AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(11): 20383–20400.
DOI: 10.3934/math.20221117
Received: 08 July 2022
Revised: 24 August 2022
Accepted: 06 September 2022
Published: 19 September 2022

Research article

Oscillatory behavior of solution for fractional order fuzzy neutral predator-prey system

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Abstract: The goal of this study is to see if there is a solution for the fuzzy delay predator-prey system (FDPPS) with Caputo derivative. To begin, we use Schaefer's fixed point theorem to obtain results for the existence theorem of at least one solution in a Caputo FDPPS where the initial condition is also represented by a fuzzy number on fuzzy number space. We also determine the necessary and sufficient conditions of solutions for the system. Several examples are also presented to explain the oscillatory property and the existence of a solution.

Keywords: fuzzy variable; level-wise continuous function; fuzzy solution; fuzzy delay predator-prey system; oscillation solution
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

In real-world occurrences, a wide range of physical processes exhibit fractional-order behavior that can alter through time and space. The operations of differentiation and integration of fractional order are authorized by fractional calculus. The fractional order can be used on both imaginary and real integers [1–3]. Due to vast variety of fractional calculus of applications in disciplines such as physics, chemistry, biology, electronics, thermal systems, electrical engineering, mechanics, signal processing, weapon systems, electro hydraulics, population modeling, robotics, and control, and many others, the theory of fuzzy sets continues to attract researchers' attention [4–10]. As a result, over the last few years, it has caught the interest of scholars. In the investigation of population dynamics, the basis of many models used nowadays is formed by the predator-prey model. For mathematical ecology, it is one of the most popular systems.

In 1920, the predator-prey model was presented by Volterra and Lokta for the study of population dynamics. Extensions and variations in this model are beginning to be done for many years. The study of this model in relation to oscillatory and stabilities behavior is very famous nowadays. The essential aspects of predator-prey models, which have a strong biological foundation, have been highlighted.

In the PP model, the study on population was expanded by integrating harvesting and time delay. The rate of population change is not entirely determined by the current population. However, when considering time delay, it is also dependent on the previous population. When using the harvesting model, some research establishes a link between population and economic difficulties. There is a lot of work on the subject of delayed Predator-prey system [11]. Modeling dynamical systems in a state of flux with fuzzy differential equations is a natural choice. These systems will give a more accurate description of modern-world problems.

The amount of work being done in this area is fast increasing these days. The notion of fuzzy derivative was first introduced by Chang and Zadeh [12]. In 1982, Dubois et al. [13] followed up, using the extension idea in their method. The concept of the fuzzy differential equation was introduced to the analysis of fuzzy dynamical concerns by Kandel and Byatt. Many researchers worked on fuzzy differential equation theory and its application to real-world problems [14–16].

The existence and uniqueness theorems are the most essential and fundamental theorems in classical differential equation theory. Theorems on fuzzy-set functions have been studied in several publications. Some of these are cited as [17–19], and [20]. Ge et al. [17] developed the concept of uncertain delay differential equations. Using Banach's fixed point theorem, he proved the existence-uniqueness theorem for the equation under linear growth and Lipschitz conditions. For fuzzy differential equations, Chen et al. [18] designed a new existence-uniqueness theorem. To distinguish the theorem from preceding tasks, they apply the Liu procedure [18]. Fuzzy delay differential equations with the nonlocal condition were as shown by Balachandran and Prakash [19] existence of solutions. Park et al. [20] established the existence-uniqueness theorem for fuzzy differential equations by applying successive approximations on $\mathbf{E^m}$. The existence theorem was applied to a particular type of fuzzy differential equation. Abbas et al. [21, 22] worked on a partial differential equation. Niazi et al. [23], Iqbal et al. [24], Shafqat et al. [25], Abuasbeh et al. [26] and Alnahdi et al. [27] existence-uniqueness of the FFEE were investigated.

In 2014, Barzinji et al. demonstrated the existence of a solution for FDPP with fuzzy initial conditions on (E^n, D) in [28]. The DPP system is

$$\begin{cases} \dot{X}(\omega) = X(1-X) - cyX \\ \dot{y}(\omega) = cbe^{-d_j\tau}Y(\omega-\tau)X(\omega-\tau) - dY \\ X(0) = X_0, \\ Y(0) = Y_0, \quad -\tau \leqslant \omega \leqslant 0, \end{cases}$$

and the FDPPS in a vector form is

$$\begin{cases} \dot{u}(\omega) = f(\omega, x(\omega), x_\omega)\ \omega \in \mathcal{J} = [0, a], \\ x(0) = x_0 \quad -\tau \leqslant \omega \leqslant 0. \end{cases}$$

Ladde et al. [29] and reference [30] recently discovered the oscillation theory of delay differential equations. As a result, only a few results on the oscillatory property of distinct fuzzy differential systems have been published [31].

The existence of a solution for Caputo FDPP with fuzzy initial condition on $(\mathbf{E^m}, D)$ where $\beta \in [1, 2]$ is motivated by the previously mentioned papers. The predator-prey system is

$$\begin{cases} {}_0^cD_\omega^\beta \mathrm{u}(\omega) = \mathrm{u}(1-\mathrm{u}) - g\mathrm{uv} \\ {}_0^cD_\omega^\beta \mathrm{v}(\omega) = ghe^{-d_i\sigma}\mathrm{v}(\omega-\sigma)\mathrm{u}(\omega-\sigma) - d\mathrm{v} \\ \mathrm{u}(0) = \mathrm{u}_0 \\ \mathrm{u}'(0) = \mathrm{u}_1 \\ \mathrm{v}(0) = \mathrm{v}_0 \quad -\sigma \leqslant \omega \leqslant 0, \end{cases} \tag{1.1}$$

where x represents prey population, y represents predator population, d represents predator death rate, c represents constant predator response, σ represents the constant time required to change prey biomass into predator biomass, and x_0, y_0 represent the initial conditions.

The FDPP system in a vector form:

$$\begin{cases} {}_0^cD_\omega^\beta \mathrm{u}(\omega) = f(\omega, \mathrm{u}(h(\omega)), \mathrm{u}_\omega)\ \omega \in J = [0, a], \\ \mathrm{u}(0) = \mathrm{u}_0, \\ \mathrm{u}'(0) = \mathrm{u}_1, \quad -\sigma \leqslant \omega \leqslant 0. \end{cases} \tag{1.2}$$

To deal with a fuzzy process, the goal of this work is to investigate the existence and uniqueness of results to FDPP systems by using Caputo derivative. Some researchers discovered FDE results in the literature, though the vast majority of them were first-order differential equations. In our research, we discovered results for Caputo derivatives of order (1,2). We employ FDPPS. The theory of fuzzy sets continues to attract the interest of academics due to its wide range of applications in fields such as engineering, robotics, mechanics, control, thermal systems, electrical, and signal processing. The important points of the FDDE with the nature of the solution of a fundamental existence theorem. The oscillatory behavior of such an equation has vast importance. We will examine oscillation for the Caputo FDPP system in this work, and we will discover the sufficient and necessary criteria for all solutions to be oscillatory.

This paper is organized as follows. In Section 2, some notations, concepts and terminologies are given. In Section 3, the formulation of the fuzzy delay differential predator-prey system are presented. In Section 4, we prove the existence theorem for fuzzy delay predator-prey system. In Section 5, we discuss the oscillation solution of the fuzzy delay predator-prey system. Some examples are presented in Section 6. Finally, Section 7 provides applications in real life and Section 8 provides a brief conclusion.

2. Preliminaries

2.1. Fuzzy sets and numbers

Assume $\mathcal{M}_k(\mathbf{R^m})$ be family of all nonempty compact convex subsets of $\mathbf{R^m}$, addition and scalar multiplication are usually also defined as $\mathcal{M}_k(\mathbf{R^m})$. Consider two nonempty bounded subsets of $\mathbf{R^m}$, $\mathcal{A}$ and $\mathcal{B}$. Hausdroff metric is used to define the distance between $\mathcal{A}$ and $\mathcal{B}$ as,

$$d(\mathcal{A}, \mathcal{B}) = \max\{\sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} \|a-b\|, \sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}} \|a-b\|\},$$

where $(\|x\|)$ indicate usual Euclidean norm in $\mathbf{R^m}$.

We can have addition and scalar multiplication in fuzzy number space $\mathbf{E^m}$ using Zadeh's extension principle, as shown in:

$$[x \oplus y]^\beta = [x]^\beta \oplus [y]^\beta,\ [kx]^\beta = k[y]^\beta$$

where $x, y \in \mathbf{E^m}, k \in \mathbf{R^m}$ and $1 \leqslant \beta \leqslant 2$.

Define $D : \mathbf{E^m} \times \mathbf{E^m} \to \mathbf{R^{m+}}$ by equation

$$D(x, y) = \sup_{1 \leqslant \beta \leqslant 2} \max\{[\mathrm{u}]^\beta, [\mathrm{v}]^\beta\}$$

where d is the Hausdorff metric for a non-empty compact sets in $\mathbf{R^m}$.

It is now quite easy to see, D is a metric in $\mathbf{E^m}$. Making use of the result,

(i) $(\mathbf{E^m}, D)$ is a complete metric space.
(ii) $D(x \oplus z, y \oplus z) = D(x, y)$ for all $x, y, z \in \mathbf{E^m}$.
(iii) $D(kx, ky) = |k|D(x, y)\ \ \forall x, y \in \mathbf{E^m}$ and $k \in \mathbf{R^m}$.
(iv) $D(x \oplus y, z \oplus e) \leqslant D(x, z) \oplus D(y, e)$ for all $x, y, z, e \in \mathbf{E^m}$.

Remark 2.1. *On* $\mathbf{E^m}$*, we can define subtraction* $\ominus$*, called the H-difference as follows* $\mathrm{u} \ominus \mathrm{v}$ *has sense if there exist* $\omega \in \mathbf{E^m}$ *such that* $x = y + z$*.*

Clearly, $x - y \nexists\ \forall\ x, y \in \mathbf{E^m}$*. In what follows, we consider* $C_b = C([0, b], \mathbf{E^m})$*, space of all continuous fuzzy functions define on* $[0, b] \subset \mathbf{R^m}$ *into* $\mathbf{E^m}$*, where* $b > 0$*. For* $x, y \in C_b$*, we define the metric*

$$\mathcal{H}(x, y) = \sup_{\omega \in [0,b]} D(x(\omega), y(\omega)).$$

Then $(C_b, \mathcal{H})$ *is complete metric space.*

Consider the compact interval $T = [c, d] \subset \mathbf{R^m}$*. For the set-valued fuzzy mappings, we recall the properties of measurability and integrability [32].*

Definition 2.2. *[32] A mapping* $\mathcal{F} : \mathcal{I} \in \mathbf{E^m}$ *is a strongly measurable if for all* $\beta \in [1, 2]$ *the set-valued function* $\mathcal{G}_\beta : \mathcal{I} \to \mathcal{M}_k(\mathbf{R^m})$ *define by* $\mathcal{G}_\beta(\omega) = [\mathcal{F}(\omega)]^\beta$ *is Lebesgue measurable when* $\mathcal{M}_k(\mathbf{R^m})$ *is endowed with topology generated by the Hausdorff metric d.*

A mapping $\mathcal{G} : \mathcal{I} \in \mathbf{E^m}$ *is called an integrably bounded if there exists an integrable function* $k : \mathcal{I} \to \mathbf{R^m}_+$ *such that* $D(\mathcal{G}_0(\omega), C_0) \leqslant k(\omega)$ *for all* $\omega \in T$*.*

Definition 2.3. *[32] Let* $\mathcal{G} : \mathcal{I} \in \mathbf{E^m}$*. Then integral of* $\mathcal{G}$ *over* $\mathcal{I}$ *denoted by* $\int_\mathcal{I} \mathcal{G}(\omega)d\omega$*, is defined by equation* $[\int_\mathcal{I} \mathcal{G}(\omega)d\omega]^\beta = \int_\mathcal{I} \mathcal{G}_\beta(\omega)d\omega = \{\int_\mathcal{I} \mathcal{G}(\omega)d\omega / g : \mathcal{I} \to \mathbf{R^m}$ *is a measurable selection for* $\mathcal{G}_\beta\} \forall \beta \in [1, 2]$*.*

Also, strongly measurable and an integrably bounded mapping $\mathcal{G} : \mathcal{I} \to \mathbf{E^m}$ *is said to be integrable over* $\mathcal{I}$ *if* $\int_\mathcal{I} \mathcal{G}(\omega)d\omega \in \mathbf{E^m}$*.*

Proposition 2.4. *If* $\mathcal{G} : \mathcal{I} \in \mathbf{E^m}$ *is a strongly measurable and integrably bounded then* $\mathcal{F}$ *is integrable.*

The definitions and theorems listed here can be found in [20].

Proposition 2.5. *Assume* $\mathcal{G}, \mathcal{H} : \mathcal{I} \in \mathbf{E^m}$ *be integrable and* $c \in \mathcal{I}, \lambda \in \mathbf{R^m}$*. Now FDPP system*

(i) $\int_\mathcal{I} (\mathcal{G}(\omega) \oplus \mathcal{H}(\omega))d\omega = \int_\mathcal{I} \mathcal{G}(\omega)d\omega \oplus \int_\mathcal{I} \mathcal{H}(\omega)d\omega,$
(ii) $\int_{\omega_0}^{\omega_0+a} \mathcal{G}(\omega)d\omega = \int_{\omega_0}^{c} \mathcal{G}(\omega)d\omega + \int_{c}^{\omega_0+a} \mathcal{G}(\omega)d\omega,$

(iii) $D(\mathcal{F}, \mathcal{G})$ *is an integrable,*
(iv) $D(\int_{\mathcal{I}} \mathcal{G}(\omega)d\omega, \int_{\mathcal{I}} \mathcal{H}(\omega)d\omega) \leqslant \int_{\mathcal{I}} D(\mathcal{G}, \mathcal{H})(\omega)d\omega.$

Theorem 2.6. *[20] Assume* $G : \mathcal{I} \to \mathbf{E^m}$ *is differentiable and let derivative* G' *is integrable on* $\mathcal{I}$*. For all* $s \in \mathcal{I}$*, we now have*

$$\mathcal{G}(s) = \mathcal{G}(a) + \int_a^s \mathcal{G}'(\omega)d\omega.$$

Definition 2.7. *[20] The mapping* $g : \mathcal{I} \times \mathbf{E^m} \to \mathbf{E^m}$ *is said to be level-wise continuous at a point* $(\omega_0, \mathrm{u}_0) \in \mathcal{I} \times \mathbf{E^m}$ *provided for any fixed* $\beta \in [1, 2]$ *and arbitrary* $\epsilon > 0$*, there exist* $\xi(\epsilon, \beta) > 0$ *, then*

$$d([g(\omega, u)]_\beta, [g(\omega_0, u_0)]_\beta) < \epsilon$$

when $|\omega - \omega_0| < \xi(\epsilon, \beta)$ *and* $d([x]_\beta, [x_0]_\beta) < \xi(\epsilon, \beta)$ *for all* $\omega \in \mathcal{I}, \mathrm{u} \in \mathbf{E^m}$.

Corollary 2.8. *[32] Given that* $\mathcal{G} : \mathcal{I} \times \mathbf{E^m} \to \mathbf{E^m}$ *is continuous. Then there's the function.*

$$\mathcal{H}(\omega) = \int_a^\omega \mathcal{G}(\omega)ds, \omega \in I$$

is differentiable and $\mathcal{H}'(\omega) = G(\omega)$*. Then, if* $\mathcal{G}$ *is continuously differentiable on* $\mathcal{I}$*, The following is the mean value theorem,*

$$D(\mathcal{G}(b), \mathcal{G}(a)) \leqslant (b - a) \sup\{D(\mathcal{G}'(\omega), \tilde{0}), \omega \in \mathcal{I}\}.$$

As a result, have

$$D(\mathcal{H}(b), \mathcal{H}(a)) \leqslant (b - a) \sup\{D(\mathcal{G}(\omega), \tilde{0}), \omega \in \mathcal{I}\}.$$

Theorem 2.9. *[20] Assume V is any metric space and U is a compact metric space. If and only if* Ω *is equi-continuous on U, and* $\Omega(u) = \{\varphi(u) : \varphi \in \Omega\}$ *is totally bounded subset of V for each* $u \in U$*, the subset* ω *of* $C(U, V)$ *of continuous mapping of U into V is totally bounded in metric of uniform convergence.*

Consider a system with a delay differential,

$$\frac{d}{d\omega}[{}_0^c D_\omega^\beta \mathrm{u}(\omega) + \sum_{i=1}^m Q_i \mathrm{u}(\omega - \kappa_i)] + R_0 \mathrm{u}(\omega) + \sum_{j=1}^n R_j \mathrm{u}(\omega - \psi_j) = 0. \tag{2.1}$$

Definition 2.10. *[30] A solution of the system (2.1)* $\mathrm{u}(\omega) = [\mathrm{u}_1(\omega), ..., \mathrm{u}_n(\omega)]$ *is said to oscillate if every component* $\mathrm{u}_i(\omega)$ *of solution has an arbitrarily large zeros. On the other hand, it is called a non-oscillatory solution.*

Theorem 2.11. *[30] Suppose the coefficients* Q_i *and* $\mathbf{R^m}_i$ *of Eq (2.1) are real* $n \times n$ *matrices and delays* κ_i *and* φ_i *are positive numbers. Assume* $\mathrm{u}(\omega)$ *be a solution of Eq (1.1) on* $[0, \infty)$*. Then there exist a positive constant* $\mathcal{M}$ *and* ζ *such that* $\|\mathrm{u}(\omega)\| \leqslant \mathcal{M}e^{\zeta\omega}$ *for* $\omega \geqslant 0$.

Theorem 2.12. *[30] Assume* $u \in C\{[0, \infty), \mathbf{R^m}\}$ *and suppose that there exist positive constants* ζ *and* ψ *such that* $|\mathrm{u}(\omega)| \leqslant \mathcal{M}e^{\zeta\omega}$ *for* $\omega \geqslant 0$*. Then abscissa of convergence* ψ_0 *of Laplace transform* $\mathcal{U}(s)$ *of* $\mathrm{u}(\omega)$ *satisfies* $\psi_0 \leqslant \psi$*. In addition,* $\mathcal{U}(s)$ *exists and is an analytic function of s for* $Res < \psi_0$.

Lemma 2.13. *[30] Consider the nonlinear delay differential system:*

$$ {}_{0}^{c}D_{\omega}^{\beta}\mathrm{u}(\omega) + g(\mathrm{u}(\omega - \zeta)) = 0. \tag{2.2}$$

As $\omega \to 0$, every non-oscillatory solution of the Eq (2.2) tends to zero.

Definition 2.14. *[30] A solution for the system (2.2) $\mathrm{u}(\omega) = [\mathrm{u}_1(\omega), ..., \mathrm{u}_n(\omega)]^T$ is said to oscillate if every component $u_i(\omega)$ of solution has arbitrarily large zeros. On the other hand, it is called a non-oscillatory solution.*

Theorem 2.15. *[30] Consider a differential delay system:*

$$ {}_{0}^{c}D_{\omega}^{\beta}\mathrm{u}(\omega) + q\mathrm{u}(\omega - \zeta) = 0,\ \omega \geqslant 0. \tag{2.3}$$

Here are several statements that are equivalent:

(i) *The delay differential system (2.3) has a positive solution.*
(ii) *The delay differential inequality:*

$$ {}_{0}^{c}D_{\omega}^{\beta}\mathrm{v}(\omega) + q\mathrm{v}(\omega - \zeta) \leqslant 0,\ \omega \geqslant 0 \tag{2.4}$$

has a positive solution.

In this situation, the existence and uniqueness of theorems for delay differential equations will be shown.

Theorem 2.16. *[33] (Existence) Assume*

$$ {}_{0}^{c}D_{\omega}^{\beta}\mathrm{u}(\omega) = f(\omega, \mathrm{u}(\omega), \mathrm{u}(\omega - \tau))\mathrm{u}(\theta) = \psi,\quad \omega \geqslant 0. \tag{2.5}$$

Suppose Ω is an open subset in $\mathbf{R^m} \times \mathcal{B}$ and g is a continuous on ω. If $(\psi, \mu) \in \omega$, then there is a solution of (2.3) passing through (ψ, μ).

$g(\omega, \mu)$ is Lipschitz in μ in compact set $\mathcal{M}$ of $\mathbf{R^m} \times \mathcal{B}$ if there is a constant $k > 0$ that is for $(\omega, \mu_i) \in M$, for $i = 1, 2|g(\omega, \mu_1) - g(\omega, \mu_2)| \leq k|\mu_1 - \mu_2|$.

Theorem 2.17. *[33] (Uniqueness) Assume Ω is an open set in $\mathbf{R^m} \times B, g : \Omega \to \mathbf{R^m}$ is continuous, and $\mathcal{G}(\omega, \psi)$ is Lipschitz in ψ in each set in Ω. If $(\theta, \psi) \in \Omega$, there is a unique solution of (2.5) through (κ, φ).*

3. Fuzzy delay predator-prey system

In this part, we define a basic system called the FDPP system. Consider a PP system with a time delay:

$$\begin{cases} {}_{0}^{c}D_{\omega}^{\alpha}\mathrm{u}(\omega) = \mathrm{u}(1 - \mathrm{u}) - g\mathrm{u}\mathrm{v}, \\ {}_{0}^{c}D_{\omega}^{\alpha}\mathrm{v}(\omega) = ghe^{-d_i\sigma}\mathrm{v}(\omega - \sigma)\mathrm{u}(\omega - \sigma) - d\mathrm{v}, \\ \mathrm{u}(0) = \mathrm{u}_0, \\ \mathrm{u}'(0) = \mathrm{u}_1, \\ \mathrm{v}(0) = \mathrm{v}_0, \quad -\sigma \leqslant \omega \leqslant 0. \end{cases} \tag{3.1}$$

where x represents prey population, y represents predator population, d represents predator death rate, c represents constant predator response, σ represents the constant time required to change prey biomass into predator biomass, and x_0, y_0 represent the initial conditions.

The linear component and $x(\omega)$, $y(\omega)$ of system (3.1) are then fuzzified using fuzzy symmetric triangular number and parametric from representation of β-cut and $x(\omega)$, $y(\omega)$ are non negative fuzzy functions:

$$\tilde{1} = (1 - (1-\beta)\kappa_1, 1 + (1-\beta)\kappa_1)$$
$$\tilde{d} = (1 - (1-\beta)\kappa_2, d + (1-\beta)\kappa_2)$$

where $1 \leqslant \beta \leqslant 2$.

The FDPP system can be written as a vector:

$$\begin{cases} {}_0^cD_\omega^\beta \mathrm{u}(\omega) = f(\omega, \mathrm{u}(h(\omega)), \mathrm{u}_\omega)\ \omega \in \mathcal{J} = [0, a], \\ \mathrm{u}(0) = \mathrm{u}_0, \\ \mathrm{u}'(0) = \mathrm{u}_1, \ -\sigma \leqslant \omega \leqslant 0 \end{cases} \tag{3.2}$$

where

$$f(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega) = \mathcal{A}\mathrm{u}(\omega) + \mathcal{B}(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega)$$
$$\dot{\mathrm{u}}(\omega) = \begin{bmatrix} \dot{\mathrm{u}}(\omega) \\ \dot{\mathrm{v}}(\omega) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix}, \quad \mathrm{u}(\omega) = \begin{bmatrix} \mathrm{u}(\omega) \\ \mathrm{u}(\omega) \end{bmatrix}$$
$$\mathcal{B}(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega) = \begin{bmatrix} -\mathrm{u}^2 - g\mathrm{uv} \\ ghe^{-d_i\sigma}\mathrm{v}(\omega-\sigma)x(\omega-\sigma) \end{bmatrix}$$
$$\mathrm{u}_0 = \begin{bmatrix} \mathrm{u}_0 \\ \mathrm{v}_0 \end{bmatrix}$$

$-\sigma \leqslant \omega \leqslant 0$.

Where f is fuzzy mapping from $\mathbf{E^m} \to \mathbf{E^m}$, $\mathrm{u}(\omega)$ and $\mathrm{u}_\omega = \mathrm{u}(\omega-\sigma)$ are nonnegative fuzzy functions of ω in $\mathbf{E^m}$. Matrix $\mathcal{A}$ has members that are called fuzzy numbers. ${}_0^cD_\omega^\beta \mathrm{u}(\omega)$ is fuzzy Caputo derivative of $\mathrm{u}(\omega)$ where u_0 and u_1 are fuzzy number.

4. Existence of the solution

Definition 4.1. *Solution to problem (3.2) refers to the mapping* $\mathrm{u}(\omega) : \mathcal{G} \to \mathbf{E^m}$. *if it is continuous at all levels and obeys the integral equation:*

$$\begin{aligned} \mathrm{u}(\omega) &= C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1\omega + \int_0^\omega f(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds \\ \mathrm{u}(\omega) &= C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1\omega + \int_0^\omega (\mathcal{A}\mathrm{u}(s) + \mathcal{B}(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds\ \forall\omega \in \mathcal{G}. \end{aligned} \tag{4.1}$$

Now let $\mathcal{L} = \zeta \in \mathbf{E^m} : \mathcal{H}(\zeta, \mathrm{u}_0) \leqslant b$ *be a space of a continuous function with*

$$\mathcal{H}(\zeta, \varphi) = \sup_{0 \leqslant \omega \leqslant \delta} D(\zeta(\omega), \varphi(\omega))$$

and b positive number. The following is how we present the existence and uniqueness theorem for the FDPP system (3.2).

Theorem 4.2. *Assume $\mathcal{A}$ and $\mathcal{B}$ are level-wise continuous on $\mathcal{G}$ implies that mapping $g : \mathcal{G}\times\mathcal{L} \to \mathbf{E^m}$ is level-wise continuous on $\mathcal{G}$ and there exists constant $\mathcal{J}_0$ that is*

$$D\Big(g(\omega, \mathrm{u}(h(\omega)), \mathrm{u}_\omega), g(\omega, \mathrm{v}(\omega, \mathrm{v}(h(\omega)), \mathrm{v}_\omega)\Big) \leqslant D(f(\mathrm{u}, \mathrm{v})) \leqslant J_0 D(\mathrm{u}, \mathrm{v})$$

for all $\mathrm{u}, \mathrm{v} \in \mathbf{E^m}$ *and* $\omega \in \mathcal{G}$.

Then there's an another solution $u(\omega)$ *of (3.2) defined on interval* $[0, \delta]$ *where*

$$\delta = \left\{a, \frac{b}{\mathcal{P}}, \frac{1}{\mathcal{J}_0}\right\}$$

and

$$\mathcal{P} = \max D\Big(g(\omega, \mathrm{u}(h(\omega)), \mathrm{u}_\omega), \tilde{0}\Big),\ \tilde{0} \in \mathbf{E^m}.$$

Proof. Consider the definition of the operator $\psi : L \to \mathcal{L}$ as

$$\begin{aligned} \psi\mathrm{u}(\omega) &= C_q(\omega)\mathrm{u}_0 + K_q(\omega)\mathrm{u}_1 + \int_0^\omega f(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds \\ &= C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1 + \int_0^\omega (\mathcal{A}\mathrm{u}(s) + \mathcal{B}(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds. \end{aligned} \tag{4.2}$$

First, we demonstrate that $\psi : \mathcal{L} \to \mathcal{L}$ is continuous when $\zeta \in \mathcal{L}$ and $\mathcal{H}(\psi\zeta, \mathrm{u}_0) \leqslant b$.

$$\begin{aligned} \mathcal{P} &= \max D\Big(g(\omega, \mathrm{u}(h(\omega)), \mathrm{u}_\omega), \tilde{0}\Big), \\ D(\psi\zeta(\omega+h), \psi\zeta(\omega)) &= D\Big(C_q(\omega)\mathrm{u}_0 + K_q(\omega)\mathrm{u}_1 + \int_0^{\omega+h} g(s, \zeta(h(s)), \xi_s), C_q(\omega)\mathrm{u}_0 + K_q(\omega)\mathrm{u}_1 \\ &\quad + \int_0^\omega g(s, \zeta(h(s)), \xi_s)\Big) \\ &\leqslant D\Big(\int_0^{\omega+h} g(s, \zeta(h(s)), \xi_s), \int_0^\omega g(s, \zeta(h(s)), \xi_s)\Big) \\ &\leqslant \int_0^{\omega+h} D(g(s, \zeta(h(s)), \xi_s), \tilde{0})ds \\ &= l\mathcal{P} \to 0\ as\ l \to 0. \end{aligned}$$

As a result, the mapping ψ is continuous. Now

$$\begin{aligned} D(\psi\zeta(\omega), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1) &= D\Big(\int_0^\omega g(s, \zeta(h(s)), \xi_s), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1\Big) \\ &\leqslant \int_0^\omega D(g(s, \zeta(h(s)), \xi_s), \tilde{0})ds \\ &= \mathcal{P}\omega \end{aligned}$$

and so

$$\mathcal{H}(\psi\zeta, C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1) = \sup_{0\leqslant\omega\leqslant\delta} D(\psi\zeta(\omega), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1)$$

$$\leqslant \; P\delta$$
$$\leqslant \; b.$$

After that, ψ maps $\mathcal{L}$ to $\mathcal{L}$. Because $C([0,\delta],\mathbf{E^m})$ is complete metric space with metric $\mathcal{H}$, we can now prove that $\mathcal{L}$ is closed subset of $C([0,\delta],\mathbf{E^m})$, implying that $\mathcal{L}$ is complete metric space. Assume ϕ_n is sequence in $\mathcal{L}$ that is $\phi_n \to \phi \in C([0,\delta],\mathbf{E^m})$ as $n \to \infty$. Then

$$D(\phi(\omega), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1) \leqslant D(\phi(\omega), \phi_n(\omega)) + D(\phi_n(\omega), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1),$$

and also,

$$\begin{aligned}
\mathcal{H}(\phi, C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1) &= \sup_{0\leqslant\omega\leqslant\delta} D(\psi(\omega), C_q(\omega)\mathrm{u}_0, \mathcal{K}_q(\omega)\mathrm{u}_1)\\
&\leqslant \mathcal{H}(\phi,\phi_n) + \mathcal{H}(\phi, C_q(\omega)\mathrm{u}_0) + \mathcal{H}(\phi, \mathcal{K}_q(\omega)\mathrm{u}_1)\\
&\leqslant \epsilon + b + b\\
&\leqslant \epsilon + 2b
\end{aligned}$$

for sufficiently large n and an arbitrary $\epsilon > 0$. Hence, $\phi \in \mathcal{L}$. This demonstrates that $\mathcal{L}$ is a closed subset of $C([0,\delta],\mathbf{E^m})$. As a result, $\mathcal{L}$ is the complete metric space.
We'll show that ψ represents contraction mapping, using Proposition 2.5 and the assumption of the theorem. For $\zeta, \phi \in L$,

$$\begin{aligned}
D(\psi\zeta(\omega), \psi\phi(\omega)) &= D\Big(C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1 + \int_0^\omega g(s,\zeta(h(s)),\zeta_s)ds, C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1\\
&\quad + \int_0^\omega g(s,\phi(h(s)),\phi_s)ds\Big)\\
&\leqslant \int_0^\omega D\Big(g(s,\zeta(h(s)),\zeta_s), g(s,\phi(h(s)),\phi_s)\Big)ds\\
&\leqslant \int_0^\omega \mathcal{M}_0 D(\zeta(s),\phi(s))ds.
\end{aligned}$$

We conclude

$$\begin{aligned}
\mathcal{H}(\psi\zeta(\omega), \psi\phi(\omega)) &\leqslant \sup_{\omega\in\delta}\Big\{\int_0^\omega \mathcal{M}_0 D(\zeta(s),\phi(s))ds\Big\}\\
&\leqslant \delta\mathcal{M}_0 D(\zeta(\omega),\phi(\omega))\\
&\leqslant \delta\mathcal{M}_0\mathcal{H}(\zeta,\phi).
\end{aligned}$$

Since $\delta\mathcal{M}_0 < 2, \psi$ is contraction mapping. Now, ψ has unique fixed point $u \in C([0,\delta],\mathbf{E^m})$ that is $\psi u = u$, and

$$\begin{aligned}
\mathrm{u}(\omega) &= C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1 + \int_0^\omega f(s,\mathrm{u}(h(s)),\mathrm{u}_s)ds\\
&= C_q(\omega)\mathrm{u}_0 + \mathcal{K}_q(\omega)\mathrm{u}_1 + \int_0^\omega (\mathcal{A}\mathrm{u}(s) + \mathcal{B}(s,u(h(s)),\mathrm{u}_s)ds. \qquad (4.3)
\end{aligned}$$

□

Theorem 4.3. *Consider that g and* u_0 *as in Theorem 4.2. And let* $u(\omega, u_0), v(\omega, v_0)$ *be solutions of system (3.2) corresponding to* u_0, v_0*, respectively. Then there's a constant* $r > 1$ *that implies*

$$\mathcal{H}(u(\omega, u_0), v(\omega, v_0)) \leqslant rD(u_0, v_0)$$

for any $u_0, v_0 \in \mathbf{E^m}$ *and* $r = \frac{1}{(1-r\mathcal{M}_0)}$.

Proof. Assume that $u(\omega, u_0), v(\omega, v_0)$ are solutions of the Eq (3.2) corresponding to u_0, v_0, respectively. Then

$$\begin{aligned} D\Big(u(\omega, u_0), v(\omega, v_0)\Big) &= D\Big(C_q(\omega)u_0 + \mathcal{K}_q(\omega)u_1 + \int_0^{\omega} g(s, u(h(s)), u_s)ds, C_q(\omega)v_0 + \mathcal{K}_q(\omega)v_1 \\ &\quad + \int_0^{\omega} g(s, u(h(s))ds\Big) \\ D\Big(u(\omega, u_0), v(\omega, v_0)\Big) &\leqslant D(C_q(\omega)u_0, \mathcal{K}_q(\omega)v_0) + D(u_1, v_1) + \int_0^{\omega} D\Big(g(s, u(h(s)), u_s), g(s, v(h(s)), v_s)\Big) \\ D\Big(u(\omega, u_0), v(\omega, v_0)\Big) &\leqslant D(u_0, v_0) + D(u_1, v_1) + \int_0^{\omega} \mathcal{M}_0 D(u(h(s)), v(h(s))). \end{aligned}$$

Therefore,

$$\mathcal{H}(x(\omega, u_0, u_1), v(\omega, v_0, u_1)) \leqslant D(u_0, v_0) + D(u_1, v_1) + \delta\mathcal{M}_0\mathcal{H}(u(\omega, u_0), v(\omega, v_0)),$$

and

$$\mathcal{H}(x(\omega, u_0, u_1), v(\omega, v_0, u_1)) \leqslant \frac{1}{(1-\delta\mathcal{M}_0)} D(u_0, v_0).$$

As a result, the theorem's proof is complete. For the FDPP system with starting value (3.2), we present a generalization of Theorem 4.3. □

Theorem 4.4. *If* $g : \mathcal{G} \times \mathbf{E^m} \to \mathbf{E^m}$ *is level-wise continuous and bounded, then initial value problem (3.2) has at least one solution on the interval* $\mathcal{G}$.

Proof. When g is both continuous and bounded, there is a $q \leqslant 1$ that is

$$D(g(\omega, u(h(\omega)), u_\omega), \tilde{0}) \leqslant q, \ \omega \in \mathcal{G}, \ u \in \mathbf{E}^2.$$

Assume $\mathcal{B}$ is bounded set in $C(\mathcal{G}, \mathbf{E^m})$. The set $\psi\mathcal{B} = \{\psi u : u \in \mathcal{B}\}$ is totally bounded if and only if it is equi-continuous and for every $\omega \in \mathcal{G}$, set $\psi\mathcal{B} = \{\psi u(\omega) : \omega \in \mathcal{G}\}$ is totally bounded subset of $\mathbf{E^m}$. For $\omega_0, \omega_1 \in \mathcal{G}$ with $\omega_0 \leqslant \omega_1$, and $u \in \mathcal{B}$ we get that

$$\begin{aligned} D(\psi u(\omega_0), \psi u(\omega_1)) &= D\Big(C_q(\omega)u_0 + \mathcal{K}_q(\omega)u_1 + \int_0^{\omega_0} g(s, u(h(s)), u_s)ds, C_q(\omega)u_0 + K_q(\omega)u_1 \\ &\quad + \int_0^{\omega_1} g(s, u(h(s)), u_s)ds\Big) \\ &\leqslant D\Big(\int_0^{\omega_0} g(s, u(h(s)), u_s)ds, \int_0^{\omega_1} g(s, u(h(s)), u_s)ds\Big) \\ &\leqslant D(g(s, u(h(s)), u_s), \tilde{0})ds \end{aligned}$$

$$\leqslant \ |\omega_0 - \omega_1| * \sup\{D(g(s, \mathrm{u}(h(s)), \mathrm{u}_s), \tilde{0})\ \omega \in \mathcal{G}\}$$
$$\leqslant \ |\omega_0 - \omega_1| * q.$$

This shows that $\psi\mathcal{B}$ is equi-continuous. Now, for fixed $\omega \in \mathcal{G}$. Now

$$D(\psi\mathrm{u}(\omega), \psi\mathrm{u}(\omega')) \leqslant |\omega_0 - \omega_1| * q,\ \ for\ every\ \omega' \in \mathcal{G}, \mathrm{u} \in \mathcal{B}.$$

We have come to the conclusion that the set $\{\psi\mathrm{u}(\omega) : \mathrm{u} \in \mathcal{B}\}$ is totally bounded in $\mathbf{E^m}$, and so $\psi\mathcal{B}$ is relatively compact subset of $C(\mathcal{G}, \mathbf{E^m})$. Since, ψ is compact, ψ bounded sets are transformed into relatively compact sets. We notice, u is the operator's fixed point ψ defined by Eq (3.2) if and only if $\mathrm{u} \in C(\mathcal{G}, \mathbf{E^m})$ is solution of (3.2).
Then, in metric space, we consider ball $(C(\mathcal{G}, \mathbf{E^m}), \mathcal{H})$,

$$\mathcal{B} = \{\zeta \in C(\mathcal{G}, \mathbf{E^m}), \mathcal{H}(\zeta, \tilde{0} \leqslant p)\},\ \ p = a * q.$$

Now, $\psi\mathcal{B} \subset \mathcal{B}$. For $\mathrm{u} \in C(\mathcal{G}, \mathbf{E^m})$,

$$\begin{aligned} D(\psi\mathrm{u}(\omega), \psi(u(0))) &= D\Big(C_q(\omega)\mathrm{u}_0 + \int_0^\omega g(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds, \mathrm{u}_0\Big) \\ &\leqslant \int_0^\omega D(g(s, \mathrm{u}(h(s)), \mathrm{u}_s), \tilde{0}) \\ &\leqslant |\omega| * q \\ &\leqslant a * q \end{aligned}$$

$$\begin{aligned} D(\psi\mathrm{u}(\omega), \psi(\mathrm{u}'(0))) &= D\Big(\mathcal{K}_q(\omega)\mathrm{u}_1 + \int_0^\omega g(s, \mathrm{u}(h(s)), \mathrm{u}_s)ds, \mathrm{u}_1\Big) \\ &\leqslant \int_0^\omega D(g(s, \mathrm{u}(h(s)), \mathrm{u}_s), \tilde{0}) \\ &\leqslant |\omega| * q \\ &\leqslant a * q \end{aligned}$$

Therefore, we define $\tilde{0} : \mathcal{G} \to \mathbf{E^m}, \tilde{0}(\omega) = \tilde{0}, \omega \in \mathcal{G}$ so, we have

$$H(\psi\mathrm{u}, \psi 0) = \sup\{D(\psi\mathrm{u}(\omega), \psi 0(\omega)) : \omega \in \mathcal{G}\}$$

$$H(\psi\mathrm{u}, \psi' 0) = \sup\{D(\psi\mathrm{u}(\omega), \psi' 0(\omega)) : \omega \in \mathcal{G}\}.$$

As a result, ψ is compact and consequently it has fixed point $\mathrm{u} \in \mathcal{B}$. The initial value problem (3.2) is solved with this fixed point. □

5. Oscillation of fuzzy delay predator-prey system

The oscillation of all FDPPS solutions is discussed in this section. Suppose the following system (3.2). We also present the following f hypotheses, which will only be accepted if they are stated explicitly:

$$\liminf_{\mathrm{u}\to 0} \frac{f(\mathrm{u})}{\mathrm{u}} \geqslant 2, \tag{5.1}$$

$$\lim_{u\to 0}\frac{f(u)}{u}=2. \tag{5.2}$$

When condition (5.1) or (5.2) is satisfied, the following linear equation is satisfied:

$${}_0^cD_\omega^\beta u(\omega)-\mathcal{C}u(\omega)-\mathcal{D}u(\omega-\sigma)=0 \tag{5.3}$$

will be referred to as the system's linearized equation (3.2). C and D are fuzzy matrices with characteristic equations,

$$det(\varpi\mathcal{I}-C-\mathcal{D}e^{-\varpi\tau})=0. \tag{5.4}$$

To establish our oscillation theorem, we must first suppose that the non-linear FDPPS theorem has the same oscillating behavior as the equivalent linear system.

Theorem 5.1. *Assume that each linearized equation (5.3) solution is oscillatory. Then every (3.2) solution oscillates as well.*

Proof. Suppose that Eq (3.2) has a non-oscillatory solution $u(\omega)$ for the sake of contradiction. We suppose that $u(\omega)$ will be positive at some point. The case where $u(\omega)$ becomes negative in the end is identical and will be ignored. We know that $\lim_{\omega\to\infty}u(\omega)=0$ owing to Lemma 2.13. As a result of (5.1),

$$\lim_{\omega\to\infty}\inf\frac{f(u(\omega-\tau))}{u(\omega-\tau)}\geqslant 2.$$

Let $\epsilon\in(1,2)$. Then there exists T_ϵ such that $\omega\geqslant T_\epsilon$ and

$$f(u(\omega-\tau))\geqslant(1-\epsilon)2u(\omega-\tau).$$

As a result of Eq (3.2),

$${}_0^cD_\omega^\beta u(\omega)+(1-\epsilon)2u(\omega-\tau)\leqslant 1,\omega\geqslant T_\epsilon.$$

Equation (5.3) has a positive solution, according to Theorem 2.15. This contradicts the claim that all solutions to Eq (5.3) are oscillatory and that proof is complete.
The solution of a linearized system's oscillation theorem is given. □

Theorem 5.2. *The following propositions are identical if you consider the linearized system (5.3). Componentwise, every solution of Eq (5.3) oscillates. There are no real roots in the characteristic equation (5.4).*

Proof. For $(a)\to(b)$, the proof is easy. There exists non-zero vector v that is $(\varpi_0\mathcal{I}-C-De-\varpi_0\tau v)=0$ if ϖ_0 is real root of characteristic equation (5.4). Now, $u(\omega)=e^{\varpi_0\omega}v$ is clearly non-oscillatory solution of Eq (5.3).
For $(b)\to(a)$. Assume (b) holds that Eq (5.3) has non-oscillatory solution $u(\omega)=[u(\omega),v(\omega)]^T$ for sake of contradiction. For $\omega\geqslant\tau$, we suppose that the components of $u(\omega)$ are positive. We know, that $u(\omega)$ is of an exponential order because of the Theorem 2.15, and hence there exists $\eta\in\mathbf{R^m}$ that is Laplace transformations of both sides of the Eq (5.3) yield.

$$\mathcal{F}(s)\mathcal{U}(s)=\phi(s),\ Res>\eta \tag{5.5}$$

where

$$\mathcal{F}(s) = s\mathcal{I} - C - \mathcal{D}e^{-s\tau} \tag{5.6}$$

and

$$\phi(s) = C_q(\omega)\mathrm{u}(0) + \mathcal{K}_q(\omega)\mathrm{u}'(0) - C - De^{-s\tau}\int_{-\tau}^{0} e^{-s\omega}u(\omega)d\omega. \tag{5.7}$$

According to the hypothesis, for any $s \in \mathbf{R^m}$, $det[\mathcal{F}(s)] \neq 0$. In addition,

$$\lim_{s\to\infty}(det[\mathcal{F}(s)]) = \infty \tag{5.8}$$

and

$$det[\mathcal{F}(s)] > 0 \;\forall\; s \in \mathbf{R^m}. \tag{5.9}$$

Suppose $\mathrm{u}(s)$ represents the Laplace transform of the solution's first component $\mathrm{u}(\omega)$. Then, according to the Cramer rule,

$$\mathcal{U}(s) = \frac{det[\mathcal{M}(s)]}{det[\mathcal{F}(s)]}, Res > \eta \tag{5.10}$$

where

$$A = \begin{bmatrix} \phi_1(s) & \mathcal{F}_{12}(s) \\ \phi_2(s) & \mathcal{F}_{22}(s) \end{bmatrix}$$

ϕ_1 is i^{th} component of vector $\phi(s)$ and $\mathcal{F}_{ij}(s)$ is $(i,j)^{\omega h}$ component of matrix $\mathcal{F}(s)$. Obviously, for all $i, j = 1, 2$ functions $\phi_1(s)$ and $\mathcal{F}_{ij}(s)$ are entire and thus $det[\mathcal{M}(s)]$ and $det[\mathcal{F}(s)]$ are also complete functions. Assume σ_0 be abscissa of convergence of $\mathrm{u}(s)$, now,

$$\phi_0 = \inf\{\sigma \in \mathbf{R^m} : U(\sigma)\ exists\}.$$

According to Theorem 2.12, we find $\sigma_0 = -\infty$ and (5.8) becomes

$$\mathcal{U}(s) = \frac{det[\mathcal{M}(s)]}{det[\mathcal{F}(s)]}\ \forall s \in \mathbf{R^m}. \tag{5.11}$$

As $u(\omega) > 1$ then $\mathcal{U}(s) > 1 \forall s \in \mathbf{R^m}$ and by (5.7) and (5.9), $det[\mathcal{M}(s)] > 1$ for $s \in \mathbf{R^m}$. There are positive constants $\mathcal{K}$, γ, and s_0, as defined by $M(s)$ and (5.6) and (5.7), respectively.

$$det[\mathcal{M}(s)] \leqslant \mathcal{K}e^{-\gamma s}\ for\ s \leqslant -s_0. \tag{5.12}$$

Also, given (5.8), (5.9) and fact that $\det[\mathcal{F}(s)]$ is variable s and $e^{-s\tau}$, positive number m exists that is

$$det[\mathcal{F}(s)] \geqslant m\ for\ s \in \mathbf{R^m}. \tag{5.13}$$

It may be concluded from (5.11)–(5.13) that

$$\mathcal{U}(s) = \int_0^\infty e^{-s\omega}d\omega \geqslant \int_T^\infty e^{-s\omega}\mathrm{u}(s)d\omega \geqslant e^{-sT}\int_T^\infty e^{-s\omega}\mathrm{u}(s)d\omega > 1$$

and so

$$1 < \int_T^\infty e^{-s\omega}\mathrm{u}(s)d\omega \leqslant \frac{\mathcal{K}}{m}e^{s(T-\gamma)} \to 1\ as \to -\infty.$$

For $\omega \geqslant T$, this means that $\mathrm{u}(\omega) = 1$, which is contraction. The proof is done. □

6. Example

Example 6.1. *Consider a delay predator-prey system where $\mathcal{G} = [0, 3]$ is the initial value.*

$$\begin{cases} {}^{c}_{0}D^{\alpha}_{\omega}\mathrm{u}(\omega) = \mathrm{u}(1-\mathrm{u}) - \mathrm{uv}, \\ {}^{c}_{0}D^{\alpha}_{\omega}\mathrm{v}(\omega) = \mathrm{v} + e^{-3}\mathrm{v}(\omega-2)\mathrm{u}(\omega-2) - d\mathrm{v}, \\ \mathrm{u}(0) = \mathrm{u}_0, \\ \mathrm{v}(0) = \mathrm{v}_0, \quad -\sigma \leqslant \omega \leqslant 0. \end{cases} \tag{6.1}$$

The following is a vector representation of the system (5.12):

$$\begin{cases} {}^{c}_{0}D^{\beta}_{\omega}\mathrm{u}(\omega) = f(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega)\ \omega \in \mathcal{J} = [0, a], \\ \mathrm{u}(0) = \mathrm{u}_0, \ -\sigma \leqslant \omega \leqslant 0 \end{cases} \tag{6.2}$$

where

$$f(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega) = \mathcal{A}\mathrm{u}(\omega) + \mathcal{B}(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega).$$

According to Lemma 2.13 and Eq (5.1), $\lim_{\omega\to\infty} \mathrm{u}(\omega) = 0$,

$$\lim_{\omega\to\infty}\inf \frac{f(\mathrm{u}(\omega-\tau))}{\mathrm{u}(\omega-\tau)} \geqslant 2.$$

As a result, f is fuzzy mapping $f : \mathcal{G} \times \mathbf{E^m} \to \mathbf{E^m}$, $\mathrm{u}_\omega = \mathrm{u}(\omega-1)$ are a positive fuzzy functions of ω in $\mathbf{E^m}$, and x_0 is fuzzy number because $\mathcal{A}$ is fuzzy matrix. The mapping f is a level-wise continuous and bounded in $\mathbf{E^m}$ because $\mathcal{A}$ and $\mathcal{B}$ are a level-wise continuous and bounded on $\mathcal{G}$. f satisfies condition of Theorem 4.4, and so initial value issue (6.1) has a solution on $\mathcal{J}$, according to Theorem 4.4. Consider the following linearized system of (6.2):

$$ {}^{c}_{0}D^{\beta}_{\omega}\mathrm{u}(\omega) - C\mathrm{u}(\omega) - D\mathrm{u}(\omega-\tau) = 0. \tag{6.3}$$

The fuzzy matrices C and D have the following characteristic equation:

$$\varpi^4 + \mathcal{A}\varpi^3 + \mathcal{B}\varpi^2 + \mathcal{C}\varpi + \mathcal{D} + e^{-(1+\varpi)}(\mathcal{E}\varpi^3 + \mathcal{F}\varpi^2 + \mathcal{G}\varpi) + e^{-2(1+\varpi)}(\mathcal{H}\varpi^2 + \mathcal{I}\varpi + \mathcal{J}) = 0 \tag{6.4}$$

where

$$\begin{aligned} &\mathcal{A} = -2a_1 + 4,\ \mathcal{B} = a_1^2 - 4a_1 + 4(1-(1-\beta)^2\theta_1^2),\ \mathcal{C} = a_2^2, \\ &\mathcal{D} = 2a_2^2(1 + (1-\beta)\theta_1) - a_2^2 a_1,\ \mathcal{E} = -2, \mathcal{F} = 4a_1 - 8, \\ &\mathcal{G} = -2a_1^2 + 8a_1 - 8(1-(1-\beta)^2\theta_1^2),\ \mathcal{H} = (1-(1-\beta)^2\theta_1^2), \\ &\mathcal{I} = -2a_1(1-(1-\beta)^2\theta_1^2) + 4(1-(1-\beta)^2\theta_1^2), \\ &\mathcal{J} = a_1^2(1-(1-\beta)^2\theta_1^2) - 4a_1(1-(1-\beta)^2\theta_1^2) + 4(1-(1-\beta)^2\theta_1^2)^2. \end{aligned} \tag{6.5}$$

There are no real roots in the characteristic equation (6.4). The linearized system oscillates as a result of Theorem 5.2. The system (6.2) oscillates as well, according to the Theorem 5.1.

Example 6.2. *Consider FDPPS with $\mathcal{G} = [0, 3]$ as the initial value.*

$$\begin{cases} {}^{c}_{0}D^{\alpha}_{\omega}\mathrm{u}(\omega) = \mathrm{u}(1-\mathrm{u}) - \mathrm{uv}, \\ {}^{c}_{0}D^{\alpha}_{\omega}\mathrm{v}(\omega) = \mathrm{v} + 4e^{-2}\mathrm{v}(\omega-4)\mathrm{u}(\omega-4) - d\mathrm{v}, \\ \mathrm{u}(0) = \mathrm{u}_0, \\ \mathrm{v}(0) = \mathrm{v}_0, \quad -\sigma \leqslant t \leqslant 0. \end{cases} \tag{6.6}$$

The following is a vector representation of the system (6.6):

$$\begin{cases} {}_0^cD_\omega^\beta \mathrm{u}(\omega) = f(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega)\ \omega \in \mathcal{J} = [0, a], \\ \mathrm{u}(0) = \mathrm{u}_0, \ -\sigma \leqslant \omega \leqslant 0 \end{cases} \tag{6.7}$$

where

$$f(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega) = \mathcal{A}\mathrm{u}(\omega) + \mathcal{B}(\omega, \mathrm{u}(\omega), \mathrm{u}_\omega).$$

According to Lemma 2.13 and Eq (5.1), $\lim_{\omega\to\infty} \mathrm{u}(\omega) = 0$,

$$\lim_{\omega\to\infty} \inf \frac{f(\mathrm{u}(\omega-\tau))}{\mathrm{u}(\omega-\tau)} \geqslant 2.$$

As a result, f is fuzzy mapping $f : \mathcal{G} \times \mathbf{E^m} \to \mathbf{E^m}$, $\mathrm{u}_\omega = \mathrm{u}(\omega - 1)$ are positive fuzzy functions of ω in $\mathbf{E^m}$, and x_0 is fuzzy number because $\mathcal{A}$ is a fuzzy matrix. The mapping f is a level-wise continuous and bounded in $\mathbf{E^m}$ because $\mathcal{A}$ and $\mathcal{B}$ are a level-wise continuous and bounded on $\mathcal{G}$. f satisfies condition of Theorem 4.4, so initial value problem (6.7) has solution on $\mathcal{J}$, according to Theorem 4.4. Consider (6.7) as a linearized system:

$${}_0^cD_\omega^\beta \mathrm{u}(\omega) - C\mathrm{u}(\omega) - D\mathrm{u}(\omega - \tau) = 0. \tag{6.8}$$

The fuzzy matrices C and D have the following characteristic equation:

$$\varpi^4 + \mathcal{A}\varpi^3 + \mathcal{B}\varpi^2 + C\varpi + \mathcal{D} + e^{-2(2+\varpi)}(\mathcal{E}\varpi^3 + \mathcal{F}\varpi^2 + \mathcal{G}\varpi) + e^{-4(4+\varpi)}(\mathcal{H}\varpi^2 + \mathcal{I}\varpi + \mathcal{J}) = 0 \tag{6.9}$$

where

$$\begin{aligned} &\mathcal{A} = -2a_1 + 4,\ \mathcal{B} = a_1^2 - 4a_1 + 4(1 - (1-\beta)^2\theta_1^2),\ C = a_2^2, \\ &\mathcal{D} = 2a_2^2(1 + (1-\beta)\theta_1) - a_2^2 a_1,\ \mathcal{E} = -2, \mathcal{F} = 8a_1 - 16, \\ &\mathcal{G} = -4a_1^2 + 16a_1 - 16(1 - (1-\beta)^2\theta_1^2),\ \mathcal{H} = 4(1 - (1-\beta)^2\theta_1^2), \\ &\mathcal{I} = -8a_1(1 - (1-\beta)^2\theta_1^2) + 16(1 - (1-\beta)^2\theta_1^2), \\ &\mathcal{J} = 4a_1^2(1 - (1-\beta)^2\theta_1^2) - 16a_1(1 - (1-\beta)^2\theta_1^2) + 16(1 - (1-\beta)^2\theta_1^2)^2. \end{aligned} \tag{6.10}$$

There are no real roots in the characteristic equation (6.9). The linearized system oscillates as a result of Theorem 5.2. The system (6.7) oscillates as well, according to the Theorem 5.1. The solution curve of the oscillatory property of the system 6.6 is as shown in Figure 1:

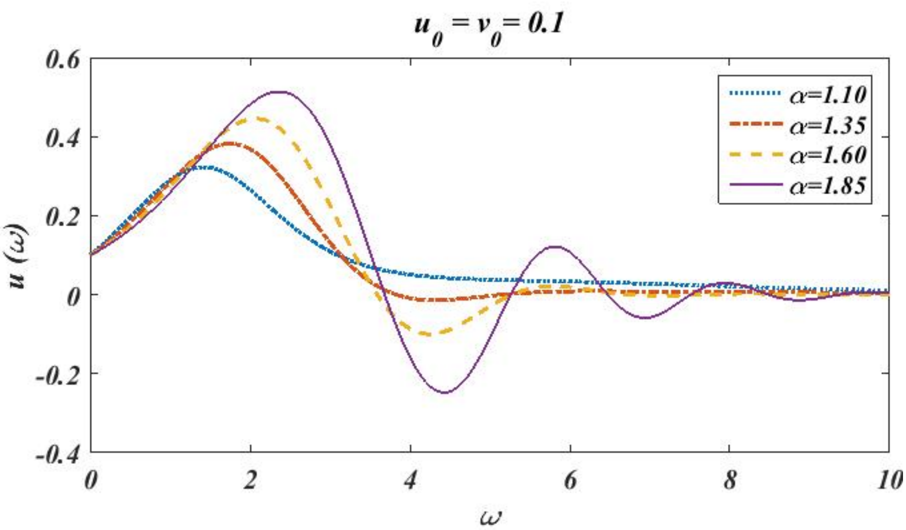

Figure 1. Curve of the oscillatory property of the system 6.6 for order (1,2).

7. Applications in real life

Predator-prey models play a crucial role in studying population dynamics and the management of renewable resources. Very rich and interesting dynamical behaviors, such as Hopf bifurcation, limit cycles, and homoclinic loops, have been observed. Time delay can be incorporated into a predator-prey model in four different ways. It can induce oscillations via Hopf bifurcation in all four types of models. May-type and Wangersky-Cunningham-type models exhibit switch of stability when the time delay takes a sequence of critical values. Constant-rate harvesting could induce more complex dynamics in delayed predator-prey systems, depending on which species is harvested. When the prey is selectively harvested, the dynamics are similar to that of the models without harvesting. Hopf bifurcation usually occurs and over-harvesting can always drive both species to extinction. The results of a study on the collapse of Atlantic cod stocks in the Canadian Grand Banks may be useful in designing fishing policies for the fishery industry, according to researchers at the University of British Columbia and the Canadian Department of Fisheries and Oceans. In general the functional response p(x) is a monotone function, but there are experiments that indicate that nonmonotonic responses occur at the microbial level. This is often seen when micro-organisms are used for waste decomposition or for water purification. A system of delayed differential equations has been proposed to explain why there is a time delay between changes in substrate concentration and corresponding changes in the growth rate of microorganism.

8. Conclusions

We introduced an FDPPS for the Caputo derivative in this research. On the interval $[1, 2]$, we successfully demonstrated the existence-uniqueness of the FDPPS. On $\mathcal{J}$, we additionally generalized Theorem for the existence theorem of the solution of an FDPPS with a fuzzy initial condition. We also covered the oscillation theorem for FDPPS solutions. The examples provided demonstrate how the results can be applied. Future work could also include expanding on the concept introduced in this paper and introducing observability and generalizing previous efforts. This is a productive field with a wide range of research initiatives that can result in a wide range of applications and theories.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Project No. GRANT794], King Faisal University (KFU), Ahsa, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of DSR at KFU.

Conflict of interest

The authors declare no conflict of interest.

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