiplicity $\ge nl + k - 1$. Hence, by Lemma 2, we have

$$\begin{aligned}
&\overline{N}(r,f) = \overline{N}(r,g) \le \frac{1}{n+k-1}N\left(r,\frac{1}{\phi}\right)\\
\le&\frac{1}{n+k-1}T(r,\phi) + O(1)\\
=&\frac{1}{n+k-1}N(r,\phi) + \frac{1}{n+k-1}m(r,\phi) + O(1)\\
\le&\frac{1}{n+k-1}\left[\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right)\right] + S(r,f) + S(r,g).
\end{aligned} \tag{3.10}$$

It follows from Lemma 4 that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\\
=&N\left(r,\frac{1}{(f^n)^{(k)}}\right)-\left[N\left(r,\frac{1}{(f^n)^{(k)}}\right)-\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\right]\\
\le&N\left(r,\frac{1}{f^n}\right)+k\overline{N}(r,f)-\left[N\left(r,\frac{1}{(f^n)^{(k)}}\right)-\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\right]+S(r,f)\\
\le&(k+1)\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\\
\le&(2k+1)T(r,f)+S(r,f). \qquad (3.11)
\end{aligned}$$

Similarly, we have

$$\overline{N}\left(r,\frac{1}{G}\right)\le(2k+1)T(r,g)+S(r,g). \qquad (3.12)$$

By (3.10)–(3.12), we get

$$\overline{N}(r,f)=\overline{N}(r,g)\le\frac{2k+1}{n+k-1}[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \qquad (3.13)$$

Set

$$\varphi=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}. \qquad (3.14)$$

Suppose $\varphi\not\equiv 0$. Let z_0 be a common simple zeros of $F(z)-1$ and $G(z)-1$, by a simple computation, we see that $\varphi(z_0)=0$. Thus, by Nevanlinna's first fundamental theorem and Lemma 2, we have

$$N_{1)}\left(r,\frac{1}{F-1}\right)=N_{1)}\left(r,\frac{1}{G-1}\right)\le N\left(r,\frac{1}{\varphi}\right)\le T(r,\varphi)+O(1)\le N(r,\varphi)+S(r,F)+S(r,G), \qquad (3.15)$$

where $N_{1)}\left(r,\frac{1}{F-1}\right)$ is the counting function of simple zeros of $F(z)-1$.

It follows from F and G share 1 CM and (3.14) that

$$N(r,\varphi)\le\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right), \qquad (3.16)$$

where $N_0\left(r,\frac{1}{F'}\right)$ is the counting function for which $F'(z)=0$ and $f(z)(F(z)-1)\ne 0$.

Since F and G share 1 CM, then we get

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)=2\overline{N}\left(r,\frac{1}{F-1}\right)\le N_{1)}\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{F-1}\right)\\
=&N_{1)}\left(r,\frac{1}{F-1}\right)+\frac{1}{2}\left[N\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)\right]. \qquad (3.17)
\end{aligned}$$

By Lemma 1, we have

$$T(r,f^n)\le\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f^n}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)-N_0\left(r,\frac{1}{F'}\right)+S(r,f), \qquad (3.18)$$

$$T(r,g^n) \leq \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g^n}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,g). \tag{3.19}$$

It follows from (3.15)-(3.19) that

$$\begin{aligned} T(r,f^n) + T(r,g^n) \leq & 2\overline{N}(r,f) + 2\overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{f^n}\right) + N_{k+1}\left(r,\frac{1}{g^n}\right) + \overline{N}\left(r,\frac{1}{f}\right) \\ & + \overline{N}\left(r,\frac{1}{g}\right) + \frac{1}{2}\left[N\left(r,\frac{1}{F-1}\right) + N\left(r,\frac{1}{G-1}\right)\right] + S(r,f) + S(r,g). \end{aligned} \tag{3.20}$$

By Nevanlinna's first fundamental theorem and Lemma 4, we get

$$\begin{aligned} & N\left(r,\frac{1}{F-1}\right) + N\left(r,\frac{1}{G-1}\right) \leq T(r,(f^n)^{(k)}) + T(r,(g^n)^{(k)}) + O(1) \\ \leq & T(r,f^n) + T(r,g^n) + k\overline{N}(r,f) + k\overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.21}$$

We note that

$$N_{k+1}\left(r,\frac{1}{f^n}\right) = (k+1)\overline{N}\left(r,\frac{1}{f}\right),\ N_{k+1}\left(r,\frac{1}{g^n}\right) = (k+1)\overline{N}\left(r,\frac{1}{g}\right). \tag{3.22}$$

Since f and g share ∞ IM, then by (3.20)-(3.22), we obtain

$$\begin{aligned} & \frac{n}{2}[T(r,f) + T(r,g)] \leq \left(\frac{k}{2}+2\right)[\overline{N}(r,f) + \overline{N}(r,g)] \\ & +(k+2)\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right] + S(r,f) + S(r,g). \end{aligned} \tag{3.23}$$

Next, we consider two subcases.

If $\Theta_{\min} \geq \frac{2}{k+4}$. By Lemma 9, we get $S(r,f) = S(r,g)$.

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.

It follows from (3.23) that

$$[n-2(k+2)]T(r,f) \leq (k+4)\overline{N}(r,f) + S(r,f),\ r \in I. \tag{3.24}$$

By (3.24), we get

$$\begin{aligned} n-2(k+2) \leq & (k+4)\overline{\lim_{\substack{r\to\infty \\ r\in I}}}\frac{\overline{N}(r,f)}{T(r,f)} + \lim_{\substack{r\to\infty \\ r\in I}}\frac{S(r,f)}{T(r,f)} \leq (k+4)\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)} \\ \leq & (k+4)(1-\Theta(\infty,f)) \leq (k+4)(1-\Theta_{\min}). \end{aligned} \tag{3.25}$$

Hence, it follows from $n > 3k+8-\Theta_{\min}(k+4)$ and (3.25) that we get a contradiction.

Otherwise, by (3.13) and (3.23), we have

$$\begin{aligned}
&\left[\frac{n}{2}-(k+2)\right][T(r,f)+T(r,g)]\\
\leq&\frac{(k+4)(2k+1)}{n+k-1}[T(r,f)+T(r,g)]+S(r,f)+S(r,g).
\end{aligned}\tag{3.26}$$

It follows from $n > 3k+6$ and (3.26) that

$$T(r,f)+T(r,g)\leq S(r,f)+S(r,g),$$

a contradiction.

Thus, we get $\varphi\equiv 0$, that is

$$\frac{F''}{F'}-2\frac{F'}{F-1}\equiv\frac{G''}{G'}-2\frac{G'}{G-1}.\tag{3.27}$$

By solving this equation, we obtain

$$\frac{1}{F-1}=\frac{a}{G-1}+b,\tag{3.28}$$

where $a(\neq 0), b$ are two finite complex numbers. Next, we consider two subcases.

Case 2.1. $b\neq 0$. Since f and g share ∞ IM, we know that F and G share ∞ IM, it follows from (3.28) that $F\neq\infty, G\neq\infty$. Hence $\frac{1}{F-1}\neq 0$, thus by (3.28) we deduce that $G\neq\frac{b-a}{b}$.

Now, we consider two subcases.

Case 2.1.1. $b=a$. It follows from $\frac{a}{G-1}\neq 0$ and (3.28) that $F\neq 1+\frac{1}{b}$. In the following, we consider two subcases.

Case 2.1.1.1. $b\neq -1$. Then we have $1+\frac{1}{b}\neq 0$. By Lemma 1, we obtain

$$\begin{aligned}
nT(r,f)\leq&\overline{N}(r,f)+N\left(r,\frac{1}{f^n}\right)+N\left(r,\frac{1}{F-(1+\frac{1}{b})}\right)-N\left(r,\frac{1}{F'}\right)+S(r,f)\\
\leq&\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F-(1+\frac{1}{b})}\right)+S(r,f)\\
\leq&(k+1)T(r,f)+S(r,f).
\end{aligned}$$

It follows from either $n>3k+6$ or $n>3k+8-\Theta_{\min}(k+4)\geq 2k+4$ that $T(r,f)\leq S(r,f)$, a contradiction.

Case 2.1.1.2. $b=-1$, Thus $a=-1$. By (3.28), we deduce that $FG\equiv 1$, that is

$$(f^n)^{(k)}(g^n)^{(k)}\equiv 1.\tag{3.29}$$

Since f and g share ∞ IM, then by (3.29), we deduce that $f\neq\infty,\ g\neq\infty$. It follows from (3.29) that $(f^n)^{(k)}\neq 0,\ (g^n)^{(k)}\neq 0, f\neq 0,\ g\neq 0$.

If $k\geq 2$, then by Lemma 10, we get $f(z)=c_1e^{cz}, g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying$(-1)^k(c_1c_2)^n(nc)^{2k}=1$.

If $k=1$, then by Lemma 11, we get $f(z)=c_1e^{cz}, g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$.

Case 2.1.2. $b \neq a$. Hence, we have $\frac{b-a}{b} \neq 0$, $G - \frac{b-a}{b} \neq 0$. In this case, by using the same argument as used in Case 2.1.1.1, we get a contradiction.

Case 2.2. $b = 0$. Thus by (3.28), we have $\frac{1}{F-1} = \frac{a}{G-1}$, that is

$$aF - G = a - 1. \tag{3.30}$$

If $a = 1$, then by (3.30), we have $F \equiv G$. That is $(f^n)^{(k)} \equiv (g^n)^{(k)}$. By using the same argument as used in Case 1.1, we get $f \equiv tg$, where t is a constant such that $t^n = 1$.

If $a \neq 1$, then by (3.30), we get $a(f^n)^{(k)} - (g^n)^{(k)} = a - 1$, that is $(af^n - g^n)^{(k)} = a - 1$. Thus, we obtain $af^n - g^n = p$, where p is a polynomial of degree k. Then by using the same argument as used in Case 1.1, we get a contradiction.

This completes the proof of Theorem 1.

3.2. *Proof of Theorem 3*

Imitating the proof of Theorem 1, we can easily to prove Theorem 3 only by replacing (3.16) with the following formule:

$$N(r, \varphi) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right).$$

Thus, we omit the details.

3.3. *Proof of Theorem 5*

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then

$$H = \frac{(f^n)^{(k)} - z}{(g^n)^{(k)} - z}, \tag{3.31}$$

where $H(\not\equiv 0, \infty)$ is a meromorphic function.

From (3.31), we get

$$\overline{N}(r, H) \leq \overline{N}_L(r, f),\ \overline{N}\left(r, \frac{1}{H}\right) \leq \overline{N}_L(r, g). \tag{3.32}$$

Let

$$f_1 = \frac{(f^n)^{(k)}}{z},\ f_2 = H,\ f_3 = -\frac{H(g^n)^{(k)}}{z}, \tag{3.33}$$

then $\sum_{i=1}^{3} f_i \equiv 1$.

It follows from Lemma 4 that

$$T(r, f_1) + T(r, f_2) + T(r, f_3) \leq O(T(r, f) + T(r, g)).$$

We first suppose that both f_2 and f_3 are not constants.

If f_1, f_2, f_3 are linearly independent, then by (3.32) and (3.33) and Lemma 6 we have

$$
\begin{aligned}
T(r,f_1) \le & N_2\left(r,\frac{1}{f_1}\right)+N_2\left(r,\frac{1}{f_2}\right)+N_2\left(r,\frac{1}{f_3}\right)+\overline{N}(r,f_1)+\overline{N}(r,f_2)+\overline{N}(r,f_3)+o(T(r))\\
\le & N_2\left(r,\frac{z}{(f^n)^{(k)}}\right)+N_2\left(r,\frac{1}{H}\right)+N_2\left(r,\frac{z}{H(g^n)^{(k)}}\right)+\overline{N}\left(r,\frac{(f^n)^{(k)}}{z}\right)\\
& +\overline{N}(r,H)+\overline{N}\left(r,\frac{H(g^n)^{(k)}}{z}\right)+S(r,f)+S(r,g)\\
\le & N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+2\overline{N}_L(r,g)+N_2\left(r,\frac{1}{(g^n)^{(k)}}\right)+2\overline{N}(r,f)+\overline{N}_L(r,f)\\
& +2\log r+S(r,f)+S(r,g).
\end{aligned} \tag{3.34}
$$

By (3.34), we obtain

$$
\begin{aligned}
& T(r,(f^n)^{(k)}) \le T(r,f_1)+\log r\\
\le & N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+N_2\left(r,\frac{1}{(g^n)^{(k)}}\right)+2\overline{N}_L(r,g)+2\overline{N}(r,f)\\
& +\overline{N}_L(r,f)+3\log r+S(r,f)+S(r,g)\\
= & N\left(r,\frac{1}{(f^n)^{(k)}}\right)-\left[N_{(3}\left(r,\frac{1}{(f^n)^{(k)}}\right)-2\overline{N}_{(3}\left(r,\frac{1}{(f^n)^{(k)}}\right)\right]\\
& +N\left(r,\frac{1}{(g^n)^{(k)}}\right)-\left[N_{(3}\left(r,\frac{1}{(g^n)^{(k)}}\right)-2\overline{N}_{(3}\left(r,\frac{1}{(g^n)^{(k)}}\right)\right]\\
& +2\overline{N}_L(r,g)+2\overline{N}(r,f)+\overline{N}_L(r,f)+3\log r+S(r,f)+S(r,g).
\end{aligned} \tag{3.35}
$$

Let z_0 be a zero of f with multiplicity p. Then z_0 is a zero of $(f^n)^{(k)}$ with multiplicity $np-k \ge 3$. So, we have

$$
N_{(3}\left(r,\frac{1}{(f^n)^{(k)}}\right)-2\overline{N}_{(3}\left(r,\frac{1}{(f^n)^{(k)}}\right)\ge(n-k-2)N\left(r,\frac{1}{f}\right). \tag{3.36}
$$

Similarly, we have

$$
N_{(3}\left(r,\frac{1}{(g^n)^{(k)}}\right)-2\overline{N}_{(3}\left(r,\frac{1}{(g^n)^{(k)}}\right)\ge(n-k-2)N\left(r,\frac{1}{g}\right). \tag{3.37}
$$

By Nevanlinna's first fundamental theorem, we have

$$
m\left(r,\frac{1}{f^n}\right)\le m\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f)=T(r,(f^n)^{(k)})-N\left(r,\frac{1}{(f^n)^{(k)}}\right)+S(r,f). \tag{3.38}
$$

By (3.35)–(3.38) and Lemma 4, we get

$$
\begin{aligned}
nT(r,f) \le & (k+2)N\left(r,\frac{1}{f}\right)+(k+2)N\left(r,\frac{1}{g}\right)+2\overline{N}_L(r,g)\\
& +(k+2)\overline{N}(r,f)+\overline{N}_L(r,f)+3\log r+S(r,f)+S(r,g).
\end{aligned} \tag{3.39}
$$

Similarly, we obtain

$$\begin{aligned} nT(r,g) \leq &(k+2)N\left(r,\frac{1}{g}\right)+(k+2)N\left(r,\frac{1}{f}\right)+2\overline{N}_L(r,f)\\ &+(k+2)\overline{N}(r,g)+\overline{N}_L(r,g)+3\log r+S(r,f)+S(r,g). \end{aligned} \tag{3.40}$$

Noting that

$$\overline{N}_L(r,f)+\overline{N}_L(r,g)\leq \overline{N}(r,f)=\overline{N}(r,g). \tag{3.41}$$

It follows from (3.39)–(3.41) that

$$\begin{aligned} &[n-2(k+2)](T(r,f)+T(r,g))\\ \leq &\left(k+\frac{7}{2}\right)[\overline{N}(r,f)+\overline{N}(r,g)]+6\log r+S(r,f)+S(r,g). \end{aligned} \tag{3.42}$$

Next, we consider two cases.
Case 1. If f and g have poles, since f and g share ∞ IM, then

$$\overline{N}(r,f)=\overline{N}(r,g)\geq \log r. \tag{3.43}$$

Set

$$F=\frac{(f^n)^{(k)}}{z},\ G=\frac{(g^n)^{(k)}}{z}. \tag{3.44}$$

It follows from $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM that F and G share 1 CM almost.
By Lemma 4 and (3.44), we obtain

$$\begin{aligned} T(r,F)\leq &T(r,(f^n)^{(k)})+\log r\\ \leq &T(r,f^n)+k\overline{N}(r,f)+\log r+S(r,f)\\ \leq &(n+k+1)T(r,f)+S(r,f). \end{aligned}$$

It follows $S(r,F)=S(r,f)$. Similarly, we get $S(r,G)=S(r,g)$.
Set

$$\phi=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}. \tag{3.45}$$

In the following, we consider two subcases.
Case 1.1. $\phi\equiv 0$. Then by (3.45), we have

$$\frac{F-1}{F}\equiv C\frac{G-1}{G}, \tag{3.46}$$

where C is a nonzero finite complex number. Next, we consider two subcases.

Case 1.1.1. $C=1$. It follows from (3.46) that $F\equiv G$, that is $(f^n)^{(k)}\equiv (g^n)^{(k)}$. Next, by using the same argument as used in proof of Theorem 1, we have $f\equiv tg$, where t is a constant such that $t^n=1$.

Case 1.1.2. $C\neq 1$. Then by (3.46), we obtain

$$\frac{1}{F}-\frac{C}{G}=1-C. \tag{3.47}$$

Next, using the same argument as used in the proof of Theorem 1, we get a contradiction.

Case 1.2. $\phi \not\equiv 0$.

Let z_0 be a pole of f with multiplicity l_1. Then by f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2. Set $l = \min\{l_1, l_2\}$, by (3.45), we get z_0 is a zero of ϕ with multiplicity $\geq nl + k - 1$. Hence, by Lemma 2, we have

$$\begin{aligned}
&\overline{N}(r,f) = \overline{N}(r,g) \leq \frac{1}{n+k-1} N\left(r, \frac{1}{\phi}\right) \\
\leq &\frac{1}{n+k-1} T(r,\phi) + O(1) \\
= &\frac{1}{n+k-1} N(r,\phi) + \frac{1}{n+k-1} m(r,\phi) + O(1) \\
\leq &\frac{1}{n+k-1}\left[\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right)\right] + S(r,f) + S(r,g).
\end{aligned} \tag{3.48}$$

It follows from Lemma 4 that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F}\right) = \overline{N}\left(r,\frac{z}{(f^n)^{(k)}}\right) = \overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right) \\
= &N\left(r,\frac{1}{(f^n)^{(k)}}\right) - \left[N\left(r,\frac{1}{(f^n)^{(k)}}\right) - \overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\right] \\
\leq &N\left(r,\frac{1}{f^n}\right) + k\overline{N}(r,f) - \left[N\left(r,\frac{1}{(f^n)^{(k)}}\right) - \overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\right] + S(r,f) \\
\leq &(k+1)\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\
\leq &(2k+1)T(r,f) + S(r,f).
\end{aligned} \tag{3.49}$$

Similarly, we have

$$\overline{N}\left(r,\frac{1}{G}\right) \leq (2k+1)T(r,g) + S(r,g). \tag{3.50}$$

By (3.48)–(3.50), we get

$$\overline{N}(r,f) = \overline{N}(r,g) \leq \frac{2k+1}{n+k-1}[T(r,f) + T(r,g)] + S(r,f) + S(r,g). \tag{3.51}$$

Next, we consider two subcases.

If $\Theta_{\min} \geq \frac{2.5}{k+6.5}$. By Lemma 8, we get $S(r,f) = S(r,g)$.

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.

It follows from (3.42) and (3.43) that

$$2[n-2(k+2)]T(r,f) \leq (2k+13)\overline{N}(r,f) + S(r,f), \ r \in I. \tag{3.52}$$

By (3.52), we obtain

$$2[n-2(k+2)] \leq (2k+13)\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{\overline{N}(r,f)}{T(r,f)} + \lim_{\substack{r\to\infty\\ r\in I}}\frac{S(r,f)}{T(r,f)} \leq (2k+13)\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}$$
$$\leq (2k+13)(1-\Theta(\infty,f)) \leq (2k+13)(1-\Theta_{\min}). \tag{3.53}$$

Hence, it follows from $n > 3k + 10.5 - \Theta_{\min}(k + 6.5)$ and (3.53) that we get a contradiction.

Otherwise, by (3.42), (3.43) and (3.51), we get

$$[n-2(k+2)][T(r,f)+T(r,g)]$$
$$\leq \frac{(2k+13)(2k+1)}{n+k-1}[T(r,f)+T(r,g)] + S(r,f) + S(r,g). \tag{3.54}$$

It follows from $n > 3k + 8$ that

$$T(r,f) + T(r,g) \leq S(r,f) + S(r,g),$$

a contradiction.

Thus, we deduce that f_1, f_2, f_3 are linearly dependent, so there exists three constants $(c_1, c_2, c_3) \neq (0,0,0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0. \tag{3.55}$$

Assuming that $c_1 = 0$, by (3.55), we get $c_2 f_2 + c_3 f_3 \equiv 0$ and $c_3 \neq 0$, that is

$$(g^n)^{(k)} = \frac{c_2}{c_3} z,$$

thus, g is a polynomial, which is a contradiction.

Therefore, we get $c_1 \neq 0$. we deduce that $(c_2, c_3) \neq (0,0)$. Suppose that $c_2 \neq 0$, by (3.55) and $\sum_{i=1}^{3} f_i \equiv 1$, we obtain

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1,$$

and $c_1 \neq c_2$, $c_1 \neq c_3$.

Thus, it follows from (3.31) and (3.33) that

$$\left(1 - \frac{c_3}{c_1}\right) \frac{(g^n)^{(k)}}{z} + \frac{(g^n)^{(k)} - z}{(f^n)^{(k)} - z} = 1 - \frac{c_2}{c_1}. \tag{3.56}$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM and f and g have poles, then by (3.56), we get a contradiction.

Therefore, $c_2 = 0$, $c_3 \neq 0$, by (3.55) and $\sum_{i=1}^{3} f_i \equiv 1$, we obtain

$$\left(1 - \frac{c_1}{c_3}\right) f_1 + f_2 = 1,$$

similarly, we have a contradiction.

Hence, we deduce that either f_2 or f_3 is a constant. Next, we consider two subcases.

Case 1.2.1. $f_2 = C$. By (3.33) and $\sum_{i=1}^{3} f_i \equiv 1$, we have

$$\frac{(f^n)^{(k)}}{z} - C\frac{(g^n)^{(k)}}{z} = 1 - C. \tag{3.57}$$

If $C \neq 1$, then by (3.57), we get $(f^n)^{(k)} - C(g^n)^{(k)} = (1-C)z$, that is $(f^n - Cg^n)^{(k)} = (1-C)z$. Thus, we obtain $f^n - Cg^n = p$, where p is a polynomial of degree $k+1$. Then by using the same argument as used in proof of Theorem 1, we get a contradiction.

Hence, $C = 1$, we have $(f^n)^{(k)} \equiv (g^n)^{(k)}$. Next, by using the same argument as used in proof of Theorem 1, we have $f \equiv tg$, where t is a constant such that $t^n = 1$.

Case 1.2.2. $f_3 = C$. By (3.31), (3.33) and $\sum_{i=1}^{3} f_i \equiv 1$, we obtain

$$\frac{(f^n)^{(k)}}{z} + \frac{(f^n)^{(k)} - z}{(g^n)^{(k)} - z} = 1 - C. \tag{3.58}$$

If $C \neq 1$, since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM and f and g have poles, then by (3.58), we get a contradiction.

So, we have $C = 1$. By (3.33) and $\sum_{i=1}^{3} f_i \equiv 1$, we obtain

$$\frac{(g^n)^{(k)}}{z} = -\frac{1}{H}, \ \frac{(f^n)^{(k)}}{z} = -H.$$

Hence, we have $(f^n)^{(k)}(g^n)^{(k)} \equiv z^2$. We know that f and g share ∞ IM, thus, $f \neq \infty,\ g \neq \infty$, we get a contradiction.

Case 2. If f and g are two entire functions, from the assumption and Lemma 8, we deduce that either both f and g are two transcendental entire functions or both f and g are two polynomials. In the following, we consider two subcases.

Case 2.1. f and g are two transcendental entire functions. By the arguments similar to the proof of Case 1, we easily get either $(f^n)^{(k)} \equiv (g^n)^{(k)}$ or $(f^n)^{(k)}(g^n)^{(k)} \equiv z^2$.

If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, then by using the same argument as used in the proof of Theorem 1, we get $f \equiv tg$, where t is a constant such that $t^n = 1$.

If $(f^n)^{(k)}(g^n)^{(k)} \equiv z^2$, it follows from either $n > k+3$ that $f \neq 0,\ g \neq 0$, that is $f^n \neq 0,\ g^n \neq 0$. Furthermore, we see that either $(f^n)^{(k)}$ or $(g^n)^{(k)}$ has at most two zeros.

If $k = 1$, then by Lemma 12, we get $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$.

If $k \geq 2$, since f and g are two nonconstant entire functions and have no zeros, then by Lemma 13, we deduce that $f = e^p$, $g = e^q$, where p and q are two polynomials.

Hence, we have

$$(f^n)^{(k)} = A[(p')^k + P_{k-1}(p')]e^{np},\ (g^n)^{(k)} = A[(q')^k + P_{k-1}(q')]e^{nq}, \tag{3.59}$$

where A is a nonzero constant, $P_{k-1}(h)$ is a differential polynomial in h of degree at most $k-1$ and $h = \{p', q'\}$.

Thus, by (3.59), we get

$$A^2[(p')^k + P_{k-1}(p')][(q')^k + P_{k-1}(q')]e^{n(p+q)} \equiv z^2. \tag{3.60}$$

From (3.60), we deduce $p + q = a$, where a is a constant. Then $p' = -q'$, hence, we obtain

$$A_1(p')^{2k} = z^2 + R_{2k-1}(p'). \tag{3.61}$$

where A_1 is a nonzero constant, $R_{2k-1}(p')$ is a differential polynomial in p' of degree at most $2k - 1$.

It follows (3.61) and $k \geq 2$ that we get a contradiction.

Case 2.2. f and g are two polynomials. Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, we have

$$(f^n)^{(k)} - z = c((g^n)^{(k)} - z), \tag{3.62}$$

where c is a nonzero constant.

If $c \neq 1$, it follows from (3.62) that $(f^n)^{(k)} - c(g^n)^{(k)} = (1 - c)z$, that is $(f^n - cg^n)^{(k)} = (1 - c)z$. Thus, we obtain $f^n - cg^n = p$, where p is a polynomial of degree $k + 1$. Next, by using the same argument as used in proof of Theorem 1, we get a contradiction. Thus $c = 1$, we get $(f^n)^{(k)} \equiv (g^n)^{(k)}$. Next, by using the same argument as used in proof of Theorem 1, we have $f \equiv tg$, where t is a constant such that $t^n = 1$.

This completes the proof of Theorem 5.

3.4. *Proof of Theorem 7*

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ CM, then

$$e^h = \frac{(f^n)^{(k)} - z}{(g^n)^{(k)} - z},$$

where h is a nonconstant entire function.

Next, using the same argument as used in the proof Theorem 5, Theorem 7 is proved. Thus, we omit the details.

4. Conclusions

In this work, we study a uniqueness question of meromorphic functions concerning fixed points. By using deficiencies, we extend and improve some results. We establish the relationship between the uniqueness of meromorphic functions and entire functions.

Acknowledgments

This research was funded by the National Natural Science Foundation of China (Grant No 12171127) and the Natural Science Foundation of Zhejiang Province (Grant No LY21A010012).

Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

Appendix

The exact mistakes in the proof of Theorem 2 in Bhoosnurmath and Dyavanal [2]

In 2007, Bhoosnurmath and Dyavanal made a defective reasoning in the proof of Theorem 2 ([2], p.1200). We now analyze the defective reasoning as follows:

Bhoosnurmath and Dyavanal ([2], p.1200) wrote: Suppose that f has a zero z_0 of order p, then z_0 is a zero of $(f^n)^{(k)}$ of order $(3k+k_1)p-k = 3pk+k_1p-k$, and z_0 is a pole of order $(3k+k_1)q+k = 3qk+k_1q+k$, where $3k + k_1 = n$, $k_1 > 8$. By the assumption

$$(f^n)^{(k)}(g^n)^{(k)} = 1, \tag{4.1}$$

we have

$$3pk + k_1p - k = 3qk + k_1q + k,$$

i.e.,

$$3k(p - q) + k_1(p - q) = 2k,$$

i.e.,

$$(3k + k_1)(p - q) = 2k,$$

which is impossible since $p,\ q$ are integers and $k_1 > 8$. Therefore $f \neq 0$ and $g \neq 0$.

Similarly,

$$f \neq \infty \ \ and \ \ g \neq \infty. \tag{4.2}$$

Obviously, the reasoning of the lines before the claim (4.2) is right. But the claim (4.2) is not easy to obtain. By the context of the claim (4.2), we can find the following meanings of the authors of the reference [2]: Every pole of f satisfying (4.1) must be a zero of g, and every pole of g must be a zero of f. But this is a defective reasoning. In fact, if z_0 is a pole of f satisfying (4.2), then z_0 is a zero of $(g^n)^{(k)}$. But z_0 is not necessarily a zero of g^n and g. Similarly, if z_1 is a pole of g satisfying (4.2), then z_1 is a zero of $(f^n)^{(k)}$. But z_1 is not necessarily a zero of f^n and f. Therefore, the reasoning of the claim (4.2) in the reference [2] is invalid.

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