AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(12): 20461–20489.
DOI: 10.3934/math.20221121
Received: 16 July 2022
Revised: 30 August 2022
Accepted: 02 September 2022
Published: 19 September 2022

Research article

Some new integral inequalities for a general variant of polynomial convex functions

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Abstract: In this study, the concept of $(m, n)-$polynomial (p_1, p_2)- convex functions on the co-ordinates has been established with some basic properties. Dependent on this new concept, a new Hermite-Hadamard type inequality has been proved, then some new integral inequalities have been obtained for partial differentiable $(m, n)-$polynomial (p_1, p_2)- convex functions on the co-ordinates. Several special cases that some of them proved in earlier works have been considered.

Keywords: Hadamard-type inequality; co-ordinates; $(m, n)-$polynomial convex functions
Mathematics Subject Classification: 26D10, 26D15

1. Introduction

While the concept of convex function comes to the forefront with its applications in many branches of mathematics, it is a frequently used concept especially in inequality theory studies. The Hermite-Hadamard inequality, a classical inequality that produces bounds on the Cauchy mean value of a convex function, is one of the most famous inequalities proven in this sense. We will now start by recalling this inequality.

Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined in the interval I of $\mathbb{R}$ where $a, b \in I$ such that $a < b$. The statement below:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities are reversed if f is concave.

Although the concept of convex function has various types and generalizations, it has also been carried to different spaces. In [1], Dragomir touched upon the issue of transferring convex functions to multiple dimensions. This modification is very attractive due to its wide usage in many inequalities and applications in different fields of mathematics, especially convex programming.

Definition 1.1. Let us consider a bi-dimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Giving modifications of the convex function concept in multidimensional spaces undoubtedly revealed the fact that concerning Hadamard type inequalities will be proved for these new function classes. In [1], Dragomir has performed some integral inequalities for double integrals as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$ as following:

Theorem 1.2. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\ \leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\ & \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.1}$$

The above inequalities are sharp.

Numerous variants of this inequality were obtained for convexity and other types of convex functions in co-ordinates by several researchers (see the papers [2–11]). Also, we can state that the authors have established new integral identities in order to prove new inequalities in these papers as following:

Lemma 1.3. *[10] Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:*

$$\frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$
\begin{aligned}
&-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right.\\
&\left.+\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right]\\
=\;&\frac{(b-a)(d-c)}{4}\\
&\times\int_0^1\int_0^1(1-2t)(1-2s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\,dsdt.
\end{aligned}
$$

We shall proceed to recall an interesting class of functions that is called $n-$polynomial convex functions as follows:

Definition 1.4. (See [12]) Let $n\in\mathbb{N}$, $f:I\subset\mathbb{R}\to\mathbb{R}$ is an $n-$polynomial convex function, if

$$f(tx+(1-t)y)\le\frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})f(x)+\frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})f(y)$$

is valid for each $x,y\in I$ and $t\in[0,1]$.

We will indicate by $POLC(I)$ at the interval I as the class of all $n-$ polynomial convex functions. Recently, a lot of developments are done for functions of such classes (see [13–15]) and references therein. In [12], the following Hadamard type of inequality have been demonstrated by Toplu et al. for $n-$polynomial convex functions.

Theorem 1.5. *Let $f\in POLC(I)$, if $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequality holds:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{n}\sum_{\mu=1}^{n}\frac{\mu}{\mu+1}.$$

Theorem 1.6. *(See [16]) Let $f:[a,b]\to\mathbb{R}$ be an $n-$polynomial $p-$convex function. If $a<b$ and $f\in[a,b]$, then the following Hermite-Hadamard type inequalities holds:*

$$
\begin{aligned}
&\frac{1}{2}\Big(\frac{n}{n+2^{-n}-1}\Big)f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big) \qquad (1.2)\\
\le\;&\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\le\frac{f(a)+f(b)}{n}\sum_{\mu=1}^{n}\frac{\mu}{\mu+1}.
\end{aligned}
$$

${}_2F_1$ hypergeometric function which will be used in order to prove the main findings can be defined as (see [17]):

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\quad c>b>0,|z|<1.$$

One of the effective ways to find precise and optimal boundaries for Hadamard type inequalities is to use different kinds of convex functions. As a product of this effort, a new concept $(m,n)-$polynomial $(p_1,p_2)-$convex function will be constructed on the co-ordinates and properties in this article. Also, new integral inequalities of Hadamard type will be proved with this new function class. Considering some special cases of the results, scientific knowledge in this field will be contributed.

2. Main results

We start by introducing the following new class of function unifying convexity and harmonic convexity on the co-ordinates:

Definition 2.1. Let $m, n \in \mathbb{N}$ and $\Delta = [a, b] \times [c, d]$ be a bi-dimensional interval. A non-negative real valued function $f : \Delta \to \mathbb{R}$ is said to be $(m, n)-$ polynomial (p_1, p_2)- convex function on Δ on the co-ordinates, if the following inequality holds:

$$\begin{aligned}
& f\left([tx^{p_1} + (1-t)z^{p_1}]^{\frac{1}{p_1}}, [sy^{p_2} + (1-s)w^{p_2}]^{\frac{1}{p_2}}\right) \\
\leq & \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-(1-s)^{\nu})f(x, y) \\
& + \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})f(x, w) \\
& + \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-(1-s)^{\nu})f(z, y) + \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})f(z, w)
\end{aligned}$$

where $(x, y), (x, w), (z, y), (z, w) \in \Delta$, $p_1, p_2 \in \mathbb{R}$ and $t, s \in [0, 1]$.

Remark 2.2. *If we choose $m = n = 1$, it is easy to see that the definition of $(m, n)-$polynomial (p_1, p_2)-convex function reduces to the class of (p_1, p_2) convex functions.*

Remark 2.3. *If we choose $p_1 = p_2 = 1$ and $p_1 = p_2 = -1$, the definitions of $(m, n)-$polynomial (p_1, p_2)-convex function can be easily declared to be reduce to the class of $(m, n)-$polynomial convex function and $(m, n)-$Harmonically polynomial convex function on co-ordinates on Δ, respectively.*

Remark 2.4. *The $(2, 2)-$polynomial (p_1, p_2) convex functions satisfy the following inequality*

$$\begin{aligned}
& f\left([tx^{p_1} + (1-t)z^{p_1}]^{\frac{1}{p_1}}, [sy^{p_2} + (1-s)w^{p_2}]^{\frac{1}{p_2}}\right) \\
\leq & \frac{3t-t^2}{2}\frac{3s-s^2}{2}f(x, y) + \frac{3t-t^2}{2}\frac{2-s-s^2}{2}f(x, w) \\
& + \frac{2-t-t^2}{2}\frac{3s-s^2}{2}f(z, y) + \frac{2-t-t^2}{2}\frac{2-s-s^2}{2}f(z, w)
\end{aligned}$$

where $(x, y), (x, w), (z, y), (z, w) \in \Delta$ and $t, s \in [0, 1]$.

Example 2.5. *Assume that $f_\alpha : (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a family of $(m, n)-$polynomial (p_1, p_2)-convex functions, and $f(x, y) = \sup f_\alpha(x, y)$. If*

$$K = \{u \in [a, b] \subset (0, \infty), v \in [c, d] \subset (0, \infty)\}$$

is nonempty, then K is a bi-dimensional interval and f is an $(m, n)-$polynomial (p_1, p_2)-convex function on K.

Theorem 2.6. *Assume that $b > a > 0$, $d > c > 0$, $f_\alpha : [a,b] \times [c,d] \to [0,\infty)$ be a family of the $(m,n)-$polynomial (p_1,p_2)-convex function on Δ and $f(u,v) = \sup f_\alpha(u,v)$. Then, f is $(m,n)-$polynomial (p_1,p_2)-convex function on the co-ordinates, if $K = \{x,y \in [a,b]\times[c,d] : f(x,y) < \infty\}$ is bidimensional interval.*

Proof. For $t, s \in [0,1]$ and $(x,y),(x,w),(z,y),(z,w) \in \Delta$, we can write

$$\begin{aligned}
& f\Big([tx^{p_1} + (1-t)z^{p_1}]^{\frac{1}{p_1}}, [sy^{p_2} + (1-s)w^{p_2}]^{\frac{1}{p_2}}\Big) \\
&= \sup f_\infty\Big([tx^{p_1} + (1-t)z^{p_1}]^{\frac{1}{p_1}}, [sy^{p_2} + (1-s)w^{p_2}]^{\frac{1}{p_2}}\Big) \\
&\le \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-(1-s)^\nu)\sup f_\infty(x,y) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-s^\nu)\sup f_\infty(x,w) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-(1-s)^\nu)\sup f_\infty(z,y) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-s^\nu)\sup f_\infty(z,w) \\
&= \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-(1-s)^\nu)f(x,y) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-s^\nu)f(x,w) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-(1-s)^\nu)f(z,y) \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^\mu)\frac{1}{m}\sum_{\nu=1}^{n}(1-s^\nu)f(z,w).
\end{aligned}$$

Which completes the proof. □

Lemma 2.7. *Every $(m,n)-$polynomial (p_1,p_2)-convex function on Δ is $(m,n)-$polynomial (p_1,p_2)-convex function on the co-ordinates.*

Proof. Consider the function $f : \Delta \to \mathbb{R}$ is $(m,n)-$polynomial (p_1,p_2)-convex function on Δ. Then, the partial mapping $f : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ is valid. Then, we can write

$$\begin{aligned}
& f_x\Big([tw^{p_2} + (1-t)v^{p_2}]^{\frac{1}{p_2}}\Big) \\
&= f\Big(x, [tw^{p_2} + (1-t)v^{p_2}]^{\frac{1}{p_2}}\Big) \\
&= f\Big([tx^{p_1} + (1-t)x^{p_1}]^{\frac{1}{p_1}}, [tw^{p_2} + (1-t)v^{p_2}]^{\frac{1}{p_2}}\Big) \\
&\le \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^\mu)f(x,w) + \frac{1}{n}\sum_{\mu=1}^{n}(1-t^\mu)f(x,v)
\end{aligned}$$

$$= \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})f_x(w) + \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})f_x(v).$$

for $\forall\ t \in [0,1]$ and $v, w \in [c,d]$. This shows the $n-$polynomial p_2-convexity of f_x. By a similar argument, one can see the $m-$polynomial p_1-convexity of f_y. We omit the details. □

Now, we will establish associated Hadamard type inequality for $(m,n)-$polynomial (p_1,p_2)- convex function on the co-ordinates.

Theorem 2.8. *Suppose that $f : \Delta \to \mathbb{R}$ is $(m,n)-$polynomial (p_1,p_2)- convex on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{4}\Big(\frac{m}{m+2^{-m}-1}\Big)\Big(\frac{n}{n+2^{-n}-1}\Big)\\
&\times f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\\
&\leq \frac{1}{4}\Big[\Big(\frac{n}{n+2^{-n}-1}\Big)\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f\Big(\Big[\frac{a^{p_1}+b^{p_1}}{2}\Big]^{\frac{1}{p_1}},y\Big)}{y^{1-p_2}}dy\\
&+\Big(\frac{m}{m+2^{-m}-1}\Big)\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)}{x^{1-p_1}}dx\Big]\\
&\leq \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d \frac{f(x,y)}{x^{1-p_1}y^{1-p_2}}dxdy\\
&\leq \frac{1}{2}\Big[\frac{1}{n}\Big(\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(a,y)}{y^{1-p_2}}dy + \frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(b,y)}{y^{1-p_2}}dy\Big)\sum_{t=1}^{n}\frac{t}{t+1}\\
&+\frac{1}{m}\Big(\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,c)}{x^{1-p_1}}dx + \frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,d)}{x^{1-p_1}}dx\Big)\sum_{s=1}^{n}\frac{s}{s+1}\Big]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sum_{s=1}^{m}\frac{s}{s+1}\sum_{t=1}^{n}\frac{t}{t+1}
\end{aligned} \tag{2.1}$$

where for $\forall\ t,s \in [0,1]$ and $p_1,p_2 \in \mathbb{R}$.

Proof. Since f is $(m,n)-$polynomial (p_1,p_2)- convex function on the coordinates, it follows that the mapping h_x and h_y are $(m,n)-$polynomial (p_1,p_2)- convex functions. Therefore, by using the inequality (1.2) for the partial mappings, we can write

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{m}{m+2^{-m}-1}\Big)h_x\Big(\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big)\\
\leq\ &\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(x,y)}{y^{1-p_2}}dy \leq \frac{h_x(c)+h_x(d)}{m}\sum_{s=1}^{m}\frac{s}{s+1}.
\end{aligned} \tag{2.2}$$

Namely,

$$\frac{1}{2}\Big(\frac{m}{m+2^{-m}-1}\Big)f\Big(x,\Big[\frac{c^{p_2}+d^{p_2}}{2}\Big]^{\frac{1}{p_2}}\Big) \tag{2.3}$$

$$\leq \frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(x,y)}{y^{1-p_2}}dy \leq \frac{f(x,c)+f(x,d)}{m}\sum_{s=1}^{m}\frac{s}{s+1}.$$

Dividing both sides of (2.3) by $\frac{1}{b^{p_1}-a^{p_1}}$ and integrating the resulting inequality over $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{2}(\frac{m}{m+2^{-m}-1})\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f\Big(x,\big[\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big)}{x^{1-p_1}}dx \\
&\leq \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d \frac{f(x,y)}{x^{1-p_1}y^{1-p_2}}dxdy \\
&\leq \Big[\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,c)}{x^{1-p_1}}+\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,d)}{x^{1-p_1}}\Big]\frac{1}{m}\sum_{s=1}^{m}\frac{s}{s+1}.
\end{aligned} \tag{2.4}$$

By a similar argument for (2.3), but now for dividing both sides by $\frac{1}{d^{p_2}-c^{p_2}}$ and integrating over $[c,d]$ and by using the mapping h_y is (m,n)−polynomial (p_1,p_2)- convexity we get

$$\begin{aligned}
&\frac{1}{2}(\frac{n}{n+2^{-n}-1})\frac{p_2}{b^{p_1}-a^{p_1}}\int_c^d \frac{f\Big(\big[\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},y\Big)}{y^{1-p_2}}dy \\
&\leq \frac{p_1p_2}{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}\int_a^b\int_c^d \frac{f(x,y)}{x^{1-p_1}y^{1-p_2}}dxdy \\
&\leq \Big[\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(a,y)}{y^{1-p_2}}+\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(b,y)}{y^{1-p_2}}\Big]\frac{1}{n}\sum_{t=1}^{n}\frac{t}{t+1}.
\end{aligned} \tag{2.5}$$

By summing the inequalities (2.4) and (2.5) side by side, we obtain the second and third inequalities of (2.1).
By the inequality (1.2), we also have

$$\begin{aligned}
&\frac{1}{2}(\frac{m}{m+2^{-m}-1})f\Big(\big[\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big) \\
&\leq \frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f\Big([\frac{a^{p_1}+b^{p_1}}{2}]^{\frac{1}{p_1}},y\Big)}{y^{1-p_2}}dy
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}(\frac{n}{n+2^{-n}-1})f\Big(\big[\frac{a^{p_1}+b^{p_1}}{2}\big]^{\frac{1}{p_1}},\big[\frac{c^{p_2}+d^{p_2}}{2}\big]^{\frac{1}{p_2}}\Big) \\
&\leq \frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f\Big(x,[\frac{c^{p_2}+d^{p_2}}{2}]^{\frac{1}{p_2}}\Big)}{x^{1-p_1}}dx.
\end{aligned}$$

Which gives the first inequality of (2.1) by addition.
Finally, by using the inequality (1.2), we obtain

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(a,y)}{y^{1-p_2}}dy \leq \frac{f(a,c)+f(a,d)}{m}\sum_{s=1}^{m}\frac{s}{s+1}$$

$$\frac{p_2}{d^{p_2}-c^{p_2}}\int_c^d \frac{f(b,y)}{y^{1-p_2}}dy \leq \frac{f(b,c)+f(b,d)}{m}\sum_{s=1}^{m}\frac{s}{s+1}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,c)}{x^{1-p_1}}dx \leq \frac{f(a,c)+f(a,d)}{n}\sum_{t=1}^{n}\frac{t}{t+1}$$

$$\frac{p_1}{b^{p_1}-a^{p_1}}\int_a^b \frac{f(x,d)}{x^{1-p_1}}dx \leq \frac{f(a,d)+f(b,d)}{n}\sum_{t=1}^{n}\frac{t}{t+1}.$$

We can provide the last inequality of (2.1) by addition.

□

Remark 2.9. *If we choose $p_1 = p_2 = 1$, then we get the Hermite- Hadamard type inequality for $(m,n)-$polynomial convex function on coordinates on Δ.*

$$\begin{aligned}
&\frac{1}{4}\left(\frac{m}{m+2^{-m}-1}\right)\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;& \frac{1}{4}\Bigg[\left(\frac{m}{m+2^{-m}-1}\right)\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&+\left(\frac{n}{n+2^{-n}-1}\right)\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\Bigg]\\
\leq\;& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\
\leq\;& \frac{1}{2}\Bigg[\frac{1}{n}\left(\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right)\sum_{\mu=1}^{n}\frac{\mu}{\mu+1}\\
&+\frac{1}{m}\left(\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right)\sum_{\nu=1}^{m}\frac{\nu}{\nu+1}\Bigg]\\
\leq\;& \left(\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{nm}\right)\left(\sum_{\mu=1}^{n}\frac{\mu}{\mu+1}\sum_{\nu=1}^{m}\frac{\nu}{\nu+1}\right).
\end{aligned}$$

Remark 2.10. *If we choose $p_1 = p_2 = -1$, then we get the $(m,n)-$Harmonically polynomial convex function on Δ (See [18]).*

Lemma 2.11. *Let $f : \Delta \to \mathbb{R}$ be a twice partial differentiable function on $\Delta = [a,b]\times[c,d] \subset (0,\infty)\times(0,\infty)$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t\partial s} \in L_1(\Delta)$, then we have*

$$\begin{aligned}
&M(p_1,p_2;f)\\
=\;& \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\\
&\times\int_0^1\int_0^1 \Big(\frac{1-2t}{[ta^{p_1}+(1-t)b^{p_1}]^{1-\frac{1}{p_1}}}\Big)\Big(\frac{1-2s}{[sc^{p_2}+(1-s)d^{p_2}]^{1-\frac{1}{p_2}}}\Big)\\
&\times\frac{\partial^2 f}{\partial t\partial s}\Big([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}},[sc^{p_2}+(1-s)d^{p_2}]^{\frac{1}{p_2}}\Big)dtds
\end{aligned}$$

where

$$\begin{aligned}
&M(p_1, p_2; f)\\
&= \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4}\\
&- \frac{1}{2}\left\{ \frac{p_1}{b^{p_1} - a^{p_1}} \left\{ \int_a^b \frac{f(x,c)}{x^{1-p_1}} dx + \frac{f(x,d)}{x^{1-p_1}} dx \right\} \right.\\
&\left. + \frac{p_2}{d^{p_2} - c^{p_2}} \left\{ \int_c^d \frac{f(a,y)}{y^{1-p_2}} dy + \frac{f(b,y)}{y^{1-p_2}} dy \right\} \right\}\\
&+ \frac{p_1 p_2}{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})} \int_a^b \int_c^d \frac{f(x,y)}{x^{1-p_1} y^{1-p_2}} dx dy
\end{aligned}$$

for $p_1, p_2 \in \mathbb{R}$.

Proof. It suffices to note that,

$$\begin{aligned}
&M(p_1, p_2; f)\\
&= \frac{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})}{4 p_1 p_2}\\
&\times \int_0^1 \left(\frac{1-2t}{[ta^{p_1} + (1-t) b^{p_1}]^{1-\frac{1}{p_1}}} \right) \left\{ \int_0^1 \left(\frac{1-2r}{[rc^{p_2} + (1-r) d^{p_2}]^{1-\frac{1}{p_2}}} \right) \right.\\
&\left. \times \frac{\partial^2 f}{\partial t \partial r} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, [rc^{p_2} + (1-r) d^{p_2}]^{\frac{1}{p_2}} \right) dr \right\} dt.
\end{aligned} \tag{2.6}$$

We will denote

$$\begin{aligned}
I_1 = &\int_0^1 \left(\frac{1-2r}{[rc^{p_2} + (1-r) d^{p_2}]^{1-\frac{1}{p_2}}} \right)\\
&\times \frac{\partial^2 f}{\partial t \partial r} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, [rc^{p_2} + (1-r) d^{p_2}]^{\frac{1}{p_2}} \right) dr.
\end{aligned}$$

Now, by applying integrating by parts for I_1, we have

$$\begin{aligned}
I_1 = &\frac{p_2}{(d^{p_2} - c^{p_2})} \frac{\partial f}{\partial t} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, c \right)\\
&+ \frac{p_2}{(d^{p_2} - c^{p_2})} \frac{\partial f}{\partial t} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, d \right)\\
&- \frac{2p_2}{(d^{p_2} - c^{p_2})} \int_0^1 \frac{\partial f}{\partial t} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, [rc^{p_2} + (1-r) d^{p_2}]^{\frac{1}{p_2}} \right) dr.
\end{aligned}$$

Integrating again the equality above and using also the (2.6), we get

$$I_2 = \int_0^1 \left(\frac{1-2t}{[ta^{p_1} + (1-t) b^{p_1}]^{1-\frac{1}{p_1}}} \right) \frac{\partial f}{\partial t} \left([ta^{p_1} + (1-t) b^{p_1}]^{\frac{1}{p_1}}, c \right) dt$$

$$= \frac{p_1}{(b^{p_1}-a^{p_1})}\{f(a,c)+f(b,c)\} - \frac{2p_1^2}{(b^{p_1}-a^{p_1})^2}\int_a^b \frac{f(x,c)}{x^{1-p_1}}dx.$$

$$I_3 = \int_0^1 \left(\frac{1-2t}{[ta^{p_1}+(1-t)b^{p_1}]^{1-\frac{1}{p_1}}}\right)\frac{\partial f}{\partial t}\left([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}}, d\right)dt$$
$$= \frac{p_1}{(b^{p_1}-a^{p_1})}\{f(a,d)+f(b,d)\} - \frac{2p_1^2}{(b^{p_1}-a^{p_1})^2}\int_a^b \frac{f(x,d)}{x^{1-p_1}}dx.$$

$$I_4 = \int_0^1 \left\{\int_0^1 \left(\frac{1-2t}{[ta^{p_1}+(1-t)b^{p_1}]^{1-\frac{1}{p_1}}}\right)\right.$$
$$\left.\times\frac{\partial f}{\partial t}\left([ta^{p_1}+(1-t)b^{p_1}]^{\frac{1}{p_1}}, [rc^{p_2}+(1-r)d^{p_2}]^{\frac{1}{p_2}}\right)dt\right\}dr.$$

Summing up of above I_1 to I_4 and changing of the variables, we complete the proof of the lemma. □

Theorem 2.12. *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable function on $\Delta=[a,b]\times[c,d]\in(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial t\partial s}$ is (m,n)–polynomial (p_1,p_2)- convex functions on Δ such that $\frac{\partial^2 f}{\partial t\partial s}\in L_1(\Delta)$, then one has the inequality:*

$$\left|M(p_1,p_2;f)\right|$$
$$\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\left[C_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| + C_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right.$$
$$\left.+C_3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| + C_4\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right]$$

for $p_1,p_2\in\mathbb{R}$ where

$$C_1 = \frac{1}{n}\sum_{\mu=1}^{n}\left[\frac{1}{b^{(p_1-1)}}\left({}_2F_1\left(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\right) - {}_2F_1\left(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\right)\right.\right.$$
$$+{}_2F_1\left(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\right) + \frac{1}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\right)$$
$$\left.-\frac{2}{(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\right)\right)$$
$$+\frac{1}{a^{(p_1-1)}}\left(\frac{2}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{b^{p_1}}{a^{p_1}}\right)\right.$$
$$-\frac{4}{\mu+2}{}_2F_1\left(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{b^{p_1}}{a^{p_1}}\right)$$
$$\left.\left.-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right)\right]$$
$$\frac{1}{m}\sum_{\nu=1}^{m}\left[\frac{1}{d^{(p_2-1)}}\left({}_2F_1\left(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\right)\right.\right.$$

$$
\begin{aligned}
&- {}_2F_1\left(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\right)+{}_2F_1\left(1-\frac{1}{p_2},1;3;\frac{1}{2}\left(1-\frac{c^{p_2}}{d^{p_2}}\right)\right)\\
&+\frac{1}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\right)\Bigg)\\
&-\frac{2}{(\nu+1)(\nu+2)}\frac{1}{p_2}.{}_2F_1\left(1-\frac{1}{p_2},2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\right)\\
&+\frac{1}{c^{(p_2-1)}}\left(\frac{2}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{d^{p_2}}{c^{p_2}}\right)\right.\\
&-\frac{4}{\nu+2}{}_2F_1\left(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{d^{p_2}}{c^{p_2}}\right)\\
&\left.\left.-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
C_2 = &\frac{1}{n}\sum_{\mu=1}^{n}\left[\frac{1}{b^{(p_1-1)}}\left({}_2F_1\left(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\right)-{}_2F_1\left(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\right)\right.\right.\\
&+{}_2F_1\left(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\right)+\frac{1}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\right)\Bigg)\\
&-\frac{2}{(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\right)\\
&+\frac{1}{a^{(p_1-1)}}\left(\frac{2}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{b^{p_1}}{a^{p_1}}\right)\right.\\
&-\frac{4}{\mu+2}{}_2F_1\left(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{b^{p_1}}{a^{p_1}}\right)\\
&\left.\left.-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}\left(1-\frac{b^{p_1}}{a^{p_1}}\right)\right)\right)\right]\\
&\frac{1}{m}\sum_{\nu=1}^{m}\left[\frac{1}{d^{(p_2-1)}}\left({}_2F_1\left(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\right)-{}_2F_1\left(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\right)\right.\right.\\
&+{}_2F_1\left(1-\frac{1}{p_2},1;3;1-\frac{c^{p_2}}{d^{p_2}}\right)+\frac{1}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\right)\\
&-\frac{2}{(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\right)\\
&\left.\left.-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
C_3 = &\frac{1}{n}\sum_{\mu=1}^{n}\left[\frac{1}{b^{(p_1-1)}}\left({}_2F_1\left(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\right)-{}_2F_1\left(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\right)\right.\right.\\
&+{}_2F_1\left(1-\frac{1}{p_1},1;3;\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)+\frac{1}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\right)
\end{aligned}
$$

$$
-\frac{2}{(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\right)\Bigg)
$$
$$
-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\right)\Bigg)\Bigg]
$$
$$
\times\frac{1}{m}\sum_{\nu=1}^{m}\left[\frac{1}{d^{(p_2-1)}}\left({}_2F_1\left(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\right)-{}_2F_1\left(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\right)\right.\right.
$$
$$
+{}_2F_1\left(1-\frac{1}{p_2},1;3;1-\frac{c^{p_2}}{d^{p_2}}\right)+\frac{1}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\right)
$$
$$
-\frac{2}{(\nu+1)(\nu+2)}\frac{1}{p_2}{}_2F_1\left(1-\frac{1}{p_2},2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\right)
$$
$$
+\frac{1}{c^{(p_2-1)}}\left(\frac{2}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\right)-\frac{4}{\nu+2}{}_2F_1\left(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\right)\right.
$$
$$
\left.\left.-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right)\right],
$$

$$
C_4=\frac{1}{n}\sum_{\mu=1}^{n}\left[\frac{1}{b^{(p_1-1)}}\left({}_2F_1\left(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\right)-{}_2F_1\left(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\right)\right.\right.
$$
$$
+{}_2F_1\left(1-\frac{1}{p_1},1;3;\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)-\frac{2}{(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\right)
$$
$$
-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)
$$
$$
+{}_2F_1\left(1-\frac{1}{p_1},1;3;\frac{1}{2}\left(1-\frac{a^{p_1}}{b^{p_1}}\right)\right)+\frac{1}{\mu+1}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\right)\Bigg)
$$
$$
-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\left(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\right)\Bigg)\Bigg]
$$
$$
\times\frac{1}{m}\sum_{\nu=1}^{m}\left[\frac{1}{d^{(p_2-1)}}\left({}_2F_1\left(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\right)-{}_2F_1\left(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\right)\right.\right.
$$
$$
+{}_2F_1\left(1-\frac{1}{p_2},1;3;1-\frac{c^{p_2}}{d^{p_2}}\right)
$$
$$
+\frac{1}{\nu+1}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\right)-\frac{2}{(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\right)
$$
$$
\left.\left.-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\left(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}\left(1-\frac{d^{p_2}}{c^{p_2}}\right)\right)\right)\right]
$$

and $A_t=[ta^{p_1}+(1-t)b^{p_1}]$, $B_s=[sc^{p_2}+(1-s)d^{p_2}]$ *for fixed* $t,s\in[0,1]$.

Proof. From the definition of $(m,n)-$polynomial (p_1,p_2)- convex functions, we can write

$$
\left|\frac{\partial^2 f}{\partial t\partial s}(A_t{}^{\frac{1}{p_1}},B_s{}^{\frac{1}{p_2}})\right|
$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-(1-s)^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-(1-s)^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|.
\end{aligned}
$$

By using the above inequality with lemma, we have

$$
\begin{aligned}
&\Big|M(p_1,p_2;f)\Big| \\
&\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\int_0^1\int_0^1 \frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}\Big|\frac{\partial^2 f}{\partial t\partial s}\Big(A_t^{\frac{1}{p_1}}B_s^{\frac{1}{p_2}}\Big)\Big|dtds \\
&\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\int_0^1\int_0^1 \frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}} \\
&\times\Big[\frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-(1-s)^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-(1-t)^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big| \\
&+ \frac{1}{n}\sum_{\mu=1}^{n}(1-t^{\mu})\frac{1}{m}\sum_{\nu=1}^{m}(1-s^{\nu})\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|\Big]dtds.
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\Big|M(p_1,p_2;f)\Big| \\
&\leq \frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Big[\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-(1-t)^{\mu})}{A_t^{1-\frac{1}{p_1}}}dt \\
&\times\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2t|(1-(1-s)^{\nu})}{B_s^{1-\frac{1}{p_2}}}ds \\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-(1-t)^{\mu})}{A_t^{1-\frac{1}{p_1}}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-s^{\nu})}{B_s^{1-\frac{1}{p_2}}}ds \\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-t^{\mu})}{A_t^{1-\frac{1}{p_1}}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-s^{\nu})}{B_s^{1-\frac{1}{p_2}}}ds
\end{aligned}
\tag{2.7}
$$

$$+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-t^{\mu})}{A_t^{1-\frac{1}{p_1}}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-s^{\nu})}{B_s^{1-\frac{1}{p_2}}}ds\Bigg].$$

By computing the above integrals, we can easily see that

$$\begin{aligned}&\Big|M(p_1,p_2;f)\Big| \\ \leq\ &\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Bigg[C_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+C_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\ &+C_3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+C_4\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Bigg].\end{aligned}\tag{2.8}$$

This completes the proof. □

Corollary 2.13. If we set $m = n = 1$ in (2.8), we have the following inequality for (p_1, p_2)-convex function on Δ.

$$\begin{aligned}&\Big|M(p_1,p_2;f)\Big| \\ \leq\ &\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Bigg[C_{11}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+C_{22}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\ &+C_{33}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+C_{44}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Bigg]\end{aligned}$$

where

$$\begin{aligned}C_{11}=&\Bigg[\frac{1}{b^{(p_1-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)-{}_2F_1\Big(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big) \\ &+{}_2F_1\Big(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)+\frac{1}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big) \\ &-\frac{2}{(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big] \\ &+\frac{1}{a^{(p_1-1)}}\Big[\frac{2}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{b^{p_1}}{a^{p_1}}\Big) \\ &-\frac{4}{\mu+2}{}_2F_1\Big(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{b^{p_1}}{a^{p_1}}\Big) \\ &-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{b^{p_1}}{a^{p_1}})\Big)\Big]\Bigg] \\ &\cdot\Bigg[\frac{1}{d^{(p_2-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big) \\ &-{}_2F_1\Big(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)+{}_2F_1\Big(1-\frac{1}{p_2},1;3;\frac{1}{2}(1-\frac{c^{p_2}}{d^{p_2}})\Big) \\ &+\frac{1}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\end{aligned}$$

$$-\frac{2}{(\nu+1)(\nu+2)}\frac{1}{p_2}{}_2F_1\Big(1-\frac{1}{p_2},2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big]$$
$$+\frac{1}{c^{(p_2-1)}}\Big[\frac{2}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{d^{p_2}}{c^{p_2}}\Big)-$$
$$\frac{4}{\nu+2}{}_2F_1\Big(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{d^{p_2}}{c^{p_2}}\Big)$$
$$-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}(1-\frac{d^{p_2}}{c^{p_2}})\Big)\Big]\Big],$$

$$C_{22}=\Big[\frac{1}{b^{(p_1-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)$$
$$-{}_2F_1\Big(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)+{}_2F_1\Big(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)$$
$$+\frac{1}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)$$
$$-\frac{2}{(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]$$
$$+\frac{1}{a^{(p_1-1)}}\Big[\frac{2}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{b^{p_1}}{a^{p_1}}\Big)$$
$$-\frac{4}{\mu+2}{}_2F_1\Big(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{b^{p_1}}{a^{p_1}}\Big)$$
$$-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{b^{p_1}}{a^{p_1}})\Big)\Big]\Big]$$
$$\Big[\frac{1}{d^{(p_2-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)$$
$$-{}_2F_1\Big(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)+{}_2F_1\Big(1-\frac{1}{p_2},1;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)$$
$$+\frac{1}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)$$
$$-\frac{2}{(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\Big)$$
$$-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}(1-\frac{d^{p_2}}{c^{p_2}})\Big)\Big]\Big],$$

$$C_{33}=\Big[\frac{1}{b^{(p_1-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)$$
$$-{}_2F_1\Big(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)+{}_2F_1\Big(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)$$
$$+\frac{1}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)$$

$$
\begin{aligned}
&- \frac{2}{(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)\Big]\Big]\\
&\Big[\frac{1}{d^{(p_2-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)-{}_2F_1\Big(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&+{}_2F_1\Big(1-\frac{1}{p_2},1;3;\frac{1}{2}(1-\frac{c^{p_2}}{d^{p_2}})\Big)+\frac{1}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{2}{(\nu+1)(\nu+2)}\frac{1}{2}F_1\Big(1-\frac{1}{p_2},2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big]\\
&+\frac{1}{c^{(p_2-1)}}\Big[\frac{2}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{d^{p_2}}{c^{p_2}}\Big)-\\
&\frac{4}{\nu+2}{}_2F_1\Big(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{d^{p_2}}{c^{p_2}}\Big)\\
&-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}(1-\frac{d^{p_2}}{c^{p_2}})\Big)\Big]\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{44}=&\Big[\frac{1}{b^{(p_1-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-{}_2F_1\Big(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)+{}_2F_1\Big(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)\\
&+\frac{1}{\mu+1}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{2}{(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+2;\mu+3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(1-\frac{1}{p_1},\mu+1;\mu+3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)\Big]\Big]\\
&\Big[\frac{1}{d^{(p_2-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-{}_2F_1\Big(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)+{}_2F_1\Big(1-\frac{1}{p_2},1;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&+\frac{1}{\nu+1}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{2}{(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+2;\nu+3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(1-\frac{1}{p_2},\nu+1;\nu+3;\frac{1}{2}(1-\frac{d^{p_2}}{c^{p_2}})\Big)\Big].
\end{aligned}
$$

Corollary 2.14. If we set $p_1 = p_2 = 1$ in (2.8), then we have the following inequality for $(m, n)-$polynomial convex function on Δ.

$$\begin{aligned}
&\Big|M(1,1;f)\Big| \\
\leq\ & \frac{(b-a)(d-c)}{mn}\sum_{\mu=1}^{n}\Big[\frac{2^{\mu}(\mu^2+\mu+2)-2}{2^{\mu+1}(\mu+1)(\mu+2)}\Big]\sum_{\nu=1}^{m}\Big[\frac{2^{\nu}(\nu^2+\nu+2)-2}{2^{\nu+1}(\nu+1)(\nu+2)}\Big] \\
&\times\Bigg[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
&M(1,1;f) \\
=\ & \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&-\frac{1}{2}\Bigg[\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy \\
&+\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\Bigg]
\end{aligned}$$

for $t,s\in[0,1]$.

Corollary 2.15. If we set $p_1 = p_2 = -1$ in (2.8) then we have the following inequality for $(m,n)-$harmonically polynomial convex function on Δ.

$$\begin{aligned}
&\Big|M(-1,-1;f)\Big| \\
\leq\ & \frac{abcd(b-a)(d-c)}{4}\Bigg[C_1^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+C_2^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&+C_3^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+C_4^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
&|M(-1,-1;f) \\
=\ & \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)}{(xy)^2}dydx \\
&-\frac{1}{2}\Bigg[\frac{cd}{d-c}\int_c^d \frac{f(a,y)}{y^2}dy+\frac{cd}{d-c}\int_c^d \frac{f(b,y)}{y^2}dy \\
&+\frac{ab}{b-a}\int_a^b \frac{f(x,c)}{x^2}dx+\frac{ab}{b-a}\int_a^b \frac{f(x,d)}{x^2}dx\Bigg],
\end{aligned}$$

$$C_1^{\star} = \frac{1}{n}\sum_{\mu=1}^{n}\Bigg[a^2\Big[{}_2F_1\Big(2,2;3;1-\frac{b}{a}\Big)$$

$$
\begin{aligned}
& - {}_2F_1\Big(2,1;2;1-\frac{b}{a}\Big) + {}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{b}{a})\Big) \\
& + \frac{1}{\mu+1}{}_2F_1\Big(2,1;\mu+2;1-\frac{b}{a}\Big) - \frac{2}{(\mu+1)(\mu+2)}{}_2F_1\Big(2,2;\mu+3;1-\frac{b}{a}\Big)\Big] \\
& + b^2\Big[\frac{2}{(\mu+1)}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{a}{b}\Big) \\
& - \frac{4}{(\mu+2)}{}_2F_1\Big(2,\mu+2;\mu+3;1-\frac{a}{b}\Big) - \frac{1}{2^{\mu-1}(\mu+1)(\mu+2)} \\
& \times {}_2F_1\Big(2,\mu+1;\mu+3;\frac{1}{2}(1-\frac{a}{b})\Big)\Big]\Bigg] \\
& \frac{1}{m}\sum_{\nu=1}^{m}\Bigg[c^2\Big[{}_2F_1\Big(2,2;3;1-\frac{d}{c}\Big) - {}_2F_1\Big(2,1;2;1-\frac{d}{c}\Big) \\
& + {}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{d}{c})\Big) + \frac{1}{\nu+1}{}_2F_1\Big(2,1;\nu+2;1-\frac{d}{c}\Big) \\
& - \frac{2}{(\nu+1)(\nu+2)}{}_2F_1\Big(2,2;\nu+3;1-\frac{d}{c}\Big)\Big] \\
& + d^2\Big[\frac{2}{(\nu+1)}{}_2F_1\Big(2,\nu+1;\nu+2;1-\frac{c}{d}\Big) \\
& - \frac{4}{(\nu+2)}{}_2F_1\Big(2,\nu+2;\nu+3;1-\frac{c}{d}\Big) \\
& - \frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(2,\nu+1;\nu+3;\frac{1}{2}(1-\frac{c}{d})\Big)\Big]\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
C_2^\star =& \frac{1}{n}\sum_{\mu=1}^{n}\Bigg[a^2\Big[{}_2F_1\Big(2,2;3;1-\frac{b}{a}\Big) - {}_2F_1\Big(2,1;2;1-\frac{b}{a}\Big) \\
& + {}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{b}{a})\Big) + \frac{1}{\mu+1} - \frac{b}{a}\Big) + {}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{b}{a})\Big) \\
& + \frac{1}{\mu+1}{}_2F_1\Big(2,1;\mu+2;1-\frac{b}{a}\Big) - \frac{2}{(\mu+1)(\mu+2)}{}_2F_1\Big(2,2;\mu+3;1-\frac{b}{a}\Big)\Big] \\
& + b^2\Big[\frac{2}{(\mu+1)}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{a}{b}\Big) - \frac{4}{(\mu+2)}{}_2F_1\Big(2,\mu+2;\mu+3;1-\frac{a}{b}\Big) \\
& - \frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(2,\mu+1;\mu+3;\frac{1}{2}(1-\frac{a}{b})\Big)\Big]\Bigg] \\
& \frac{1}{m}\sum_{\nu=1}^{m}\Bigg[c^2\Big[{}_2F_1\Big(2,2;3;1-\frac{d}{c}\Big) - {}_2F_1\Big(2,1;2;1-\frac{d}{c}\Big) \\
& + {}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{d}{c})\Big) + \frac{1}{\nu+1}{}_2F_1\Big(2,\nu+1;\nu+2;1-\frac{d}{c}\Big) \\
& - \frac{2}{(\nu+2)}{}_2F_1\Big(2,\nu+2;\nu+3;1-\frac{d}{c}\Big) \\
& - \frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(2,\nu+1;\nu+3;\frac{1}{2}(1-\frac{d}{c})\Big)\Big]\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
C_3^\star =&\frac{1}{n}\sum_{\mu=1}^{n}\Bigg[a^2\Big[{}_2F_1\Big(2,2;3;1-\frac{b}{a}\Big)-{}_2F_1\Big(2,1;2;1-\frac{b}{a}\Big)\\
&+{}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{b}{a})\Big)\\
&+\frac{1}{\mu+1}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{b}{a}\Big)-\frac{2}{(\mu+2)}{}_2F_1\Big(2,\mu+2;\mu+3;1-\frac{b}{a}\Big)\\
&+\frac{2}{(\mu+1)}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{a}{b}\Big)\\
&-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(2,\mu+1;\mu+3;\frac{1}{2}(1-\frac{b}{a})\Big)\Big]\Bigg]\\
&\frac{1}{m}\sum_{\nu=1}^{m}\Bigg[c^2\Big[{}_2F_1\Big(2,2;3;1-\frac{d}{c}\Big)-{}_2F_1\Big(2,1;2;1-\frac{d}{c}\Big)\\
&+{}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{d}{c})\Big)+\frac{1}{\nu+1}{}_2F_1\Big(2,1;\nu+2;1-\frac{d}{c}\Big)\\
&-\frac{2}{(\nu+1)(\nu+2)}{}_2F_1\Big(2,2;\nu+3;1-\frac{d}{c}\Big)\Big]\\
&+d^2\Big[\frac{2}{(\nu+1)}{}_2F_1\Big(2,\nu+1;\nu+2;1-\frac{c}{d}\Big)\\
&-\frac{4}{(\nu+2)}{}_2F_1\Big(2,\nu+2;\nu+3;1-\frac{c}{d}\Big)\\
&-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(2,\nu+1;\nu+3;\frac{1}{2}(1-\frac{c}{d})\Big)\Big]\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
C_4^\star =&\frac{1}{n}\sum_{\mu=1}^{n}\Bigg[a^2\Big[{}_2F_1\Big(2,2;3;1-\frac{b}{a}\Big)-{}_2F_1\Big(2,1;2;1-\frac{b}{a}\Big)\\
&+{}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{b}{a})\Big)+\frac{1}{\mu+1}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{b}{a}\Big)\\
&-\frac{2}{(\mu+2)}{}_2F_1\Big(2,\mu+2;\mu+3;1-\frac{b}{a}\Big)\\
&+\frac{2}{(\mu+1)}{}_2F_1\Big(2,\mu+1;\mu+2;1-\frac{a}{b}\Big)\\
&-\frac{1}{2^{\mu-1}(\mu+1)(\mu+2)}{}_2F_1\Big(2,\mu+1;\mu+3;\frac{1}{2}(1-\frac{b}{a})\Big)\Big]\Bigg]\\
&\frac{1}{m}\sum_{\nu=1}^{m}\Bigg[c^2\Big[{}_2F_1\Big(2,2;3;1-\frac{d}{c}\Big)-{}_2F_1\Big(2,1;2;1-\frac{d}{c}\Big)\\
&+{}_2F_1\Big(2,1;3;\frac{1}{2}(1-\frac{d}{c})\Big)+\frac{1}{\nu+1}{}_2F_1\Big(2,\nu+1;\nu+2;1-\frac{d}{c}\Big)\\
&-\frac{2}{(\nu+2)}{}_2F_1\Big(2,\nu+2;\nu+3;1-\frac{d}{c}\Big)
\end{aligned}
$$

$$-\frac{1}{2^{\nu-1}(\nu+1)(\nu+2)}{}_2F_1\Big(2,\nu+1;\nu+3;\frac{1}{2}(1-\frac{d}{c})\Big)\Big]\Big].$$

Theorem 2.16. *Let* $f:\Delta=[a,b]\times[c,d]\in(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a partial differential mapping on* Δ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* (m,n)*–polynomial* (p_1,p_2)*- convex function on* Δ*, then one has the following inequality*

$$\begin{aligned}
&\Big|M(p_1,p_2;f)\Big|\\
\leq\ &\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2(r+1)^{\frac{2}{r}}}\Big[C_5\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q+C_6\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q\\
&+C_7\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q+C_8\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\Big]^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
&C_5\\
=&\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-(1-t)^\mu)A_t^{-q(1-\frac{1}{p_1})}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-(1-s)^\nu)B_s^{-q(1-\frac{1}{p_2})}ds\\
=&\frac{1}{b^{q(p_1-1)}}\frac{1}{n}\sum_{\mu=1}^{n}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{\mu+1}{}_2F_1\Big(q(1-\frac{1}{p_1}),1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times\frac{1}{d^{q(p_2-1)}}\frac{1}{m}\sum_{\nu=1}^{m}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{\nu+1}{}_2F_1\Big(q(1-\frac{1}{p_2}),1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
&C_6\\
=&\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-(1-t)^\mu)A_t^{-q(1-\frac{1}{p_1})}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-s^\nu)B_s^{-q(1-\frac{1}{p_2})}ds\\
=&\frac{1}{b^{q(p_1-1)}}\frac{1}{n}\sum_{\mu=1}^{n}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{\mu+1}{}_2F_1\Big(q(1-\frac{1}{p_1}),1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times\frac{1}{d^{q(p_2-1)}}\frac{1}{m}\sum_{\nu=1}^{m}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{\nu+1}{}_2F_1\Big(q(1-\frac{1}{p_2}),\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}$$

$$
\begin{aligned}
&C_7\\
&=\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-t^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-(1-s)^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds\\
&=\frac{1}{b^{q(p_1-1)}}\frac{1}{n}\sum_{\mu=1}^{n}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&\quad-\frac{1}{\mu+1}{}_2F_1\Big(q(1-\frac{1}{p_1}),\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\quad\times\frac{1}{d^{q(p_2-1)}}\frac{1}{m}\sum_{\nu=1}^{m}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&\quad-\frac{1}{\nu+1}{}_2F_1\Big(q(1-\frac{1}{p_2}),1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
C_8=&\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-t^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-s^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds\\
=&\frac{1}{b^{q(p_1-1)}}\frac{1}{n}\sum_{\mu=1}^{n}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{\mu+1}{}_2F_1\Big(q(1-\frac{1}{p_1}),\mu+1;\mu+2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times\frac{1}{d^{q(p_2-1)}}\frac{1}{m}\sum_{\nu=1}^{m}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{\nu+1}{}_2F_1\Big(q(1-\frac{1}{p_2}),\nu+1;\nu+2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big]
\end{aligned}
$$

where $A_t=[ta^{p_1}+(1-t)b^{p_1}]$ *and* $B_s=[sc^{p_2}+(1-s)d^{p_2}]$ *for fixed* $t,s\in[0,1]$, $r,q>1$ *and* $\frac{1}{r}+\frac{1}{q}$.

Proof. With the aid of the identity that is given in Lemma 2.11 and by using the Hölder inequality for double integrals, we get

$$
\begin{aligned}
&\Big|M(p_1,p_2;f)\Big|\\
\le&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\int_0^1\int_0^1\frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}\Big|\frac{\partial^2 f}{\partial t\partial s}\Big(A_t^{\frac{1}{p_1}}B_s^{\frac{1}{p_2}}\Big)\Big|dtds\\
\le&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Big(\int_0^1\int_0^1|1-2t|^r|1-2s|^r dtds\Big)^{\frac{1}{r}}\\
&\times\Big(\int_0^1\int_0^1 A_t^{-q(1-\frac{1}{p_1})}B_s^{-q(1-\frac{1}{p_2})}dtds\Big|\frac{\partial^2 f}{\partial t\partial s}(A_t^{\frac{1}{p_1}}B_s^{\frac{1}{p_2}})\Big|^q\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Taking into account $(m,n)-$polynomial (p_1,p_2)- convexity of $|\frac{\partial^2 f}{\partial t\partial s}|^q$, we obtain

$$\begin{aligned}
&|M(p_1,p_2;f)| \\
\leq&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4}\Big(\frac{1}{r+1}\Big)^{\frac{2}{r}} \\
&\Bigg[\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-(1-t)^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt \\
&\times\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-(1-s)^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds \\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-(1-t)^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt \\
&\times\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-s^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds \\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-t^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt \\
&\times\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-(1-s)^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds \\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1(1-t^{\mu})A_t^{-q(1-\frac{1}{p_1})}dt \\
&\times\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1(1-s^{\nu})B_s^{-q(1-\frac{1}{p_2})}ds\Bigg]^{\frac{1}{q}}.
\end{aligned}\tag{2.9}$$

We get the desired result by computing the above integral. □

Corollary 2.17. If we set $m=n=1$ in Theorem 7, we have the following inequality (p_1,p_2)-convex function on Δ.

$$\begin{aligned}
&\Big|M(p_1,p_2;f)\Big| \\
\leq\ &\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2(r+1)^{\frac{2}{r}}}\Bigg[C_{55}\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q+C_{66}\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q \\
&+C_{77}\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q+C_{88}\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\Bigg]^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
C_{55}=&\frac{1}{b^{q(p_1-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big) \\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_1}),1;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{d^{q(p_2-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_2}),1;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{66}=&\frac{1}{b^{q(p_1-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_1}),1;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times \frac{1}{d^{q(p_2-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_2}),2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{77}=&\frac{1}{b^{q(p_1-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_1}),2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times \frac{1}{d^{q(p_2-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_2}),1;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
C_{88}=&\frac{1}{b^{q(p_1-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_1}),1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_1}),2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)\Big]\\
&\times \frac{1}{d^{q(p_2-1)}}\Big[{}_2F_1\Big(q(1-\frac{1}{p_2}),1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\
&-\frac{1}{2}{}_2F_1\Big(q(1-\frac{1}{p_2}),2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)\Big].
\end{aligned}
$$

Corollary 2.18. If we set $p_1 = p_2 = 1$ in Theorem 7, then we get (m, n)−polynomial convex function on Δ.

$$
\begin{aligned}
&\Big|M(1,1;f)\Big|\\
\le&\frac{(b-a)(d-c)}{4}\Big(\frac{1}{r+1}\Big)^{\frac{2}{r}}\Big(\frac{1}{n}\sum_{\mu=1}^{n}\frac{\mu}{\mu+1}\Big)^{\frac{1}{q}}\Big(\frac{1}{m}\sum_{\nu=1}^{n}\frac{\nu}{\nu+1}\Big)^{\frac{1}{q}}
\end{aligned}
$$

$$\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right]^{\frac{1}{q}}$$

for $t, s \in [0,1]$ and $\frac{1}{r}+\frac{1}{q}=1$.

Corollary 2.19. If we set $p_1 = p_2 = -1$ in Theorem 7, then we get the $(m,n)-$harmonically polynomial convex function on Δ (See [18]).

$$\begin{aligned}&\left|M(-1,-1;f)\right|\\ \leq\ &\frac{bd(b-a)(d-c)}{4ac(r+1)^{\frac{2}{r}}}\left[C_5^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+C_6^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.\\ &\left.+C_7^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+C_8^{\star}\left|^q\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right]^{\frac{1}{q}}\end{aligned}$$

where

$$\begin{aligned}&C_5^{\star}\\ =&\frac{1}{n}\sum_{\mu=1}^{n}\left[{}_2F_1\Big(2q,1;2;1-\frac{b}{a}\Big)-\frac{1}{\mu+1}{}_2F_1\Big(2q,1;\mu+2;1-\frac{b}{a}\Big)\right]\\ &\times\frac{1}{m}\sum_{\nu=1}^{m}\left[{}_2F_1\Big(2q,1;2;1-\frac{d}{c}\Big)-\frac{1}{\nu+1}{}_2F_1\Big(2q,1;\nu+2;1-\frac{d}{c}\Big)\right],\end{aligned}$$

$$\begin{aligned}&C_6^{\star}\\ =&\frac{1}{n}\sum_{\mu=1}^{n}\left[{}_2F_1\Big(2q,1;2;1-\frac{b}{a}\Big)-\frac{1}{\mu+1}{}_2F_1\Big(2q,1;\mu+2;1-\frac{b}{a}\Big)\right]\\ &\times\frac{1}{m}\sum_{\nu=1}^{m}\left[{}_2F_1\Big(2q,1;2;1-\frac{d}{c}\Big)-\frac{1}{\nu+1}{}_2F_1\Big(2q,\nu+1;\nu+2;1-\frac{d}{c}\Big)\right],\end{aligned}$$

$$\begin{aligned}&C_7^{\star}\\ =&\frac{1}{n}\sum_{\mu=1}^{n}\left[{}_2F_1\Big(2q,1;2;1-\frac{b}{a}\Big)-\frac{1}{\mu+1}{}_2F_1\Big(2q,\mu+1;\mu+2;1-\frac{b}{a}\Big)\right]\\ &\times\frac{1}{m}\sum_{\nu=1}^{m}\left[{}_2F_1\Big(2q,1;2;1-\frac{d}{c}\Big)-\frac{1}{\nu+1}{}_2F_1\Big(2q,1;\nu+2;1-\frac{d}{c}\Big)\right],\end{aligned}$$

and

$$C_8^{\star}$$

$$=\frac{1}{n}\sum_{\mu=1}^{n}\left[{}_2F_1\Big(2q,1;2;1-\frac{b}{a}\Big)-\frac{1}{\mu+1}{}_2F_1\Big(2q,\mu+1;\mu+2;1-\frac{b}{a}\Big)\right]$$
$$\times\frac{1}{m}\sum_{\nu=1}^{m}\left[{}_2F_1\Big(2q,1;2;1-\frac{d}{c}\Big)-\frac{1}{\nu+1}{}_2F_1\Big(2q,\nu+1;\nu+2;1-\frac{d}{c}\Big)\right].$$

Theorem 2.20. *Let $f:\Delta=[a,b]\times[c,d]\in\longrightarrow\mathbb{R}$ be a partial differential mapping on Δ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is $(m,n)-$polynomial (p_1,p_2)- convex function on Δ, then one has the following inequality for $q>1$*

$$\begin{aligned}&\Big|M(p_1,p_2;f)\Big|\\ \le&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}(D)^{1-\frac{1}{q}}\\ &\left[C_1\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q+C_2\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q\right.\\ &\left.+C_3\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q+C_4\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\right]^{\frac{1}{q}}\end{aligned}\tag{2.10}$$

where

$$\begin{aligned}D=&\int_0^1\int_0^1|1-2t|A_t^{1-\frac{1}{p_1}}|1-2s|B_s^{1-\frac{1}{p_2}}dtds\\ =&\frac{1}{b^{(p_1-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_1},2;3;1-\frac{a^{p_1}}{b^{p_1}}\Big)-{}_2F_1\Big(1-\frac{1}{p_1},1;2;1-\frac{a^{p_1}}{b^{p_1}}\Big)\\ &+{}_2F_1\Big(1-\frac{1}{p_1},1;3;\frac{1}{2}(1-\frac{a^{p_1}}{b^{p_1}})\Big)\Big]\\ &\frac{1}{d^{(p_2-1)}}\Big[{}_2F_1\Big(1-\frac{1}{p_2},2;3;1-\frac{c^{p_2}}{d^{p_2}}\Big)-{}_2F_1\Big(1-\frac{1}{p_2},1;2;1-\frac{c^{p_2}}{d^{p_2}}\Big)\\ &+{}_2F_1\Big(1-\frac{1}{p_2},1;3;\frac{1}{2}(1-\frac{c^{p_2}}{d^{p_2}})\Big)\Big]\end{aligned}$$

for C_1,C_2,C_3 and C_4 are same with Theorem 2.12.

Proof. We have the following

$$\begin{aligned}&\Big|M(p_1,p_2;f)\Big|\\ \le&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\int_0^1\int_0^1\frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}\Big|\frac{\partial^2 f}{\partial t\partial s}\Big(A_t^{\frac{1}{p_1}}B_s^{\frac{1}{p_2}}\Big)\Big|dtds\\ \le&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Big(\int_0^1\int_0^1\frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}dtds\Big)^{1-\frac{1}{q}}\\ &\times\Big(\int_0^1\int_0^1\frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}\Big|\frac{\partial^2 f}{\partial t\partial s}(A_t^{\frac{1}{p_1}},B_s^{\frac{1}{p_2}})\Big|dtds\Big)^{\frac{1}{q}}.\end{aligned}$$

By applying the $(m,n)-$polynomial (p_1,p_2)- convexity of $|\frac{\partial^2 f}{\partial t\partial s}|^q$, we get

$$\begin{aligned}
&\Big|M(p_1,p_2;f)\Big| \qquad (2.11)\\
\leq&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}\Big(\int_0^1\int_0^1\frac{|1-2t||1-2s|}{A_t^{1-\frac{1}{p_1}}B_s^{1-\frac{1}{p_2}}}dtds\Big)^{1-\frac{1}{q}}\\
&\Bigg(\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-(1-t)^\mu)}{A_t^{(1-\frac{1}{p_1})}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-(1-s)^\nu)}{B_s^{(1-\frac{1}{p_2})}}ds\\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-(1-t)^\mu)}{A_t^{(1-\frac{1}{p_1})}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-s^\nu)}{B_s^{(1-\frac{1}{p_2})}}ds\\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-t^\mu)}{A_t^{(1-\frac{1}{p_1})}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-(1-s)^\nu)}{B_s^{(1-\frac{1}{p_2})}}ds\\
&+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\frac{1}{n}\sum_{\mu=1}^{n}\int_0^1\frac{|1-2t|(1-t^\mu)}{A_t^{(1-\frac{1}{p_1})}}dt\frac{1}{m}\sum_{\nu=1}^{m}\int_0^1\frac{|1-2s|(1-s^\nu)}{B_s^{(1-\frac{1}{p_2})}}ds\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

We get the desired result by computing the above integrals. □

Corollary 2.21. If we set $m=n=1$ in (2.10), we have the following inequality (p_1,p_2)-convex function on Δ.

$$\begin{aligned}
&\Big|M(p_1,p_2;f)\Big|\\
\leq&\frac{(b^{p_1}-a^{p_1})(d^{p_2}-c^{p_2})}{4p_1p_2}(D)^{1-\frac{1}{q}}\\
&\Bigg[C_{11}\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q+C_{22}\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q\\
&+C_{33}\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q+C_{44}\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\Bigg]^{\frac{1}{q}}
\end{aligned}$$

where C_{11}, C_{22}, C_{33} and C_{44} are same in Corollary 2.13.

Corollary 2.22. If we set $p_1=p_2=1$ in (2.11), then we get the $(m,n)-$polynomial convex function on Δ.

$$\begin{aligned}
&\Big|M(1,1;f)\Big|\\
\leq&\frac{(b-a)(d-c)}{4}\Big(\frac{1}{p+1}\Big)^{2-\frac{2}{q}}\Big(\frac{1}{n}\sum_{\mu=1}^{n}\Big[\frac{2^\mu(\mu^2+\mu+2)-2}{2^{\mu+1}(\mu+1)(\mu+2)}\Big]\\
&\frac{1}{m}\sum_{\nu=1}^{m}\Big[\frac{2^\nu(\nu^2+\nu+2)-2}{2^{\nu+1}(\nu+1)(\nu+2)}\Big]\Big)^{\frac{1}{q}}\\
&\times\Bigg[\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|^q\Bigg]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.23. If we set $p_1 = p_2 = -1$ in (2.10), then we get the $(m,n)-$harmonically polynomial convex function on Δ.

$$\begin{aligned}
&\left|M(-1,-1;f)\right| \\
\leq&\frac{abcd(b-a)(d-c)}{4}(D_1)^{1-\frac{1}{q}} \\
&\times\left[C_1^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+C_2^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+C_3^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+C_4^{\star}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
&D_1 \\
=&\int_0^1\int_0^1\frac{|1-2t|}{(A_t)^2}\frac{|1-2s|}{(B_s)^2}dtds \\
=&a^2\left[{}_2F_1(2,2;3;1-\frac{b}{a})-{}_2F_1(2,1;2;1-\frac{b}{a})+{}_2F_1(2,1;3;\frac{1}{2}(1-\frac{b}{a}))\right] \\
&c^2\left[{}_2F_1(2,2;3;1-\frac{d}{c})-{}_2F_1(2,1;2;1-\frac{d}{c})+{}_2F_1(2,1;3;\frac{1}{2}(1-\frac{d}{c}))\right]
\end{aligned}$$

for $C_1^{\star}, C_2^{\star}, C_3^{\star}$ and $C_4^{\star}$ are same in Corollary 2.15.

3. Conclusions

This paper will give directions to several researchers who would like to extend and generalize the main findings. We have introduced the concept of $(m,n)-$polynomial (p_1,p_2)- convex functions on the co-ordinates and proved some further properties of this interesting class of function. We have expanded the argument by giving a new variant of Hermite-Hadamard inequality on the co-ordinates. The findings have supported by giving earlier results in the special cases.

Acknowledgments

The publication has been prepared with the support of P.R.I.N.. The research of the second author has been fully supported by H.E.C. Pakistan under NRPU project 7906. This paper has been supported by TÜBİTAK-BİDEB.

Conflict of interest

The authors declare no conflict of interest.

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