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Research article

Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel

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Abstract: In the existing article, the existence of solutions to nonlinear fractional differential inclusions in the sense of the Atangana-Baleanu-Caputo ($\mathcal{ABC}$) fractional derivatives in Banach space is studied. The investigation of the main results relies on the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of non-compactness. A simulated example is proposed to explain the obtained results.

Keywords: Atangana-Baleanu fractional derivatives; measure of non-compactness; Mönch fixed point theorem
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1. Introduction

Presently, the differential systems of non-integer orders have gained wide prominence due to their great relevance in describing several real-world problems in physics, mechanics and engineering. For instance, we refer the reader to the monographs of Baleanu et al. [12], Hilfer [28], Kilbas et al. [31], Mainardi [33], Miller and Ross [34], Podlubny [37], Samko et al. [39] and the papers [22, 40].

During the study of some phenomena, a number of researchers realized the importance of fractional operators with non-singular kernels which can model practical physical phenomena well like the heat transfer model, the diffusion equation, electromagnetic waves in dielectric media, and circuit model (see [7, 9, 23, 24] and the references existing therein).

Caputo and Fabrizio in [11] studied a new kind of fractional derivative with an exponential kernel. Atangana and Baleanu [10] introduced a new operator with fractional order based upon the generalized Mittag-Leffler function. Their newly fractional operator involves kernel being nonlocal and nonsingular. The nonlocality of the kernel gives better description of the memory within the structure with different scale. Abdeljawad [4] extended this fractional derivative from order between zero and one to higher arbitrary order and formulated their associated integral operators.

On the other hand, the theory of measure of non-compactness is an essential tool in investigating the existence of solutions to nonlinear integral and differential equations, see, for example, the recent papers [6, 14, 20, 26, 38] and the references existing therein.

In [19], Benchohra et al. studied the existence of solutions for the fractional differential inclusions with boundary conditions

$$\begin{cases} {}^{C}\mathcal{D}^{r}y(t) \in G\left(t, y(t)\right), & \text{a.e. on } [0,T], \ \ 1 < r < 2, \\ y(0) = y_0, \quad y(T) = y_T, \end{cases} \tag{1.1}$$

where ${}^{C}\mathcal{D}^{r}$ is the Caputo fractional derivative, $G : [0,T] \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a multi-valued map, $y_0, y_T \in \mathbf{E}$ and $(\mathbf{E}, |\cdot|)$ is a Banach space and $\mathfrak{P}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \neq \emptyset\}$.

In the present work, we are interested in studying the existence of solutions for the following nonlinear fractional differential inclusions with $\mathcal{ABC}$ fractional derivatives

$$\begin{cases} {}^{ABC}_{\ \ a}\mathcal{D}^{\alpha}x(t) \in F\left(t, x(t)\right), & \text{a.e. on } \mathbf{J} := [a,b], \\ x(a) = x'(a) = 0, \end{cases} \tag{1.2}$$

where ${}^{ABC}_{\ \ a}\mathcal{D}^{\alpha}$ denotes the $\mathcal{ABC}$ fractional derivative of order $\alpha \in (1,2]$, $(\mathbf{E}, |\cdot|)$ is a Banach space, $\mathfrak{P}(\mathbf{E})$ is the family of all nonempty subsets of $\mathbf{E}$, and $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a given multi-valued map. We study the $\mathcal{ABC}$ fractional inclusion (1.2) in the case where the right hand side is convex-valued by means of the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of non-compactness.

Differential inclusions play an important role as a tool in the study of various dynamical processes described by equations with a discontinuous or multivalued right-hand side, occurring, in particular, in the study of dynamics of economical, social, and biological macrosystems. They also are very useful in proving existence theorems in control theory.

Due to the importance of fractional differential inclusions in mathematical modeling of problems in game theory, stability, optimal control, and so on. For this reason, many contributions have been investigated by some researchers [1–3, 5, 8, 16–18, 25, 30, 35].

It is worth noting that the results included with the fractional differential inclusions with the $\mathcal{ABC}$ fractional derivatives in Banach spaces are rather few, so the outputs of this paper are a new addition for the development of this topic.

2. Preliminaries

First at all, we recall the following definition of Riemann-Liouville fractional integral.

Definition 2.1. [31] Take $\alpha > 0$, $a \in \mathbb{R}$, and v a real-valued function defined on $[a, \infty)$. The Riemann-Liouville fractional integral is defined by

$${}_{a}I^{\alpha}v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} v(s)\, ds.$$

Next, we present the basic definitions of the new fractional operator due to Atangana and Baleanu [10] and the extended ones due to Abdeljawad [4].

Definition 2.2. [10] Take $\alpha \in [0,1]$ and $v \in H^1(a,b),\ a < b$, then the $\mathcal{ABC}$ fractional derivative is given by

$$ {}^{ABC}_{\ \ a}D^{\alpha}v(t) = \frac{\mathsf{B}(\alpha)}{1-\alpha}\int_a^t \mathbb{E}_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]v'(s)\,ds, \tag{2.1}$$

where $\mathsf{B}(\alpha) = \frac{\alpha}{2-\alpha} > 0$ denotes the normalization function such that $B(0) = B(1) = 1$ and $\mathbb{E}_{\alpha}$ denotes the Mittag-Leffler function defined by

$$\mathbb{E}_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}.$$

The associated Atangana-Baleanu ($\mathcal{AB}$) fractional integral by

$$ {}^{AB}_{\ a}I^{\alpha}v(t) = \frac{1-\alpha}{\mathsf{B}(\alpha)}v(t) + \frac{\alpha}{\mathsf{B}(\alpha)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}v(s)ds. \tag{2.2}$$

The following definitions concern with the higher order case.

Definition 2.3. [4] Take $\alpha \in (m, m+1]$, for some $m \in \mathbb{N}_0$, and v be such that $v^{(n)} \in H^1(a,b)$. Set $\beta = \alpha - m$. Then $\beta \in (0,1]$ and we define the $\mathcal{ABC}$ fractional derivative by

$$ {}^{ABC}_{\ \ a}\mathcal{D}^{\alpha}v(t) = {}^{ABC}_{\ \ a}D^{\beta}v^{(m)}(t). \tag{2.3}$$

In the light of the convention $v^{(0)}(t) = v(t)$, one has ${}^{ABC}_{\ \ a}\mathcal{D}^{\alpha}v(t) = {}^{ABC}_{\ \ a}D^{\alpha}v(t)$ for $\alpha \in (0,1]$.

The correspondent fractional integral is given by

$$ {}^{ABC}_{\ \ a}\mathcal{I}^{\alpha}v(t) = {}_aI^m\ {}^{ABC}_{\ \ a}I^{\alpha}v(t). \tag{2.4}$$

Lemma 2.4. *[4] For $u(t)$ defined on $[a,b]$ and $\alpha \in (m, m+1]$, for some $m \in \mathbb{N}_0$, we obtain that*

$$ {}^{ABC}_{\ \ a}\mathcal{I}^{\alpha}\ {}^{ABC}_{\ \ a}\mathcal{D}^{\alpha}u(t) = u(t) - \sum_{k=0}^{m}\frac{u^{(k)}}{k!}(t-a)^k.$$

Denote by $C(\mathbf{J},\mathbf{E})$ the Banach space of all continuous functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\| = \sup_{t\in\mathbf{J}}|x(t)|$. By $L^1(\mathbf{J},\mathbf{E})$, we indicate the space of Bochner integrable functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\|_1 = \int_0^b |x(t)|\,dt$.

2.1. Multi-valued maps analysis

Let the Banach space be $(\mathbf{E}, |\cdot|)$. The expressions we have used are $\mathfrak{P}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \neq \emptyset\}$, $\mathfrak{P}_{\mathbf{cl}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is closed}\}$, $\mathfrak{P}_{\mathbf{bd}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is bounded}\}$, $\mathfrak{P}_{\mathbf{cp}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is compact}\}$, $\mathfrak{P}_{\mathbf{cvx}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is convex}\}$.

- A multi-valued map $\mathfrak{U} : \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is convex (closed) valued, if $\mathfrak{U}(x)$ is convex (closed) for all $x \in \mathbf{E}$.

- $\mathfrak{U}$ is bounded on bounded sets if $\mathfrak{U}(B) = \cup_{x\in B}\mathfrak{U}(x)$ is bounded in $\mathbf{E}$ for any $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$, i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathfrak{U}(x)\}\} < \infty$.
- $\mathfrak{U}$ is called upper semi-continuous on $\mathbf{E}$ if for each $x^* \in \mathbf{E}$, the set $\mathfrak{U}(x^*)$ is nonempty, closed subset of $\mathbf{E}$, and if for each open set N of $\mathbf{E}$ containing $\mathfrak{U}(x^*)$, there exists an open neighborhood N^* of x^* such that $\mathfrak{U}(N^*) \subset N$.
- $\mathfrak{U}$ is completely continuous if $\mathfrak{U}(B)$ is relatively compact for each $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$.
- If $\mathfrak{U}$ is a multi-valued map that is completely continuous with nonempty compact values, then $\mathfrak{U}$ is u.s.c. if and only if $\mathfrak{U}$ has a closed graph (that is, if $x_n \to x_0, y_n \to y_0$, and $y_n \in \mathfrak{U}(x_n)$, then $y_0 \in \mathfrak{U}(x_0)$.

For more details about multi-valued maps, we refer to the books of Deimling [21] and Hu and Papageorgiou [29].

Definition 2.5. A multi-valued map $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is said to be Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $u \in \mathbf{E}$;

(ii) $x \mapsto F(t, x)$ is upper semi-continuous for almost all $t \in \mathbf{J}$.

We define the set of the selections of a multi-valued map F by

$$\mathcal{S}_{F,x} := \{f \in L^1(\mathbf{J}, \mathbf{E}) : f(t) \in F(t, x(t)) \text{ for a.e. } t \in \mathbf{J}\}.$$

Lemma 2.6. *[32] Let $\mathbf{J}$ be a compact real interval and $\mathbf{E}$ be a Banach space. Let F be a multi-valued map satisfying the Caratheòdory conditions with the set of L^1-selections $\mathcal{S}_{F,u}$ nonempty, and let $\Theta : L^1(\mathbf{J}, \mathbf{E}) \to C(\mathbf{J}, \mathbf{E})$ be a linear continuous mapping. Then the operator*

$$\Theta \circ \mathcal{S}_{F,x} : C(\mathbf{J}, \mathbf{E}) \to \mathfrak{P}_{\mathbf{bd,cl,cvx}}(C(\mathbf{J}, \mathbf{E})), \quad x \mapsto (\Theta \circ \mathcal{S}_{F,x})(x) := \Theta(\mathcal{S}_{F,x})$$

is a closed graph operator in $C(\mathbf{J}, \mathbf{E}) \times C(\mathbf{J}, \mathbf{E})$.

2.2. Measure of non-compactness

We specify this part of the paper to explore some important details of the Kuratowski measure of non-compactness.

Definition 2.7. [15] Let $\Lambda_{\mathbf{E}}$ be the family of bounded subsets of a Banach space $\mathbf{E}$. We define the Kuratowski measure of non-compactness $\kappa : \Lambda_{\mathbf{E}} \to [0, \infty]$ of $\mathbf{B} \in \Lambda_{\mathbf{E}}$ as

$$\kappa(\mathbf{B}) = \inf\{\epsilon > 0 : \mathbf{B} \subset \bigcup_{j=1}^{m} \mathbf{B}_j \text{ and } \operatorname{diam}(\mathbf{B}_j) \leq \epsilon\},$$

for some $m \in \mathbb{N}$ and $\mathbf{B}_j \in \mathbf{E}$.

Lemma 2.8. *[15] Let $\mathbf{C}, \mathbf{D} \subset \mathbf{E}$ be bounded, the Kuratowski measure of non-compactness possesses the next characteristics:*

i. $\kappa(\mathbf{C}) = 0 \Leftrightarrow \mathbf{C}$ *is relatively compact;*

ii. $\mathbf{C} \subset \mathbf{D} \Rightarrow \kappa(\mathbf{C}) \le \kappa(\mathbf{D})$*;*

iii. $\kappa(\mathbf{C}) = \kappa(\overline{\mathbf{C}})$*, where* $\overline{\mathbf{C}}$ *is the closure of* $\mathbf{C}$*;*

iv. $\kappa(\mathbf{C}) = \kappa(conv(\mathbf{C}))$*, where* $conv(\mathbf{C})$ *is the convex hull of* $\mathbf{C}$*;*

v. $\kappa(\mathbf{C}+\mathbf{D}) \le \kappa(\mathbf{C}) + \kappa(\mathbf{D})$*, where* $\mathbf{C}+\mathbf{D} = \{u+v : u \in \mathbf{C},\ v \in \mathbf{D}\}$*;*

vi. $\kappa(v\mathbf{C}) = |v|\kappa(\mathbf{C})$*, for any* $v \in \mathbb{R}$.

Theorem 2.9. *(Mönch's fixed point theorem) [36] Let* Ω *be a closed and convex subset of a Banach space* $\mathbf{E}$*;* $\mathcal{U}$ *a relatively open subset of* Ω*, and* $\mathcal{N} : \overline{\mathcal{U}} \to \mathfrak{P}(\Omega)$*. Assume that graph* $\mathcal{N}$ *is closed,* $\mathcal{N}$ *maps compact sets into relatively compact sets and for some* $x_0 \in \mathcal{U}$*, the following two conditions are satisfied:*

(i) $G \subset \overline{\mathcal{U}}$*,* $G \subset conv(x_0 \cup \mathcal{N}(G))$*,* $\overline{G} = \overline{C}$ *implies* $\overline{G}$ *is compact, where* C *is a countable subset of* G*;*

(ii) $x \notin (1-\mu)x_0 + \mu\mathcal{N}(x) \quad \forall u \in \overline{\mathcal{U}} \backslash \mathcal{U},\ \ \mu \in (0,1)$.

Then there exists $x \in \overline{\mathcal{U}}$ *with* $x \in \mathcal{N}(x)$.

Theorem 2.10. *[27] Let* $\mathbf{E}$ *be a Banach space and* $C \subset L^1(\mathbf{J}, \mathbf{E})$ *countable with* $|u(t)| \le h(t)$ *for a.e.* $t \in \mathbf{J}$ *, and every* $u \in C$*; where* $h \in L^1(\mathbf{J}, \mathbb{R}_+)$*. Then the function* $z(t) = \kappa(C(t))$ *belongs to* $L^1(\mathbf{J}, \mathbb{R}_+)$ *and satisfies*

$$\kappa\Big(\Big\{\int_0^b u(\tau)\,d\tau : u \in C\Big\}\Big) \le 2\int_0^b \kappa(C(\tau))\,d\tau.$$

3. Main results

We start this section with the definition of a solution of the $\mathcal{ABC}$ fractional inclusion (1.2).

Definition 3.1. A function $x \in C(\mathbf{J}, \mathbf{E})$ is said to be a solution of the $\mathcal{ABC}$ fractional inclusion (1.2) if there exist a function $f \in L^1(\mathbf{J}, \mathbf{E})$ with $f(t) \in F(t, x(t))$ for a.e. $t \in \mathbf{J}$, such that ${}^{ABC}_{\ \ a}D^{\alpha}_{t} x(t) = f(t)$ on $\mathbf{J}$, and the conditions $x(a) = x'(a) = 0$ are satisfied.

Lemma 3.2. *[4] For any* $h \in C(\mathbf{J}, \mathbb{R})$*, the solution* x *of the linear* $\mathcal{ABC}$ *fractional differential equation*

$$\begin{cases} {}^{ABC}_{\ \ a}D^{\alpha} x(t) = h(t), \ \ t \in \mathbf{J}, \\ x(a) = x'(a) = 0, \end{cases} \tag{3.1}$$

is given by the following integral equation

$$x(t) = \frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t h(s)\,ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,ds, \quad t \in \mathbf{J}. \tag{3.2}$$

Remark 3.3. The result of Lemma 3.2 is true not only for real valued functions $x \in C(\mathbf{J}, \mathbb{R})$ but also for a Banach space functions $x \in C(\mathbf{J}, \mathbf{E})$.

Lemma 3.4. *Assume that $F : \mathbf{J}\times\mathbf{E} \to \mathfrak{P}(\mathbf{E})$ satisfies Carathèodory conditions, i.e., $t \mapsto F(t,x)$ is measurable for every $x \in \mathbf{E}$ and $x \mapsto F(t,x)$ is continuous for every $t \in \mathbf{J}$. A function $x \in C(\mathbf{J},\mathbf{E})$ is a solution of the $\mathcal{ABC}$ fractional inclusion (1.2) if and only if it satisfies the integral equation*

$$x(t) = \frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t f(s)\,ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \tag{3.3}$$

where $f \in L^1(\mathbf{J},\mathbf{E})$ with $f(t) \in F(t,x(t))$ for a.e. $t \in \mathbf{J}$.

Now, we are ready to present the main result of the current paper.

Theorem 3.5. *Let $\varrho > 0$, $\mathcal{K} = \{x \in \mathbf{E} : \|x\| \le \varrho\}$, $\mathcal{U} = \{x \in C(\mathbf{J},\mathbf{E}) : \|x\| < \varrho\}$, and suppose that:*

(H1) *The multi-valued map $F : \mathbf{J}\times\mathbf{E} \to \mathfrak{P}_{\mathbf{cp,cvx}}(\mathbf{E})$ is Carathèodory.*

(H2) *For each $\varrho > 0$, there exists a function $\varphi \in L^1(\mathbf{J},\mathbb{R}_+)$ such that*

$$\|F(t,x)\|_{\mathfrak{P}} = \{|f| : f(t) \in F(t,x)\} \le \varphi(t),$$

for a.e. $t \in \mathbf{J}$ and $x \in \mathbf{E}$ with $|x| \le \varrho$, and

$$\lim_{\varrho\to\infty}\inf \frac{\int_a^b \varphi(t)dt}{\varrho} = \ell < \infty.$$

(H3) *There is a Carathèodory function $\vartheta : \mathbf{J}\times[0,2\varrho] \to \mathbb{R}_+$ such that*

$$\kappa(F(t,G)) \le \vartheta(t,\kappa(G)),$$

a.e. $t \in \mathbf{J}$ and each $G \subset \mathcal{K}$, and the unique solution $\theta \in C(\mathbf{J},[a,2\varrho])$ of the inequality

$$\theta(t) \le 2\left\{\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t \vartheta(s,\kappa(G(s)))\,ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\vartheta(s,\kappa(G(s)))\,ds\right\},\quad t \in \mathbf{J},$$

is $\theta \equiv 0$.

Then the $\mathcal{ABC}$ fractional inclusion (1.2) possesses at least one solution, provided that

$$\ell < \left[\frac{(2-\alpha)(b-a)}{\mathsf{B}(\alpha-1)} + \frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right]^{-1}. \tag{3.4}$$

Proof. Define the multi-valued map $\mathcal{N} : C(\mathbf{J},\mathbf{E}) \to \mathfrak{P}(C(\mathbf{J},\mathbf{E}))$ by

$$(\mathcal{N}x)(t) = \begin{cases} f \in C(\mathbf{J},\mathbf{E}) : f(t) = \frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w(s)\,ds \\ +\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}w(s)\,ds, \qquad w \in \mathcal{S}_{F,x}. \end{cases} \tag{3.5}$$

In accordance with Lemma 3.4, the fixed points of $\mathcal{N}$ are solutions to the $\mathcal{ABC}$ fractional inclusion (1.2). We shall show in five steps that the multi-valued operator $\mathcal{N}$ satisfies all assumptions of Mönch's fixed point theorem (Theorem 2.9) with $\overline{\mathcal{U}} = C(\mathbf{J},\mathcal{K})$.

Step 1. $\mathcal{N}(x)$ is convex, for any $x \in C(\mathbf{J}, \mathcal{K})$.

For $f_1, f_2 \in \mathcal{N}(x)$, there exist $w_1, w_2 \in \mathcal{S}_{F,x}$ such that for each $t \in \mathbf{J}$, we have

$$f_i(t) = \frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_a^t w_i(s)\, ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} w_i(s)\, ds, \ i = 1, 2.$$

Let $0 \le \mu \le 1$. Then for each $t \in \mathbf{J}$, one has

$$\begin{aligned}(\mu f_1 + (1-\mu) f_2)(t) &= \frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_a^t (\mu w_1(s) + (1-\mu) w_2(s))\, ds \\ &\quad + \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (\mu w_1(s) + (1-\mu) w_2(s))\, ds.\end{aligned}$$

Since $\mathcal{S}_{F,x}$ is convex (forasmuch F has convex values), then $\mu f_1 + (1-\mu) f_2 \in \mathcal{N}(x)$.

Step 2. $\mathcal{N}(G)$ is relatively compact for each compact $G \in \overline{\mathcal{U}}$.

Let $G \in \overline{\mathcal{U}}$ be a compact set and let $\{f_n\}$ be any sequence of elements of $\mathcal{N}(G)$. We show that $\{f_n\}$ has a convergent subsequence by using the Arzelà-Ascoli criterion of non-compactness in $C(\mathbf{J}, \mathcal{K})$. Since $f_n \in \mathcal{N}(G)$, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$, such that

$$f_n(t) = \frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_a^t w_n(s)\, ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} w_n(s)\, ds,$$

for $n \ge 1$. In view of Theorem 2.10 and the properties of the Kuratowski measure of non-compactness, we have

$$\begin{aligned}\kappa(\{f_n(t)\}) \le 2\Bigg\{ &\frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_a^t \kappa(\{w_n(s) : n \ge 1\})\, ds \\ &+ \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^t \kappa\Big(\Big\{(t-s)^{\alpha-1} w_n(s) : n \ge 1\Big\}\Big)\, ds \Bigg\}. \qquad (3.6)\end{aligned}$$

On the other hand, since G is compact, the set $\{w_n(\tau) : n \ge 1\}$ is compact. Consequently, $\kappa(\{w_n(s) : n \ge 1\}) = 0$ for a.e. $s \in \mathbf{J}$. Likewise,

$$\kappa\Big(\Big\{(t-s)^{\alpha-1} w_n(s) : n \ge 1\Big\}\Big) = (t-s)^{\alpha-1} \kappa(\{w_n(s) : n \ge 1\}) = 0,$$

for a.e. $t, s \in \mathbf{J}$. Therefore, (3.6) implies that $\{f_n(t) : n \ge 1\}$ is relatively compact in $\mathcal{K}$ for each $t \in \mathbf{J}$.

Furthermore, For each $t_1, t_2 \in \mathbf{J}$, $t_1 < t_2$, one obtain that:

$$\begin{aligned}|f_n(t_2) - f_n(t_1)| =& \Bigg| \frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_{t_1}^{t_2} w_n(s)\, ds + \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \Big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big] w_n(s)\, ds \\ &+ \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} w_n(s)\, ds \Bigg| \\ \le& \left| \frac{2-\alpha}{\mathsf{B}(\alpha-1)} \int_{t_1}^{t_2} w_n(s)\, ds \right| + \left| \frac{\alpha-1}{\mathsf{B}(\alpha-1)} \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \Big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big] w_n(s)\, ds \right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}w_n(s)\,ds\right|\\
&\leq \frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_{t_1}^{t_2}\varphi(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\varphi(s)\,ds\\
&+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|\varphi(s)\,ds
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Thus, $\{w_n(\tau) : n \geq 1\}$ is equicontinuous. Hence, $\{w_n(\tau) : n \geq 1\}$ is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Step 3. The graph of $\mathcal{N}$ is closed.

Let $x_n \to x_*$, $f_n \in \mathcal{N}(x_n)$, and $f_n \to f_*$. It must be to show that $f_* \in \mathcal{N}(x_*)$. Now, $f_n \in \mathcal{N}(x_n)$ means that there exists $w_n \in \mathcal{S}_{F,x_n}$ such that, for each $t \in \mathbf{J}$,

$$
f_n(t)=\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w_n(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}w_n(s)\,ds.
$$

Consider the continuous linear operator $\Theta : L^1(\mathbf{J}, \mathbf{E}) \to C(\mathbf{J}, \mathbf{E})$,

$$
\Theta(w)(t)\mapsto f_n(t)=\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w_n(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}w_n(s)\,ds.
$$

It is obvious that $\|f_n - f_*\| \to 0$ as $n \to \infty$. Therefore, in the light of Lemma 2.6, we infer that $\Theta \circ \mathcal{S}_F$ is a closed graph operator. Additionally, $f_n(t) \in \Theta(\mathcal{S}_{F,x_n})$. Since, $x_n \to x_*$, Lemma 2.6 gives

$$
f_*(t)=\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}w(s)\,ds,
$$

for some $w \in \mathcal{S}_{F,x}$.

Step 4. G is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Assume that $G \subset \overline{\mathcal{U}}$, $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, and $\overline{G} = \overline{C}$ for some countable set $C \subset G$. Using a similar approach as in Step 2, one can obtain that $\mathcal{N}(G)$ is equicontinuous. In accordance to $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, it follows that G is equicontinuous. In addition, since $C \subset G \subset conv\,(\{0\} \cup \mathcal{N}(G))$ and C is countable, then we can find a countable set $\mathbf{P} = \{f_n : n \geq 1\} \subset \mathcal{N}(G)$ with $C \subset conv\,(\{0\} \cup \mathbf{P})$. Thus, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$ such that

$$
f_n(t)=\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w_n(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}w_n(s)\,ds.
$$

In the light of Theorem 2.10 and the fact that $G \subset \overline{C} \subset \overline{conv}\,(\{0\} \cup \mathbf{P})$, we get

$$
\kappa\left(G(t)\right)\leq\kappa\left(\overline{C}(t)\right)\leq\kappa\left(\mathbf{P}(t)\right)=\kappa\left(\{f_n(t):n\geq 1\}\right).
$$

By virtue of (3.6) and the fact that $w_n(\tau) \in G(\tau)$, we get

$$
\kappa\left(G(t)\right)\leq 2\Bigg\{\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t\kappa\left(\{w_n(s):n\geq 1\}\right)\,ds
$$

$$+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t \kappa\left(\left\{(t-s)^{\alpha-1}w_n(s):n\geq 1\right\}\right)ds\Bigg\}$$

$$\leq 2\left\{\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t \kappa\left(G(s)\right)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\kappa\left(G(s)\right)\,ds\right\}$$

$$\leq 2\left\{\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t \vartheta\left(s,\kappa\left(G(s)\right)\right)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\vartheta\left(s,\kappa\left(G(s)\right)\right)\,ds\right\}.$$

Also, the function θ given by $\theta(t)=\kappa\left(G(t)\right)$ belongs to $C(\mathbf{J},[a,2\varrho])$. Consequently by **(H3)**, $\theta\equiv 0$, that is $\kappa\left(G(t)\right)=0$ for all $t\in\mathbf{J}$.

Now, by the Arzelà-Ascoli theorem, G is relatively compact in $C(\mathbf{J},\mathcal{K})$.

Step 5. Let $f\in\mathcal{N}(x)$ with $x\in\overline{\mathcal{U}}$. Since $x(\tau)\leq\varrho$ and **(H2)**, we have $\mathcal{N}(\overline{\mathcal{U}})\subset\overline{\mathcal{U}}$, because if it is not true, there exists a function $x\in\overline{\mathcal{U}}$ but $\|\mathcal{N}(x)\|>\varrho$ and

$$f(t)=\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t w(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}w(s)\,ds,$$

for some $w\in\mathcal{S}_{F,x}$. On the other hand, we have

$$\begin{aligned}\varrho<\|\mathcal{N}(x)\|&\leq\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^t |w(s)|\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|w(s)|\,ds\\&\leq\frac{2-\alpha}{\mathsf{B}(\alpha-1)}\int_a^b \varphi(s)\,ds+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\varphi(s)\,ds\\&\leq\left[\frac{(2-\alpha)(b-a)}{\mathsf{B}(\alpha-1)}+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right]\int_a^b \varphi(s)\,ds.\end{aligned}$$

Dividing both sides by ϱ and taking the lower limit as $\varrho\to\infty$, we infer that

$$\left[\frac{(2-\alpha)(b-a)}{\mathsf{B}(\alpha-1)}+\frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right]\ell\geq 1,$$

which contradicts (3.4). Hence $\mathcal{N}(\overline{\mathcal{U}})\subset\overline{\mathcal{U}}$.

As a consequence of Steps 1–5 together with Theorem 2.9, we infer that $\mathcal{N}$ possesses a fixed point $x\in C(\mathbf{J},\mathcal{K})$ which is a solution of the $\mathcal{ABC}$ fractional inclusion (1.2). □

4. Example

Consider the fractional differential inclusion

$$\begin{cases}{}^{ABC}_{\ \ 0}\mathcal{D}^{\frac{3}{2}}x(t)\in F\left(t,x(t)\right), & \text{a.e. on } [0,1],\\ x(0)=x'(1)=0,\end{cases}\tag{4.1}$$

where $\alpha=\frac{3}{2}, a=0, b=1$, and $F:[0,1]\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ is a multi-valued map given by

$$x\mapsto F\left(t,x\right)=\left(e^{-|x|}+\sin t, 3+\frac{|x|}{1+x^2}+5t^3\right).$$

For $f \in F$, one has

$$|f| = \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \leq 9, \quad x \in \mathbb{R}.$$

Thus

$$\begin{aligned}\|F(t,x)\|_{\mathfrak{P}} &= \{|f| : f \in F(t,x)\} \\ &= \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \leq 9 = \varphi(t),\end{aligned}$$

for $t \in [0,1],\ x \in \mathbb{R}$. Obviously, F is compact and convex valued, and it is upper semi-continuous.

Furthermore, for $(t,x) \in [0,1] \times \in \mathbb{R}$ with $|x| \leq \varrho$, one has

$$\lim_{\varrho\to\infty} \inf \frac{\int_0^1 \varphi(t)dt}{\varrho} = 0 = \ell.$$

Therefore, the condition (3.4) implies that

$$\left[\frac{(2-\alpha)(b-a)}{\mathsf{B}(\alpha-1)} + \frac{\alpha-1}{\mathsf{B}(\alpha-1)}\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right]^{-1} \approx 0.77217 > 0.$$

Finally, we assume that there exists a Carathèodory function $\vartheta : [0,1] \times [0,2\varrho] \to \mathbb{R}_+$ such that

$$\kappa\left(F(t,G)\right) \leq \vartheta(t,\kappa(G)),$$

a.e. $t \in [0,1]$ and each $G \subset \mathcal{K} = \{x \in \mathbb{R} : |x| \leq \varrho\}$, and the unique solution $\theta \in C([0,1],[0,2\varrho])$ of the inequality

$$\theta(t) \leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\vartheta\left(\tau,\kappa\left(G(\tau)\right)\right)\,d\tau\,du\right\},\quad t \in \mathbf{J},$$

is $\theta \equiv 0$.

Hence all the assumptions of Theorem 3.5 hold true and we infer that the $\mathcal{ABC}$ fractional inclusion (4.1) possesses at least one solution on $[0,1]$.

5. Conclusions

In this paper, we extend the investigation of fractional differential inclusions to the case of the $\mathcal{ABC}$ fractional derivatives in Banach space. Based on the set-valued version of Mönch fixed point theorem together with the Kuratowski measure of non-compactness, the existence theorem of the solutions for the proposed $\mathcal{ABC}$ fractional inclusions is founded. An clarified example is suggested to understand the theoretical finding.

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Conflict of interest

The authors declare no conflict of interest.

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