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Research article

On the generalized Gronwall inequalities involving ψ-fractional integral operator with applications

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Abstract: In this paper, a Gronwall inequality involving ψ-fractional integral operator is obtained as a generalization of [23]. An example is listed to show the applications.

Keywords: generalized Gronwall inequality; ψ-fractional integral operator; existence results
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1. Introduction

The fractional Gronwall inequalities are effective tools to study the qualitative and quantitative properties of solution for fractional differential and integral equations [1–21] by giving the explicit bounds of solutions. Further detail on fractional Gronwall inequalities mainly involving the Riemann-Liouville fractional integrals [2–16], the Caputo fractional integrals [17], the Hadamard fractional integrals [18], the Katugampola fractional integrals [19,20], and the generalized proportional fractional integrals [21].

In [22], the authors produced the ψ-Hilfer fractional derivative as the Riemann-Liouville fractional derivative and the Caputo fractional derivative. In [23], considering the continuous dependence of the solution on the order and the initial condition of ψ-fractional differential equations, the authors presented the following theorem involving the ψ-fractional integral operator.

Theorem 1.1. *[23] Let u, v be two integrable functions and g continuous with domain $[a, b]$. Let $\psi \in C^1([a, b])$ be an increasing function such that $\psi'(t) \neq 0, t \in [a, b]$. Assume that* (1) *$u$ and v are nonnegative;* (2) *g is nonnegative and nondecreasing. If*

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds.$$

Then

$$u(t) \le v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{k\alpha-1} v(s)ds,\ t \in [a,b].$$

More inequalities related to ψ-fractional integral operator, see [24–26] for details.

As the generalizations of the classical fractional calculus operators, the ψ-fractional operator (i.e. the fractional derivative and integral of a function f with respect to another function ψ) has wide applications and properties [27–34] according to the choice of the ψ-function, which makes the Riemann-Liouville, Hadamard, Katugampola, etc fractional integral operators and the properties of above operators can be unified and considered as a whole.

Motivated by [23], in order to release the limitation of the number of nonlinear terms, new generalized forms of Theorem 1.1 are presented in this article, which is effective in dealing with neutral fractional differential equations involving ψ-fractional integral operator.

The organization of this paper is: In Section 2, we give some preliminaries. In Section 3, main results are obtained. In Section 4, the applications of (1.1) are given. In Section 5, an example is given to illustrate our result. In Section 6, the paper is concluded.

2. Preliminaries

We introduce some basic definitions and properties of the calculus theory, please see the details in [27, 34].

Definition 2.1. *[27, 34] Let f be an integrable function defined on $[a,b]$ and $\psi \in C^1([a,b])$ be an increasing function with $\psi'(t) \neq 0, t \in [a,b]$. The left ψ-Riemann-Liouville fractional integral operator of order γ of a function f is defined by*

$$({}_{t_0}I_{\psi}^{\gamma} f)(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^t (\psi(t) - \psi(s))^{\gamma-1} \psi'(s) f(s)ds, \gamma > 0.$$

Definition 2.2. *[27, 34] Let $\gamma \in (n-1, n), f \in C^n([a,b])$ and $\psi \in C^n([a,b])$ be an increasing function with $\psi'(t) \neq 0, t \in [a,b]$. The left ψ-Caputo fractional derivative of order γ of a function f is defined by*

$$({}_{t_0}^{C}D_{\psi}^{\gamma} f)(t) = ({}_{t_0}I_{\psi}^{n-\gamma} f^{[n]})(t) = \frac{1}{\Gamma(n-\gamma)} \int_{t_0}^t (\psi(t) - \psi(s))^{n-\gamma-1} \psi'(s) f^{[n]}(s)ds,$$

where $n = [\alpha] + 1, f^{[n]}(s) := (\frac{1}{\psi'(t)} \frac{d}{dt})^n f(t)$ on $[a,b]$.

Theorem 2.1. *[35] Let X be a Banach space, $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{y \in X : y = lFy$ for some $l \in [0,1]\}$ is bounded, then F has at least a fixed point.*

3. Results

Theorem 3.1. *Assume that x, a are integrable and nonnegative functions and $b_j, j = 1, 2, \cdots, m$ are continuous integrable and nonnegative functions with $t \in [a,b]$. Let $\psi \in C^1([a,b])$ be an increasing function with $\psi'(t) \neq 0, t \in [a,b]$. If*

$$x(t) \le a(t) + \sum_{j=1}^{m} b_j(t) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha_j - 1} x(s)ds, \tag{3.1}$$

then

$$x(t) \le a(t) + \sum_{k=1}^{\infty} b^k(t) \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_k=1}^{m} \frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\left(\sum_{\nu=1}^{k}\alpha_{j_\nu}\right)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\left(\sum_{\nu=1}^{k}\alpha_{j_\nu}-1\right)} a(\tau)d\tau, \tag{3.2}$$

provided that there exist a constant $M > 0$ such that

$$\lim_{n\to\infty} \frac{M \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} \sum_{j_{n+1}=1}^{m} \frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})\Gamma(\alpha_{j_{n+1}})}{\Gamma\left(\sum_{\nu=1}^{n+1}\alpha_{j_\nu}\right)}}{\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} \frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})}{\Gamma\left(\sum_{\nu=1}^{n}\alpha_{j_\nu}\right)}} = \rho \in [0,1),$$

where $\alpha_{j_n} \in \{\alpha_1, \alpha_2, \cdots, \alpha_m\}, n \in N, b(t) = \max\{b_j(t)\} \le M, j = 1, 2, \cdots, m.$

Proof. Let

$$Ax(t) = \sum_{j=1}^{m} b_j(t) \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_j-1} x(s)ds. \tag{3.3}$$

Then by (3.1), we get

$$x(t) \le a(t) + Ax(t). \tag{3.4}$$

By the monotonicity of the operators A and (3.1) and mathematical induction, for $t \in [a, b]$, we have

$$\begin{aligned} x(t) &\le a(t) + Ax(t) \le a(t) + A(a(t) + Ax(t)) = a(t) + Aa(t) + A^2x(t) \\ &\le a(t) + Aa(t) + A^2(a(t) + Ax(t)) \cdots \le \sum_{k=0}^{n-1} A^k a(t) + A^n x(t), \end{aligned} \tag{3.5}$$

i.e.

$$x(t) \le \sum_{k=0}^{n-1} A^k a(t) + A^n x(t), \tag{3.6}$$

where $A^0 a(t) = a(t)$.

For $t \in [a, b]$, by mathematical induction, we will show that

$$A^n x(t) \le b^n(t) \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} \frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\times\cdots\times\Gamma(\alpha_{j_n})}{\Gamma\left(\sum_{\nu=1}^{n}\alpha_{j_\nu}\right)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\left(\sum_{\nu=1}^{n}\alpha_{j_\nu}-1\right)} x(\tau)d\tau, \tag{3.7}$$

$n \in N$, and $\lim_{n\to\infty} A^n x(t) = 0$.

For $n = 1$, the conclusion in (3.7) holds naturally. Using the change of variables $\theta = \frac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}$ and the Beta function $\frac{\Gamma(\alpha_j)\Gamma(\beta_j)}{\Gamma(\alpha_j+\beta_j)} = B(\alpha_j,\beta_j)$, we have

$$\begin{aligned}
A^2x(t) &= A(Ax(t)) \\
&= \sum_{j_1=1}^{m} b_{j_1}(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_{j_1}-1}\sum_{j_2=1}^{m} b_{j_2}(s)\int_a^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha_{j_2}-1}x(\tau)d\tau ds \\
&\leq \sum_{j_1=1}^{m} b_{j_1}(t)\sum_{j_2=1}^{m} b_{j_2}(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_{j_1}-1}\int_a^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha_{j_2}-1}x(\tau)d\tau ds \\
&= \sum_{j_1=1}^{m} b_{j_1}(t)\sum_{j_2=1}^{m} b_{j_2}(t)\int_a^t \psi'(\tau)x(\tau)\int_\tau^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_{j_1}-1}(\psi(s)-\psi(\tau))^{\alpha_{j_2}-1}dsd\tau \\
&= \sum_{j_1=1}^{m} b_{j_1}(t)\sum_{j_2=1}^{m} b_{j_2}(t)\int_a^t \psi'(\tau)x(\tau)\int_\tau^t \psi'(s)(\psi(t)-\psi(\tau))^{\alpha_{j_1}-1} \\
&\quad \times\Big[1-\tfrac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}\Big]^{\alpha_{j_1}-1}(\psi(s)-\psi(\tau))^{\alpha_{j_2}-1}dsd\tau \\
&= \sum_{j_1=1}^{m} b_{j_1}(t)\sum_{j_2=1}^{m} b_{j_2}(t)\int_a^t \psi'(\tau)x(\tau)\int_0^1(1-\theta)^{\alpha_{j_1}-1}\theta^{\alpha_{j_2}-1}d\theta(\psi(t)-\psi(\tau))^{\alpha_{j_1}+\alpha_{j_2}-1}d\tau \\
&= \sum_{j_1=1}^{m}\sum_{j_2=1}^{m} b_{j_1}(t)b_{j_2}(t)\tfrac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})}{\Gamma(\alpha_j+\alpha_l)}\int_a^t \psi'(\tau)(\psi(t)-\varphi(\tau))^{\alpha_{j_1}+\alpha_{j_2}-1}x(\tau)d\tau \\
&\leq b^2(t)\sum_{j=1}^{m}\sum_{l=1}^{m}\tfrac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})}{\Gamma(\alpha_{j_1}+\alpha_{j_2})}\int_a^t \psi'(\tau)(\psi(t)-\varphi(\tau))^{\alpha_{j_1}+\alpha_{j_2}-1}x(\tau)d\tau,\ t\in[a,b].
\end{aligned} \tag{3.8}$$

For $t \in [a,b]$, we can suppose

$$A^kx(t) \leq b^k(t)\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_k=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\Big(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}\Big)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\Big(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}-1\Big)}x(\tau)d\tau. \tag{3.9}$$

For $n = k+1$, using the non-increasing properties of $b_j(t)$, $j = 1,2,\cdots,m, t \in [a,b]$, we have

$$\begin{aligned}
A^{k+1}x(t) &= A(A^kx(t)) \leq \sum_{j_{k+1}=1}^{m} b_{j_{k+1}}(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_{j_{k+1}}-1}dsb^k(s) \\
&\quad\times\sum_{j_1=1}^{m}\cdots\sum_{j_k=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\Big(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}\Big)}\int_a^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\Big(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}-1\Big)}x(\tau)d\tau \\
&\leq b^{k+1}(t)\sum_{j_1=1}^{m}\cdots\sum_{j_{k+1}=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_{k+1}})}{\Gamma\Big(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}\Big)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\Big(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\Big)}x(\tau)d\tau.
\end{aligned} \tag{3.10}$$

Since $b_j(t)$, $j = 1,2,\cdots,m$ are all continuous functions on $[a,b]$, then there exist a constant $M > 0$ such that $b(t) = \max\{b_j(t)\} \leq M$, $j = 1,2,\cdots,m$. So we have

$$A^nx(t) \leq M^n\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_n=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})}{\Gamma\Big(\sum\limits_{\nu=1}^{n}\alpha_{j_\nu}\Big)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\Big(\sum\limits_{\nu=1}^{n}\alpha_{j_\nu}-1\Big)}x(\tau)d\tau. \tag{3.11}$$

Consider the infinite series of number $\sum\limits_{n=1}^{\infty}M^n\sum\limits_{j_1=1}^{m}\sum\limits_{j_2=1}^{m}\cdots\sum\limits_{j_n=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})}{\Gamma\Big(\sum\limits_{\nu=1}^{n}\alpha_{j_\nu}\Big)}$, by virtue of the ratio test to the infinite series of number and the asymptotic approximation in [36], we get

$$\lim_{n\to\infty}\frac{M\sum\limits_{j_1=1}^{m}\sum\limits_{j_2=1}^{m}\cdots\sum\limits_{j_n=1}^{m}\sum\limits_{j_{n+1}=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})\Gamma(\alpha_{j_{n+1}})}{\Gamma\left(\sum\limits_{\nu=1}^{n+1}\alpha_{j_\nu}\right)}}{\sum\limits_{j_1=1}^{m}\sum\limits_{j_2=1}^{m}\cdots\sum\limits_{j_n=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\cdots\Gamma(\alpha_{j_n})}{\Gamma\left(\sum\limits_{\nu=1}^{n}\alpha_{j_\nu}\right)}}=\rho\in[0,1),\tag{3.12}$$

which implies that $A^n x(t)$ is convergent. Hence the conclusion in (3.2) holds.

Theorem 3.2. *Under the hypotheses of Theorem* 3.1 *and let* $a(t)$ *be a nondecreasing function for* $t\in[a,b]$. *Then*

$$x(t)\le a(t)\Big[1+\sum_{k=1}^{\infty}b^k(t)\sum_{j_1=1}^{m}\cdots\sum_{j_k=1}^{m}\frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\left(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}\right)}\int_a^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\left(\sum\limits_{\nu=1}^{k}\alpha_{j_\nu}-1\right)}d\tau\Big].\tag{3.13}$$

Proof. Since $a(t)$ ia a nondecreasing function, for α_j, $j=1,2,\cdots,m$, then we get

$$\begin{aligned}\int_a^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)}a(\tau)d\tau &\le a(s)\int_a^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)}d\tau\\ &= \frac{a(s)}{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)}(\psi(s)-\psi(a))^{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}\right)}.\end{aligned}\tag{3.14}$$

So from (3.2) and (3.14), we have

$$\begin{aligned}x(t) &\le a(t)+\sum_{k=1}^{\infty}b^k(t)\sum_{j_1=1}^{m}\cdots\sum_{j_k=1}^{m}\frac{\Gamma(\alpha_{j_1})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)}\int_a^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)-1}a(t)d\tau\\ &= a(t)\Big[1+\sum_{k=1}^{\infty}b^k(t)\sum_{j_1=1}^{m}\cdots\sum_{j_k=1}^{m}\frac{\Gamma(\alpha_{j_1})\times\cdots\times\Gamma(\alpha_{j_k})}{\Gamma\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)}\int_a^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\left(\sum\limits_{\nu=1}^{k+1}\alpha_{j_\nu}-1\right)-1}d\tau\Big].\end{aligned}\tag{3.15}$$

4. Applications

Consider the following neutral fractional equations involving ψ-fractional integral operator

$${}_{t_0}^{C}D_{\psi}^{\gamma}\Big[x(t)-\sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,x(t))\Big]=f(t,x(t)),\ t_0,t\in J=[a,b],\tag{4.1}$$

where $\gamma>0,\gamma_i>0,i=1,2,\cdots,l.$

(H_1) For the functions $f,g_i\in C(J\times R,R)$, there are some constants $c_i,c\ge 0$ such that

$$\|g_i(t,\phi)-g_i(t,\varphi)\|\le c_i\|\phi-\varphi\|,\quad \|f(t,\phi)-f(t,\varphi)\|\le c\|\phi-\varphi\|,\ t\in J.\tag{4.2}$$

(H_1') For the functions $f,g_i\in C(J\times R,R)$, there are some constants $c_i,c\ge 0$ such that

$$\|g_i(t,\phi)\|\le c_i(1+\|\phi\|),\quad \|f(t,\phi)\|\le c(1+\|\phi\|),\ t\in J.$$

(H_2) $H=\sum\limits_{i=1}^{l}\frac{c_i(\psi(b)-\psi(a))^{\gamma_i}}{\Gamma(\gamma_i+1)}+\frac{c(\psi(b)-\psi(a))^{\gamma}}{\Gamma(\gamma+1)}<1.$

By using Definitions 2.1 and 2.2, we get the following result.

Lemma 4.1. *Under the hypotheses* $(H_1),(H_2)$. $x(t)$ *satisfies* (4.1) *if and only if* $x(t)$ *satisfies the equality*

$$x(t)=X(t_0)+\sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,x(t))+{}_{t_0}I_{\psi}^{\gamma}f(t,x(t)),\ t_0,t\in J,\tag{4.3}$$

where

$$X(t_0)=x(t_0)+\sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}g_i(t_0,x(t_0)).$$

Theorem 4.1. *Under the hypotheses* $(H_1),(H_2)$. *Then* (4.1) *has a unique solution on* J.

Proof. For $x\in C(J,R)$, denote by

$$B_r=\{x\in C^1(J,R):\|x\|\le r\},r>0$$

with

$$\|X(t_0)\|+\Big[\sum_{i=1}^{l}\frac{c_i(\psi(b)-\psi(a))^{\gamma_i}}{\Gamma(\gamma_i+1)}+\frac{c(\psi(b)-\psi(a))^{\gamma}}{\Gamma(\gamma+1)}\Big]r\le r.$$

On B_r, we define the operator Γx as

$$(\Gamma x)(t)=X(t_0)+\sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,x(t))+{}_{t_0}I_{\psi}^{\gamma}f(t,x(t)),\ t_0,t\in J.\tag{4.4}$$

By $(H_1),(H_2)$, we have

$$\begin{aligned}\|(\Gamma x)\| &\le \|X(t_0)\|+\sum_{i=1}^{l}\|{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,x(t))\|+\|{}_{t_0}I_{\psi}^{\gamma}f(t,x(t))\|\\ &\le \|X(t_0)\|+\sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}c_i\|x\|+{}_{t_0}I_{\psi}^{\gamma}c\|x\|\\ &\le \|X(t_0)\|+\Big[\sum_{i=1}^{l}\tfrac{c_i(\psi(t)-\psi(t_0))^{\gamma_i}}{\Gamma(\gamma_i+1)}+\tfrac{c(\psi(t)-\psi(t_0))^{\gamma}}{\Gamma(\gamma+1)}\Big]r\\ &\le \|X(t_0)\|+\Big[\sum_{i=1}^{l}\tfrac{c_i(\psi(b)-\psi(a))^{\gamma_i}}{\Gamma(\gamma_i+1)}+\tfrac{c(\psi(b)-\psi(a))^{\gamma}}{\Gamma(\gamma+1)}\Big]r\le r,\ t_0,t\in J,\end{aligned}\tag{4.5}$$

Then for $x,y\in C(J,R)$, by (H_2), we get

$$\begin{aligned}\|\Gamma x-\Gamma y\| &= \|\sum_{i=1}^{l}[{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,x(t))-{}_{t_0}I_{\psi}^{\gamma_i}g_i(t,y(t))]+[{}_{t_0}I_{\psi}^{\gamma}f(t,x(t))-{}_{t_0}I_{\psi}^{\gamma}f(t,y(t))]\|\\ &\le \sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}\|g_i(t,x(t))-g_i(t,y(t))\|+{}_{t_0}I_{\psi}^{\gamma}\|f(t,x(t))-f(t,y(t))\|\\ &\le \sum_{i=1}^{l}{}_{t_0}I_{\psi}^{\gamma_i}c_i\|x-y\|+{}_{t_0}I_{\psi}^{\gamma}c\|x-y\|\\ &\le \Big[\sum_{i=1}^{l}c_{i\,t_0}I_{\psi}^{\gamma_i}1+c_{t_0}I_{\psi}^{\gamma}1\Big]\|x-y\|\\ &\le \Big[\sum_{i=1}^{l}\tfrac{c_i(\psi(t)-\psi(t_0))^{\gamma_i}}{\Gamma(\gamma_i+1)}+\tfrac{c(\psi(t)-\psi(t_0))^{\gamma}}{\Gamma(\gamma+1)}\Big]\|x-y\|\\ &\le H\|x-y\|<\|x-y\|,\end{aligned}\tag{4.6}$$

i.e. the operator Γ has a unique solution on J.

Theorem 4.2. *Under the hypotheses* $(H_1'), (H_2)$. *Then* (4.1) *has at least one solution on* J.

Proof. Consider the Cauchy problem (4.1). Define the operator Γ as in (4.4).

Claim 1: Γ is continuous. Let x_n be a sequence such that $x_n \to x \in C^1(J, R)$. Then since g_i, f are continuous and (H_1'), then we have

$$\begin{aligned} \|(\Gamma x_n)(t) - (\Gamma x)(t)\| &\le \sum_{i=1}^{l} \|_{t_0} I_\psi^{\gamma_i} [g_i(t, x_n(t)) - g_i(t, x(t))\| + \|_{t_0} I_\psi^{\gamma} [f(t, x_n(t)) - f(t, x(t))]\| \\ &\le \varepsilon \Big[\sum_{i=1}^{l} c_{i t_0} I_\psi^{\gamma_i} 1 + c_{t_0} I_\psi^{\gamma} 1 \Big] \|x - x_n\| \to 0, \ t \in J. \end{aligned} \tag{4.7}$$

Thus $(\Gamma x_n) \to (\Gamma x)$ in $C^1(J, R)$ and Γ is continuous.

Claim 2: Γ maps bounded sets into bounded sets in $C^1(J, R)$. Denote by B_r as in Theorem 4.1. Then as (4.5), we get that $\|(\Gamma x)\| \le r, \ t \in J$, which implies that $\|\Gamma x\| \le r$ and the operator Γ is uniformly bounded.

Claim 3: Γ maps bounded sets into equi-continuous sets of $C^1(J, R)$. For any $x \in B_r$, where B_r is defined as in Claim 2. As $t_1 \to t_2$ for $t_1, t_2 \in J$, we have

$$\begin{aligned} & |(\Gamma x)(t_2) - (\Gamma x)(t_1)| \\ \le & \sum_{i=1}^{l} |[_{t_0} I_\psi^{\gamma_i} g_i(t_2, x(t_2)) - {}_{t_0} I_\psi^{\gamma_i} g_i(t_1, x(t_1))]| + |_{t_0} I_\psi^{\gamma} f(t_2, x(t_2)) - {}_{t_0} I_\psi^{\gamma} f(t_1, x(t_1))| \\ \le & \sum_{i=1}^{l} \tfrac{1}{\Gamma(\gamma_i)} \Big[\int_{t_0}^{t_1} |((\psi(t_2) - \psi(s))^{\gamma - 1} - (\psi(t_1) - \psi(s))^{\gamma - 1}) \psi'(s) g_i(s, x(s))| ds \\ & + \int_{t_1}^{t_2} |(\psi(t_2) - \psi(s))^{\gamma - 1} \psi'(s) g_i(s, x(s))| ds \Big] \\ & + \tfrac{1}{\Gamma(\gamma)} \Big[\int_{t_0}^{t_1} |((\psi(t_2) - \psi(s))^{\gamma - 1} - (\psi(t_1) - \psi(s))^{\gamma - 1}) \psi'(s) f(s, x(s))| ds \\ & + \int_{t_1}^{t_2} |(\psi(t_2) - \psi(s))^{\gamma - 1} \psi'(s) f(s, x(s))| ds \Big] \\ \le & \sum_{i=1}^{l} \Big[\tfrac{\varepsilon}{\Gamma(\gamma_i)} \int_{t_0}^{t_1} |\psi'(s) g_i(s, x(s))| ds + \tfrac{1}{\Gamma(\gamma_i)} \int_{t_1}^{t_2} |(\psi(t_2) - \psi(s))^{\gamma_i - 1} \psi'(s) g_i(s, x(s))| ds \Big] \\ & + \tfrac{\varepsilon}{\Gamma(\gamma)} \Big[\int_{t_0}^{t_1} |\psi'(s) f(s, x(s))| ds + \tfrac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} |(\psi(t_2) - \psi(s))^{\gamma - 1} \psi'(s) f(s, x(s))| ds \Big] \\ \le & \sum_{i=1}^{l} \Big[\tfrac{\varepsilon}{\Gamma(\gamma_i)} \int_{t_0}^{t_1} \psi'(s) c_i (1 + |x(s)|) ds + \tfrac{1}{\Gamma(\gamma_i)} \int_{t_1}^{t_2} |(\psi(t_2) - \psi(s))^{\gamma_i - 1} \psi'(s) c_i (1 + |x(s)| ds \Big] \\ & + \Big[\tfrac{\varepsilon}{\Gamma(\gamma)} \int_{t_0}^{t_1} \psi'(s) c (1 + |x(s)| ds + \tfrac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{\gamma - 1} \psi'(s) c (1 + |x(s)| ds \Big] \\ \le & \sum_{i=1}^{l} \Big[\tfrac{c_i (1 + r) \varepsilon}{\Gamma(\gamma_i)} \int_{t_0}^{t_1} \psi'(s) ds + \tfrac{c_i (1 + r)}{\Gamma(\gamma_i)} \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{\gamma_i - 1} \psi'(s) ds \Big] \\ & + \Big[\tfrac{c (1 + r) \varepsilon}{\Gamma(\gamma)} \int_{t_0}^{t_1} \psi'(s) ds + \tfrac{c (1 + r)}{\Gamma(\gamma)} \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{\gamma - 1} \psi'(s) ds \Big] \\ \le & \sum_{i=1}^{l} \Big[\tfrac{c_i (1 + r) \varepsilon}{\Gamma(\gamma_i)} (\psi(t_1) - \psi(t_0)) + \tfrac{c_i (1 + r) (\psi(t_2) - \psi(t_1))^{\gamma_i}}{\Gamma(\gamma_i + 1)} \Big] \\ & + \Big[\tfrac{c (1 + r) \varepsilon}{\Gamma(\gamma)} (\psi(t_1) - \psi(t_0)) + \tfrac{c (1 + r) (\psi(t_2) - \psi(t_1))^{\gamma}}{\Gamma(\gamma + 1)} \Big] \to 0. \end{aligned} \tag{4.8}$$

Thus $\|(\Gamma x)(\hat{t}_2) - (\Gamma x)(\hat{t}_1)\| \to 0$, as $\hat{t}_1 \to \hat{t}_2$. As a consequence of Claims 1–3, it follows that $\Gamma : C^1(J, R) \to C^1(J, R)$ is continuous and completely continuous.

Claim 4: We show that the set $K = \{x \in C^1(J, R) : x = \lambda \Gamma x \text{ for some } 0 < \lambda < 1\}$ is bounded. Let $x \in K$, then $x = \lambda \Gamma x$ for some $0 < \lambda < 1$. Thus we have

$$x(t) = \lambda \Big[X(t_0) + \sum_{i=1}^{l} {}_{t_0} I_\psi^{\gamma_i} g_i(t, x(t)) + {}_{t_0} I_\psi^{\gamma} f(t, x(t)) \Big], \ t_0, t \in J. \tag{4.9}$$

By (H_1'), we have

$$\begin{aligned}\|x(t)\| &\leq \|X(t_0)\| + \sum_{i=1}^{l} \|{}_{t_0}I_{\psi}^{\gamma_i} g_i(t, x(t))\| + \|{}_{t_0}I_{\psi}^{\gamma} f(t, x(t))\| \\ &\leq \|X(t_0)\| + \sum_{i=1}^{l} {}_{t_0}I_{\psi}^{\gamma_i} \|g_i(t, x(t))\| + {}_{t_0}I_{\psi}^{\gamma} \|f(t, x(t))\| \\ &\leq \|X(t_0)\| + \sum_{i=1}^{l} c_{i\,t_0}I_{\psi}^{\gamma_i} 1 + c_{t_0}I_{\psi}^{\gamma} 1 + \sum_{i=1}^{l} {}_{t_0}I_{\psi}^{\gamma_i} c_i \|x(t)\| + {}_{t_0}I_{\psi}^{\gamma} c\|x(t)\|, \ t_0, t \in J,\end{aligned} \tag{4.10}$$

and Theorem 3.2 implies that

$$\begin{aligned}\|x(t)\| &\leq \Big(\|X(t_0)\| + \sum_{i=1}^{l} c_{i\,t_0}I_{\psi}^{\gamma_i} 1 + c_{t_0}I_{\psi}^{\gamma} 1\Big) \\ &\quad \times \Big[1 + \sum_{k=1}^{\infty} C^k \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} \frac{\Gamma(\alpha_{j_1}) \times \cdots \times \Gamma(\alpha_{j_k})}{\Gamma(\sum_{\nu=1}^{k} \alpha_{j_\nu})} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\sum_{\nu=1}^{k} \alpha_{j_\nu} - 1} d\tau\Big] \\ &= \Big(\|X(t_0)\| + \sum_{i=1}^{l} c_{i\,t_0}I_{\psi}^{\gamma_i} 1 + c_{t_0}I_{\psi}^{\gamma} 1\Big)\Big[1 + \sum_{k=1}^{\infty} C^k \sum_{j_1=1}^{m} \cdots \\ &\quad \times \sum_{j_k=1}^{m} \frac{\Gamma(\alpha_{j_1})\Gamma(\alpha_{j_2}) \times \cdots \times \Gamma(\alpha_{j_k})}{\Big(\sum_{\nu=1}^{k} \alpha_{j_\nu}\Big)\Gamma\Big(\sum_{\nu=1}^{k} \alpha_{j_\nu}\Big)} (\psi(b) - \psi(a))^{\Big(\sum_{\nu=1}^{k} \alpha_{j_\nu}\Big)}\Big],\end{aligned} \tag{4.11}$$

where $C = \max\{c_1, \cdots, c_l, c\}, \alpha_{j_k} \in \{\gamma_1, \cdots, \gamma_l, \gamma\}, k \in N$ and which shows that the set K is bounded.

By Theorem 2.1, the operator Γ has a fixed point, which is a solution of problem (4.1).

5. An example

Consider the following neutral ψ-fractional differential equation

$${}_1^C D_{\psi}^{\gamma}[x(t) - {}_1I_{\psi}^{\gamma_1} g_i(t, x(t))] = f(t, x(t)), \ t \in J = [1, 6], \tag{5.1}$$

where $\gamma = \frac{2}{3}, \gamma_1 = \frac{3}{4}, i = 1, g_1(t, x(t)) = \frac{\sqrt{t}}{5} sinx(t), f(t, x(t)) = \frac{lnt}{4} arctanx(t)$. Then g_1, f are continuous and satisfy the assumptions $(H_1), (H_2)$ with $\psi(t) = \sqrt[3]{t}, c_1 = \frac{\sqrt{6}}{5}, c = \frac{ln6}{4}$ and

$$\frac{c_1(\psi(t) - \psi(t_0))^{\gamma_1}}{\Gamma(\gamma_1 + 1)} + \frac{c(\psi(t) - \psi(t_0))^{\gamma}}{\Gamma(\gamma + 1)} = \frac{\sqrt{6}}{5} \frac{(\sqrt[3]{6} - 1)^{\frac{7}{4}}}{\Gamma(\frac{7}{4})} + \frac{ln6}{4} \frac{(\sqrt[3]{6} - 1)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})} = 0.8918 < 1.$$

Then by Theorem 4.1, (5.1) has a unique solution $x(t)$ on the interval $[1, 6]$.

By Theorem 4.2, (5.1) also has at least one solution $x(t)$ on the interval $[1, 6]$.

6. Conclusions

In this paper, we obtained a new generalized Gronwall inequality involving ψ-ractional integral operator that include the results in [23]. Furthermore, the Riemann-Liouville, the Hadamard, the Katugampola fractional integrals etc can be considered uniformly. The feasibility of the main results is checked by considering the existence of solutions of a type of neutral fractional differential equation involving ψ-fractional derivative. In the future, we will consider the stabilities for the neutral ψ-fractional differential equation.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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