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Research article

Remarks on parabolic equation with the conformable variable derivative in Hilbert scales

Phuong Nguyen Duc[1], **Ahmet Ocak Akdemir**[2], **Van Tien Nguyen**[3] **and Anh Tuan Nguyen**[4,*]

[1] Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Ho Chi Minh City, Vietnam

[2] Department of Mathematics, Faculty of Science and Letters, Ağri Ibrahim Cecen University, Agri 04100, Turkey

[3] Faculty of Math, FPT University HCM, Saigon Hi-Tech Park, Thu Duc City, Ho Chi Minh City, Vietnam

[4] Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam

* **Correspondence:** Email: nguyenanhtuan@tdmu.edu.vn.

Abstract: In this paper, we are interested in diffusion equations with conformable derivatives with variable order. We will study two different types of models: the initial value model and the nonlocal in time model. With different values of input values, we investigate the well-posedness of the mild solution in suitable spaces. We also prove the convergence of mild solution of the nonlocal problem to solutions of the initial problem. The main technique of our paper is to use the theory of Fourier series in combination with evaluation techniques for some generalized integrals. Our results are one of the first directions on the diffusion equation with conformable variable derivative in Hilbert scales.

Keywords: parabolic equations; conformable derivative; variable order fractional derivatives; variable order derivative; Sobolev embeddings
Mathematics Subject Classification: 35A05, 35A08

1. Introduction

Fractional calculus has recently attracted the attention of many researchers and has become an attractive field of study with its different application areas. Some researchers have discovered that fractional differential equations with different singular or non-singular kernel need to be determined by real-world problems in the fields of engineering and science. Some definitions/approaches, for example, Riemann-Liouville, Hadamard, Katugampola, Riesz, Caputo-Fabrizio, and Atangana-Baleanu operators, were presented and tested using a variety of theories. Many important analytical methods have been used to achieve analytical solutions to fractional diffusion equations. By replacing many differential operators of fractional order with different PDE types of integer order, we form different types of boundary value problems with fractional order. However, the types of diffusion equations with fractional derivatives in Hilbert scales space are not really abundant because of their difficulty. We can list a few interesting works on PDEs with fractional derivatives, for example, [7–9, 12–14, 16, 21, 22, 27–29] and the references therein.

Let T be a positive number. In this paper, we consider the initial value problem for the conformable heat equation (or called parabolic equation with conformable operator)

$$\begin{cases} \dfrac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} y + \mathcal{A} y(x,t) = F(x,t), & x \in \Omega, \ \ t \in (0,T), \\ y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \end{cases} \tag{1.1}$$

where $\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} y = T^0_{\beta(t)} y(t)$ is defined in Definition (2.3). Here $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with the smooth boundary $\partial\Omega$. We are interested to study two following conditions

$$y(x,0) = y_0(x), \quad x \in \Omega. \tag{1.2}$$

or nonlocal in time condition

$$y(x,0) + h y(x,T) = y_0(x), \quad h > 0, \quad x \in \Omega. \tag{1.3}$$

The condition (1.2) is also known as initial conditions, which is familiar to mathematicians in the field of PDEs. Let us provide some remarks on the condition (1.3). Non-local conditions present and explain some more realistic perspectives for some particular phenomena for which usual initial conditions are replaced by multi-time point data such as studying atomic reactors [1, 2, 26]. In terms of mathematical aspect, since these conditions provide different data from the usual initial/terminal conditions problems with associated nonlocal conditions possess particular properties. In particular, it is well-known that while the problem for the usual parabolic equation is well-posed with the initial Cauchy condition at $t = 0$ and such problem is ill-posed with given data at terminal time $t = T > 0$, the well-posedness can be witnessed for the problems involving forward parabolic equations with non-local in time conditions connecting the values at different times [5]. In fact, throughout this work, we can see that the techniques to derive well-posed results for the initial value problem and the nonlocal in time problem are quite different. The above remarks play an important role in our motivation for deciding to carry out this study. As far as we know, there is very little documentation on the solution connection boundary conditions at different points in time, for example, at the beginning and at the end. Consideration of non-local initial conditions or non-local final conditions derived from actual processes.

Before we cover our problem, we give some background on conformable derivatives. A Conformable derivative can be first stated by Khalil and his colleagues [3] for functions $f : [0, \infty] \to +\infty$, it can be considered as the general form of the classical derivative and follows the same properties as the classical derivative. Furthermore, the physical meaning of the conformable derivative is assumed to be a modification of the classical derivative of direction and magnitude. More precisely, the general conformable derivative possesses similar physical and geometrical interpretations of Newton's derivative. However, while Newton's derivative describes the velocity of a particle or slope of a tangent, the general conformable derivative can be regarded as a special velocity, its direction and strength rely on a particular function [23].

Let us take M as a Banach space, and the function $f : [0,\infty) \to M$ and $\frac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta}$ be the conformable derivative of order $0 < \beta \le 1$ locally defined by

$$\frac{{}^{\mathscr{C}}\partial^\beta f(t)}{\partial t^\beta} := \lim_{h\to 0} \frac{f(t + ht^{1-\beta}) - f(t)}{h} \quad \text{in } M, \tag{1.4}$$

for each $t > 0$. For additive information about the above definition, we refer the reader to [3,4,6,10,11, 20]. An easy observation is that if $\beta = 1$ then the definition given above is the definition of the classical derivative. To further understand the relationship between conformable and classical derivatives, we direct the reader to the interesting paper [15]. This paper can be considered as one of the first works to investigate diffusion equations with conformable derivative in the Sobolev space. According to natural development, based on the conformable derivative, mathematicians have built a good theory for conformable derivative with orders dependent on a variable.

For the reader to better understand the history of this problem, we present a number of related works. Let us provide the comments of some fractional diffusion equations associated with fractional derivative whose order is a constant, i.e., $\beta(t) = \beta$.

Now, we introduce some previous work mentioned on fractional diffusion equation with variable order. In [18], the authors considered the relaxation-type equation with fractional variable order as follows

$$\begin{cases} \dfrac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}} y(t) + By(t) = F(t), & 0 < \alpha(t) \le 1, \\ y(0) = 1, \end{cases} \tag{1.5}$$

where $\frac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}}$ is the left Caputo derivative of order $\alpha(t)$, B is the relaxation coefficient, $f(t)$ denotes the external source term. The authors investigated the cable equation with fractional variable order [19]. In [24], the authors studied a dynamical system described by the following fractional differential equation with variable order

$$\begin{cases} \dfrac{\partial^{\alpha(t,y(t))}}{\partial t^{\alpha(t)}} y(t) = F(t, y(t)), & 0 < \alpha(t, y(t)) \le 1, \\ y(c) = y_0, & , \end{cases} \tag{1.6}$$

The authors considered the following dynamical system with variable-order fractional derivative

$$\begin{cases} {}^{C}D^{q(t)} x(t) = f(t, x), \\ x(a) = 0, \end{cases}$$

where $q(t)$ is the variable-order of differentiation [25].

To the best of our knowledge, there are not any results for considering the well-posedness of two problem (3.1)–(1.2) and (3.1)–(1.3) . We draw attention to the paper [17] since it mentioned variable conformable derivative. They investigated the fundamental solutions for initial value problem for linear diffusion differential equations with the conformable variable order derivative. Their techniques are based on upper and lower solutions and monotone iterative method. One difference is that they consider (3.1) on the unbounded domain, while we consider it on the bounded domain. Our approach in this paper is different from [17] because we have to learn the ideas of Fourier series. A new point of the current paper is that we carefully examine the well-posedness of our problem.

Let us assert that the problem with the variable conformable derivative is more difficult than the derivatives of constant derivative. The main reason is the appearance of integrals with exponents as functions, for example $\int_0^t r^{\beta(r)-1}dr$ causing many difficulties in calculation and evaluation. To overcome these difficulties, we need to have skillful judgment to control the components containing these singular integrals.

The main objective of this paper is to investigate the existence and regularization of solutions for two problems. With different assumptions of the input functions F and u_0, we will show the space containing the solution. As introduced above, we have a challenge with components that contain singular integrals. Another interesting contribution is that we will examine the relationship between the solutions of two problems: nonlocal problem (3.1)–(1.3) and (3.1)–(1.2). The result is proven convergent of the mild solution to (3.1)–(1.3) when $h \to 0^+$. This proof of convergent is understood as a non-trivial task.

The structure of the paper is given as follows. Section 3 examines the well-posedness for the initial value problem (3.1)–(1.2). The existence for the mild solution to (3.1)–(1.3) is investigated in section 4. We also derive that the convergence of the mild solution to problem (3.1)–(1.3) when $h \to 0^-$.

2. Preliminary results

In this section, we introduce notations and functional settings which will be used throughout this work. Recall that the spectral problem

$$\begin{cases} \mathcal{A}\psi_j(x) = \lambda_j \psi_j(x), & x \in \Omega, \\ \psi_j(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots$ with $\lambda_j \to \infty$ as $j \to \infty$ and the corresponding set of eigenfunctions $\{\psi_j\}_{j\geq 1} \subset H_0^1(\Omega)$.

Definition 2.1. We recall the Hilbert scale space as follows

$$\mathbb{Z}^s(\Omega) = \left\{ f \in L^2(\Omega), \quad \sum_{j=1}^{\infty} \lambda_j^{2s} \Big(\int_\Omega f(x)\psi_j(x)dx \Big)^2 < \infty \right\},$$

for any $s \geq 0$. It is well-known that $\mathbb{Z}^s(\Omega)$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{Z}^s(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_j^{2s} \Big(\int_\Omega f(x)\psi_j(x)dx \Big)^2 \right)^{1/2}, \quad f \in \mathbb{Z}^s(\Omega).$$

In the following, we provide definitions of the left integral and the (left) variable order fractional derivative which are taken from [17].

Definition 2.2. Let $f : [a, \infty) \to (0, 1]$. The left integral begin at a of variable function $h : (a, \infty) \to \mathbb{R}$ is given by

$$I^a_{h(t)} f(t) = \int_a^t (s-a)^{h(s)-1} f(s)ds, \quad t > a. \tag{2.1}$$

Definition 2.3. The (left) variable order fractional derivative starting at a of a function $f : [a, \infty)$ of order $h : [a, \infty) \to (0, 1]$ is defined by

$$T^a_{h(t)} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-h(t)}) - f(t)}{\epsilon}, t > a. \tag{2.2}$$

When $a = 0$, one can write $T_{h(t)}$. Moreover if $T^a_{h(t)} f(t)$ exists on (a, ∞) then $T^a_{h(t)} f(a) = \lim_{t \to a^+} T^a_{h(t)} f(t)$.

In addition, if the fractional derivative of order $h(t) \in (0, 1]$ of f exists for all $t \in (a, \infty)$, we simply say f is $h(t)-$ differentiable.

3. Linear inhomogeneous problem with initial condition

In this section, we focus on the initial value problem

$$\begin{cases} \dfrac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} y + \mathcal{A} y(x,t) = F(x,t), & x \in \Omega, \ t \in (0,T), \\ y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where y_0 and F will be defined later. Our main purpose in this section is to study the well-posedness of Problem (3.1). We use the Fourier analysis to construct the mild solution. Let us assume that $y(x,t) = \sum_{j=1}^{\infty} \langle y(.,t), \psi_j \rangle \psi_j(x)$ where $\langle y(.,t), \psi_j \rangle := \int_\Omega y(x,t) \psi_j(x) dx$. Taking the inner product $\langle \cdot, \cdot \rangle$ of the main equation of Problem (3.1) with ψ_j gives

$$\begin{cases} \dfrac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} \langle y(.,t), \psi_j \rangle + \lambda_j \langle y(.,t), \psi_j \rangle & = \langle F(.,t), \psi_j \rangle, \qquad t \in (0,T), \\ \langle y(.,0), \psi_j \rangle & = \langle y_0, \psi_j \rangle. \end{cases} \tag{3.2}$$

By the result in [17], we obtain the following equality

$$\begin{aligned} \langle y(.,t), \psi_j \rangle &= \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr \Big) \langle y_0, \psi_j \rangle \\ &\quad + \int_0^t r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1} dz \Big) \langle F(.,r), \psi_j \rangle dr, \end{aligned} \tag{3.3}$$

where we remind that $\beta : [0, \infty) \to (0, 1]$. By the definition of Fourier series, we have the following formula of the mild solution

$$\begin{aligned} y(x,t) &= \sum_j \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr \Big) \langle y_0, \psi_j \rangle \psi_j(x) \\ &\quad + \sum_j \Big[\int_0^t r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1} dz \Big) \langle F(.,r), \psi_j \rangle dr \Big] \psi_j(x) \\ &=: J_1 + J_2. \end{aligned} \tag{3.4}$$

Lemma 3.1. *Let* $m = \min_{0\le t\le 1} |\beta(t)|$ *and* $b = \max_{0\le t\le 1} |\beta(t)|$.
i) If $0 \le t \le 1$ *then*

$$\frac{t^b}{b} \le \Big| \int_0^t r^{\beta(r)-1} dr \Big| \le \frac{t^m}{m}. \tag{3.5}$$

ii) If $t \ge 1$ *then*

$$\frac{1}{b} + \frac{t^m - 1}{m} \le \Big| \int_0^t r^{\beta(r)-1} dr \Big| \le \frac{1}{m} + \frac{t^b - 1}{b}. \tag{3.6}$$

Proof. We claim (i) as follows. Since $\beta(r) \ge m$ and $0 < \beta(r) \le 1$, we know that $0 \le 1 - \beta(r) \le 1 - m$. Since $0 \le r \le t < 1$, we know that $\frac{1}{r} > 1$. It follows that

$$\left(\frac{1}{r}\right)^{1-\beta(r)} \le \left(\frac{1}{r}\right)^{1-m}. \tag{3.7}$$

This implies that

$$\Big| \int_0^t r^{\beta(r)-1} dr \Big| = \int_0^t \left(\frac{1}{r}\right)^{1-\beta(r)} dr \le \int_0^t \left(\frac{1}{r}\right)^{1-m} dr = \frac{t^m}{m}. \tag{3.8}$$

Since $1 - \beta(r) \ge 1 - b \ge 0$, we know that

$$\left(\frac{1}{r}\right)^{1-\beta(r)} \ge \left(\frac{1}{r}\right)^{1-b}.$$

It implies the following lower bound

$$\Big| \int_0^t r^{\beta(r)-1} dr \Big| = \int_0^t \left(\frac{1}{r}\right)^{1-\beta(r)} dr \ge \int_0^t \left(\frac{1}{r}\right)^{1-b} dr = \frac{t^b}{b}. \tag{3.9}$$

We next provide the proof of (ii). Since $t \ge 1$, we derive

$$\int_0^t r^{\beta(r)-1} dr = \int_0^1 r^{\beta(r)-1} dr + \int_1^t r^{\beta(r)-1} dr. \tag{3.10}$$

Using (3.5) with $t = 1$, we obtain the following upper and lower bound

$$\frac{1}{b} \le \int_0^1 r^{\beta(r)-1} dr \le \frac{1}{m}. \tag{3.11}$$

Our next aim is to consider the term $\int_1^t r^{\beta(r)-1} dr$. It is easy to observe that

$$1 - b \le 1 - \beta(r) \le 1 - m.$$

From the fact that $0 < \frac{1}{r} < 1$, we get the upper bound below

$$\int_1^t r^{\beta(r)-1} dr = \int_1^t \left(\frac{1}{r}\right)^{1-\beta(r)} dr \le \int_1^t \left(\frac{1}{r}\right)^{1-b} dr = \frac{t^b - 1}{b}, \tag{3.12}$$

and also, the lower bound

$$\int_1^t r^{\beta(r)-1} dr = \int_1^t \left(\frac{1}{r}\right)^{1-\beta(r)} dr \ge \int_1^t \left(\frac{1}{r}\right)^{1-m} dr = \frac{t^m - 1}{m}. \tag{3.13}$$

Connecting all the above inequalities (3.11), (3.12) and (3.13) gives us the assertion (3.6). □

Lemma 3.2. *Let* $m = \min_{0\le t\le 1} |\beta(t)|$ *and* $b = \max_{0\le t\le 1} |\beta(t)|$.
i) If $0 \le r \le t \le 1$ *then*

$$\frac{t^b - r^b}{b} \le \int_r^t z^{\beta(z)-1} dz \le \frac{t^m - r^m}{m}. \tag{3.14}$$

ii) If $0 < r \le 1 \le t$, *we get*

$$\frac{1 - r^b}{b} + \frac{t^m - 1}{m} \le \int_r^t z^{\beta(z)-1} dz \le \frac{1 - r^m}{m} + \frac{t^b - 1}{b}. \tag{3.15}$$

iii) If $0 \le 1 \le r \le t$ *then*

$$\frac{t^m - r^m}{m} \le \int_r^t z^{\beta(z)-1} dz \le \frac{t^b - r^b}{b}. \tag{3.16}$$

Proof. The proof of this lemma is almost the same as that of Lemma (3.1). Our claim is divided into three cases.

• The case $0 < t \le 1$. For this case, it is easy to see that

$$\left(\frac{1}{z}\right)^{1-b} \le \left(\frac{1}{z}\right)^{1-\beta(z)} \le \left(\frac{1}{z}\right)^{1-m}.$$

This implies that

$$\int_r^t \left(\frac{1}{z}\right)^{1-b} dz \le \int_r^t z^{\beta(z)-1} dz = \int_r^t \left(\frac{1}{z}\right)^{1-\beta(z)} dz \le \int_r^t \left(\frac{1}{z}\right)^{1-m} dz. \tag{3.17}$$

It is easy to verify that

$$\int_r^t \left(\frac{1}{z}\right)^{1-b} dz = \int_r^t z^{b-1} dz = \frac{t^b - r^b}{b} \tag{3.18}$$

and

$$\int_r^t \left(\frac{1}{z}\right)^{1-m} dz = \int_r^t z^{m-1} dz = \frac{t^m - r^m}{m}. \tag{3.19}$$

Hence, we obtain that for any $0 < r \le t \le 1$

$$\frac{t^b - r^b}{b} \le \int_r^t z^{\beta(z)-1} dz \le \frac{t^m - r^m}{m}. \tag{3.20}$$

• The case $0 \le r \le t \le 1$. For this case, we get the following identity

$$\int_r^t z^{\beta(z)-1} dz = \int_r^1 z^{\beta(z)-1} dz + \int_1^t z^{\beta(z)-1} dz. \tag{3.21}$$

By setting $t = 1$ into (3.20), we arrive at

$$\frac{1 - r^b}{b} \le \int_r^1 z^{\beta(z)-1} dz \le \frac{1 - r^m}{m}. \tag{3.22}$$

This implies the following estimate

$$\frac{1-r^b}{b}+\frac{t^m-1}{m}\le\int_r^1 z^{\beta(z)-1}dz+\int_1^t z^{\beta(z)-1}dz\le\frac{1-r^m}{m}+\frac{t^b-1}{b},\tag{3.23}$$

which allows us to deduce the desired result.

• The case $0\le 1\le r\le t$. Under this case, we obtain that if $r\le z\le t$ then

$$\left(\frac{1}{z}\right)^{1-m}\le\left(\frac{1}{z}\right)^{1-\beta(z)}\le\left(\frac{1}{z}\right)^{1-b}.$$

This implies that

$$\int_r^t\left(\frac{1}{z}\right)^{1-m}dz\le\int_r^t z^{\beta(z)-1}dz\le\int_r^t\left(\frac{1}{z}\right)^{1-b}dz.\tag{3.24}$$

Hence, we find that

$$\frac{t^m-r^m}{m}\le\int_r^t z^{\beta(z)-1}dz\le\frac{t^b-r^b}{b}.\tag{3.25}$$

□

The well-posedness of Problem (3.1) is described by the following theorem.

Theorem 3.3. *i) Let $y_0\in\mathbb{Z}^{s-\varepsilon}(\Omega)$ for $\varepsilon>0$ and $F\in L^\infty(0,T;\mathbb{Z}^s(\Omega))$. Then we get*

$$\left\|y(.,t)\right\|_{\mathbb{Z}^s(\Omega)}\lesssim\left(T^{b\varepsilon}+1\right)t^{-b\varepsilon}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}+\frac{t^b+t^m}{m}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))},\quad t>0.\tag{3.26}$$

ii) Let $y_0\in\mathbb{Z}^{s-\varepsilon}(\Omega)$ for $\varepsilon>0$ and $F\in L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))$ for any $0<\delta<\frac{1}{2}$. Let us assume that $2m>b$. Then we obtain

$$\left\|y(.,t)\right\|_{\mathbb{Z}^s(\Omega)}\lesssim\left(T^{b\varepsilon}+1\right)t^{-b\varepsilon}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}+t^{m-b\delta}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}.\tag{3.27}$$

Proof. Let us recall the mild solution

$$\begin{aligned}y(x,t)&=\sum_j\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle\psi_j(x)\\&\quad+\sum_j\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\Big]\psi_j(x)\\&=J_1+J_2.\end{aligned}\tag{3.28}$$

Step 1. Estimate of the term J_1. Using the inequality $e^{-a}\le C(\varepsilon)a^{-\varepsilon}$ for any $\varepsilon>0$, we find that

$$\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\le C(\varepsilon)\lambda_j^{-\varepsilon}\left(\int_0^t r^{\beta(r)-1}dr\right)^{-\varepsilon}.\tag{3.29}$$

• If $0 < t \leq 1$ in view of (3.5), we obtain

$$\exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \leq C_1 \lambda_j^{-\varepsilon} t^{-b\varepsilon}, \tag{3.30}$$

where

$$C_1 = C(\varepsilon) b^{\varepsilon}.$$

By Parseval's equality and using (3.30), we derive that

$$\begin{aligned} &\Big\| \sum_j \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \langle y_0, \psi_j \rangle \psi_j(x) \Big\|^2_{\mathbb{Z}^s(\Omega)} \\ &= \sum_j \lambda_j^{2s} \exp\Big(-2\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \langle y_0, \psi_j \rangle^2 \leq C_1^2 t^{-2b\varepsilon} \sum_j \lambda_j^{2s-2\varepsilon} \langle y_0, \psi_j \rangle^2. \end{aligned} \tag{3.31}$$

This implies that for $t \leq 1$

$$\Big\| \sum_j \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \langle y_0, \psi_j \rangle \psi_j(x) \Big\|_{\mathbb{Z}^s(\Omega)} \leq C_1 t^{-b\varepsilon} \big\| y_0 \big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}. \tag{3.32}$$

• If $t \geq 1$ thanks to (3.6) of Lemma (3.1), we obtain

$$\exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \leq C(\varepsilon) \lambda_j^{-\varepsilon} \left(\frac{1}{b} + \frac{t^m - 1}{m} \right)^{-\varepsilon}. \tag{3.33}$$

Since $t \geq 1$, it is obvious to see that the following inequality is satisfied

$$\left(\frac{1}{b} + \frac{t^m - 1}{m} \right)^{-\varepsilon} \leq b^{\varepsilon}.$$

From the previous observations, we get that

$$\begin{aligned} \Big\| \sum_j \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \langle y_0, \psi_j \rangle \psi_j(x) \Big\|_{\mathbb{Z}^s(\Omega)} &\leq C_1 \big\| y_0 \big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} \\ &\leq C_1 T^{b\varepsilon} t^{-b\varepsilon} \big\| y_0 \big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}. \end{aligned} \tag{3.34}$$

Combining (3.32) and (3.34), we deduce the following estimate for any $0 \leq t \leq T$

$$\begin{aligned} \Big\| \sum_j \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr\Big) \langle y_0, \psi_j \rangle \psi_j(x) \Big\|_{\mathbb{Z}^s(\Omega)} &= \Big\| J_1(.,t) \Big\|_{\mathbb{Z}^s(\Omega)} \\ &\leq C_1 \Big(T^{b\varepsilon} + 1 \Big) t^{-b\varepsilon} \big\| y_0 \big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}. \end{aligned} \tag{3.35}$$

Step 2. Estimate of the term J_2.
By Parseval's equality and Hölder's inequality, we find that

$$\Big\| J_2(.,t) \Big\|^2_{\mathbb{Z}^s(\Omega)} = \sum_j \lambda_j^{2s} \Big[\int_0^t r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1} dz \Big) \langle F(.,r), \psi_j \rangle dr \Big]^2$$

$$\leq \sum_j \lambda_j^{2s}\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr\Big]$$

$$\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle^2 dr\Big]. \tag{3.36}$$

Let us now consider possible cases as follows.
Case 1: $0 < t \leq 1$ *and* $F \in L^\infty(0,T;\mathbb{Z}^s(\Omega))$.

In view of (3.14) and the fact that $\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big) \leq 1$, we derive

$$\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr \leq \int_0^t r^{\beta(r)-1}dr \leq \frac{t^m}{m}. \tag{3.37}$$

It follows from (3.36) that

$$\begin{aligned}\Big\|J_2(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} &\leq \frac{t^m}{m}\sum_j \lambda_j^{2s}\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle^2 dr\Big]\\ &\leq \frac{t^m}{m}\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\Big\|F(.,r)\Big\|^2_{\mathbb{Z}^s(\Omega)}dr\\ &\leq \frac{t^m}{m}\int_0^t r^{\beta(r)-1}\Big\|F(.,r)\Big\|^2_{\mathbb{Z}^s(\Omega)}dr \leq \frac{t^m}{m}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}\Big(\int_0^t r^{\beta(r)-1}dr\Big).\end{aligned} \tag{3.38}$$

From (3.14) we obtain that the following estimate

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \leq \frac{t^m}{m}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{3.39}$$

Case 2: $t \geq 1$ *and* $F \in L^\infty(0,T;\mathbb{Z}^s(\Omega))$.
Using (3.38) and by a similar claim in case 1, we get that

$$\begin{aligned}\Big\|J_2(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} &\leq \sum_j \lambda_j^{2s}\Big[\int_0^t r^{\beta(r)-1}dr\Big]\Big[\int_0^t r^{\beta(r)-1}\langle F(.,r),\psi_j\rangle^2 dr\Big]\\ &\leq \Big[\int_0^t r^{\beta(r)-1}dr\Big]^2\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}.\end{aligned} \tag{3.40}$$

In view of (3.6), we obtain

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \leq \Big(\int_0^t r^{\beta(r)-1}dr\Big)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))} \leq \Big(\frac{1}{m}+\frac{t^b-1}{b}\Big)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{3.41}$$

Since $b \geq m$, we have the following inequality

$$\frac{1}{m}+\frac{t^b-1}{b} \leq \frac{1}{m}+\frac{t^b-1}{m} = \frac{t^b}{m}.$$

Therefore, we derive that for any $t \geq 1$

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \leq \frac{t^b}{m}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{3.42}$$

Combining case 1 and case 2, we get the following estimate for any $t > 0$ and $F \in L^\infty(0, T; \mathbb{Z}^s(\Omega))$

$$\left\|J_2(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \leq \frac{t^b + t^m}{m}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{3.43}$$

Case 3: $0 < t \leq 1$ *and* $F \in L^\infty(0, T; \mathbb{Z}^{s-\delta}(\Omega))$.

Using the inequality $e^{-a} \leq C(\delta)a^{-\delta}$ for any $\delta > 0$, we obtain that

$$\exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1}dz\Big) \leq C(\delta)\lambda_j^{-\delta}\Big(\int_r^t z^{\beta(z)-1}dz\Big)^{-\delta}. \tag{3.44}$$

From the fact that $t \leq 1$, we use (3.14) to get

$$\Big(\int_r^t z^{\beta(z)-1}dz\Big)^{-\delta} \leq \Big(\frac{t^b - r^b}{b}\Big)^{-\delta} = b^\delta\left(t^b - r^b\right)^{-\delta}. \tag{3.45}$$

Hence, we get the following estimate

$$\begin{aligned}\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr &\leq C(\delta,b)\lambda_j^{-\delta}\int_0^t r^{\beta(r)-1}\left(t^b - r^b\right)^{\delta}dr \\ &\leq C(\delta,b)\lambda_j^{-\delta}\int_0^t r^{m-1}\left(t^b - r^b\right)^{-\delta}dr,\end{aligned} \tag{3.46}$$

where we have used (3.7). Let us now treat the integral term on the right hand side of (3.46). By applying Hölder inequality and noting that $2m > b$, we derive that

$$\begin{aligned}\int_0^t r^{m-1}\left(t^b - r^b\right)^{-\delta}dr &= \int_0^t r^{\frac{2m-b-1}{2}}r^{\frac{b-1}{2}}\left(t^b - r^b\right)^{-\delta}dr \\ &\leq \Big(\int_0^t r^{2m-b-1}dr\Big)^{1/2}\quad\Big(\int_0^t r^{b-1}\left(t^b - r^b\right)^{-2\delta}dr\Big)^{1/2} \\ &= \sqrt{\frac{t^{2m-b}}{2m-b}}\sqrt{\int_0^t r^{b-1}\left(t^b - r^b\right)^{-2\delta}dr}.\end{aligned} \tag{3.47}$$

Set $r' = r^b$, then $dr' = br^{b-1}dr$. Then, since $2\delta < 1$, we have

$$\int_0^t r^{b-1}\left(t^b - r^b\right)^{-2\delta}dr = \frac{1}{b}\int_0^{t^b}\left(t^b - r'\right)^{-2\delta}dr' = \frac{1}{b}\frac{t^{b(1-2\delta)}}{1-2\delta}. \tag{3.48}$$

Combining (3.46), (3.47) and (3.48), we get the following estimate for $t \leq 1$

$$\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr \leq \overline{C}_1\lambda_j^{-\delta}t^{m-b\delta}, \tag{3.49}$$

where

$$\overline{C}_1 = \frac{C(\delta,b)}{\sqrt{b}\sqrt{1-2\delta}\sqrt{2m-b}}.$$

This inequality together with (3.36) yields

$$\begin{aligned}\Big\|J_2(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} &\le \overline{C}_1 t^{m-b\delta}\sum_j \lambda_j^{2s-\delta}\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle^2 dr\Big]\\ &\le \overline{C}_1 C(\delta,b)t^{m-b\delta}\sum_j \lambda_j^{2s-2\delta}\int_0^t r^{m-1}\Big(t^b-r^b\Big)^{-\delta}\langle F(.,r),\psi_j\rangle^2 dr\\ &\le \overline{C}_2 t^{m-b\delta}\int_0^t r^{m-1}\Big(t^b-r^b\Big)^{-\delta}\big\|F(.,r)\big\|^2_{\mathbb{Z}^{s-\delta}(\Omega)}dr, \end{aligned}\tag{3.50}$$

where $\overline{C}_2=\overline{C}_1 C(\delta,b)$. It is obvious to see that

$$\int_0^t r^{m-1}\Big(t^b-r^b\Big)^{-\delta}\big\|F(.,r)\big\|^2_{\mathbb{Z}^{s-\delta}(\Omega)}dr \le \big\|F\big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}\int_0^t r^{m-1}\Big(t^b-r^b\Big)^{-\delta}dr. \tag{3.51}$$

In the previous claim, we showed that

$$\int_0^t r^{m-1}\Big(t^b-r^b\Big)^{-\delta}dr \le \frac{t^{m-b\delta}}{\sqrt{b}\sqrt{1-2\delta}\ \sqrt{2m-b}}. \tag{3.52}$$

Combining (3.50), (3.51) and (3.52), we obtain that for any $0<t\le 1$

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \le \overline{C}_3 t^{m-b\delta}\big\|F\big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \tag{3.53}$$

where we denote by

$$\overline{C}_3=\frac{C(\delta,b)}{(1-2\delta)(2m-b)b}.$$

Case 4: $t\ge 1$ *and* $F\in L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))$.

We need to deal with the integral term

$$\mathrm{I}=\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr.$$

To this end, we derive the following equality

$$\begin{aligned}\mathrm{I}&=\int_0^1 r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^1 z^{\beta(z)-1}dz\Big)dr+\int_1^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)dr\\ &=\mathrm{I}_1+\mathrm{I}_2.\end{aligned}\tag{3.54}$$

For the term I_1, we put $t=1$ into (3.49) to obtain

$$\mathrm{I}_1\le \overline{C}_1\lambda_j^{-\delta}. \tag{3.55}$$

For the second term I_2, we note that $1\le r\le t$. Hence $r^{\beta(r)-1}\le r^{b-1}$. In view of the inequality $e^{-a}\le C(\delta)a^{-\delta}$ for any $\delta>0$, we get

$$\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\le C(\delta)\lambda_j^{-\delta}\Big(\int_r^t z^{\beta(z)-1}dz\Big)^{-\delta}. \tag{3.56}$$

Using (3.16) and (3.56), we find that

$$\mathrm{I}_2 \leq C(\delta)\lambda_j^{-\delta} \int_1^t r^{b-1} \left(\int_r^t z^{\beta(z)-1} dz \right)^{-\delta} dr. \tag{3.57}$$

In view of (3.16), we can check easily that

$$\left(\int_r^t z^{\beta(z)-1} dz \right)^{-\delta} \leq \left(\frac{t^m - r^m}{m} \right)^{-\delta} = m^{\delta} \left(t^m - r^m \right)^{-\delta}.$$

It follows from (3.57) that

$$\mathrm{I}_2 \leq C(\delta) m^{\delta} \lambda_j^{-\delta} \int_1^t r^{b-1} \left(t^m - r^m \right)^{-\delta} dr. \tag{3.58}$$

Next, using Hölder inequality to derive that

$$\begin{aligned} \int_1^t r^{b-1} \left(t^m - r^m \right)^{-\delta} dr &= \int_1^t r^{\frac{2b-m-1}{2}} r^{\frac{m-1}{2}} \left(t^m - r^m \right)^{-\delta} dr \\ &\leq \Big(\int_1^t r^{2b-m-1} dr \Big)^{1/2} \Big(\int_1^t r^{m-1} \left(t^m - r^m \right)^{-2\delta} dr \Big)^{1/2} \\ &= \sqrt{\frac{t^{2b-m} - 1}{2b - m}} \Big(\int_1^t r^{m-1} \left(t^m - r^m \right)^{-2\delta} dr \Big)^{1/2}. \end{aligned} \tag{3.59}$$

It is not difficult to compute that

$$\int_1^t r^{m-1} \left(t^m - r^m \right)^{-2\delta} dr = \frac{1}{m} \int_1^{t^m} \left(t^m - (r')^m \right)^{-2\delta} dr' = \frac{\left(t^m - 1 \right)^{1-2\delta}}{m(1 - 2\delta)}. \tag{3.60}$$

From the above two observations, we find that

$$\int_1^t r^{b-1} \left(t^m - r^m \right)^{-\delta} dr \leq \overline{C}_4 \sqrt{t^{2b-m} - 1} \left(t^m - 1 \right)^{\frac{1}{2} - \delta}, \tag{3.61}$$

where

$$\overline{C}_4 = \frac{1}{\sqrt{2b - m} \sqrt{m(1 - 2\delta)}}.$$

This combined with (3.58) yields to the following bound

$$\begin{aligned} \mathrm{I}_2 &\leq \overline{C}_5 \lambda_j^{-\delta} \sqrt{t^{2b-m} - 1} \left(t^m - 1 \right)^{\frac{1}{2} - \delta} \\ &\leq \overline{C}_5 \lambda_j^{-\delta} t^{b - \frac{m}{2}} t^{m(\frac{1}{2} - \delta)} = \overline{C}_5 t^{m - b\delta} \lambda_j^{-\delta}, \end{aligned} \tag{3.62}$$

where $\overline{C}_5 = C(\delta) m^{\delta} \overline{C}_4$. Combining (3.54), (3.55) and (3.62) and noting that $1 \leq t^{m-b\delta}$ we derive that

$$\int_0^t r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1} dz \Big) dr \leq \overline{C}_6 t^{m-b\delta} \lambda_j^{-\delta}, \quad \overline{C}_6 = \max(\overline{C}_1, \overline{C}_5). \tag{3.63}$$

Therefore, we obtain that for $t \geq 1$

$$\begin{aligned}\Big\|J_2(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} &= \sum_j \lambda_j^{2s}\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\Big]^2\\ &= \sum_j \lambda_j^{2s}\Big[\int_0^1 r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^1 z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\Big]^2\\ &+ \sum_j \lambda_j^{2s}\Big[\int_1^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\Big]^2\\ &= I+\overline{I}(t). \end{aligned} \tag{3.64}$$

It is obvious to see that the following inequality holds

$$I = \Big\|J_2(.,1)\Big\|^2_{\mathbb{Z}^s(\Omega)} \leq |\overline{C}_3|^2\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \tag{3.65}$$

where we hace applied (3.53). This together with (3.36) and (3.56) allow us to obtain that

$$\begin{aligned}\overline{I}(t) &\leq \overline{C}_6 t^{m-b\delta}\sum_j \lambda_j^{2s-\delta}\Big(\int_1^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle^2 dr\Big)\\ &\leq \overline{C}_6 C(\delta) t^{m-b\delta}\sum_j \lambda_j^{2s-2\delta}\Big(\int_1^t r^{\beta(r)-1}\Big(\int_r^t z^{\beta(z)-1}dz\Big)^{-\delta}\langle F(.,r),\psi_j\rangle^2 dr\Big)\\ &\leq \overline{C}_6 C(\delta) t^{m-b\delta}\Big(\int_1^t r^{\beta(r)-1}m^\delta\,(t^m-r^m)^{-\delta}\,dr\Big)\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}\\ &\leq \overline{C}_7 t^{m-b\delta}\Big(\int_1^t r^{b-1}\,(t^m-r^m)^{-\delta}\,dr\Big)\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \end{aligned} \tag{3.66}$$

where $\overline{C}_7 = \overline{C}_6 C(\delta) m^\delta$. By looking at the estimate (3.61), we infer the following estimate

$$\overline{I}(t) \leq \overline{C}_7\overline{C}_4 t^{2m-2b\delta}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \tag{3.67}$$

where $\overline{C}_8 = \overline{C}_7\overline{C}_4$. Combining (3.64), (3.65) and (3.67), we derive the following bound

$$\Big\|J_2(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} \leq |\overline{C}_3|^2\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))} + \overline{C}_8 t^{2m-2b\delta}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}. \tag{3.68}$$

Since $t \geq 1$, we follows from (3.68) that

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \leq \overline{C}_9 t^{m-b\delta}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}. \tag{3.69}$$

Summarizing two cases 3 and 4, we provide the following statement

$$\Big\|J_2(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \leq \overline{C}_{10} t^{m-b\delta}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \quad t>0. \tag{3.70}$$

Hence, the proof of (3.26) is finished by combining (3.35) and (3.43). At the same time, the proof of (3.27) is derived from (3.35) and (3.70). □

4. Linear problem with nonlocal in time condition

In this section, we focus the nonlocal value problem

$$\begin{cases} \dfrac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} y + \mathcal{A} y(x,t) = F(x,t), & x \in \Omega, \;\; t \in (0,T), \\ y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ y(x,0) + h y(x,T) = y_0(x), & x \in \Omega. \end{cases} \tag{4.1}$$

Our main purpose in this section is to study the well-posedness of problem (4.1) and the convergence of the mild solution when $h \to 0^+$.

Theorem 4.1.
i) Let $y_0 \in \mathbb{Z}^{s-\varepsilon}(\Omega)$ for $\varepsilon > 0$ and $F \in L^\infty(0,T;\mathbb{Z}^s(\Omega))$. Then Problem (4.1) has a unique solution y_h such that

$$\Big\| y_h(.,t) \Big\|_{\mathbb{Z}^s(\Omega)} \lesssim t^{-b\varepsilon} \Big\| y_0 \Big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} + h t^{-b\varepsilon} \Big\| F \Big\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))} + \Big\| F \Big\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{4.2}$$

where the hidden constant depends on T, b, ε, m.
ii) Let $y_0 \in \mathbb{Z}^{s-\varepsilon}(\Omega)$ for $\varepsilon > 0$ and $F \in L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))$ for any $0 < \delta < \frac{1}{2}$. Let us assume that $2m > b$. Then we get

$$\Big\| y_h(.,t) \Big\|_{\mathbb{Z}^s(\Omega)} \lesssim t^{-b\varepsilon} \Big\| y_0 \Big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} + h t^{-b\varepsilon} \Big\| F \Big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))} + t^{m-b\delta} \Big\| F \Big\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \tag{4.3}$$

where the hidden constant depends on $T, b, \varepsilon, m, \delta$.

Proof. Let us first establish the fomula of the mild solution to nonlocal problem (4.1). Suppose that Problem (4.1) has a solution y_h. From (3.3), we get

$$\begin{aligned} \langle y_h(.,t), \psi_j \rangle &= \exp\Big(-\lambda_j \int_0^t r^{\beta(r)-1} dr \Big) \langle y_0, \psi_j \rangle \\ &\quad + \int_0^t r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^t z^{\beta(z)-1} dz \Big) \langle F(.,r), \psi_j \rangle dr \end{aligned} \tag{4.4}$$

By let $t = T$ into the above expression, we see that

$$\begin{aligned} \langle y_h(.,T), \psi_j \rangle &= \exp\Big(-\lambda_j \int_0^T r^{\beta(r)-1} dr \Big) \langle y_0, \psi_j \rangle \\ &\quad + \int_0^T r^{\beta(r)-1} \exp\Big(-\lambda_j \int_r^T z^{\beta(z)-1} dz \Big) \langle F(.,r), \psi_j \rangle dr. \end{aligned} \tag{4.5}$$

From the above two equalities and the nonlocal-in-time condition

$$y_h(x,0) + h y_h(x,T) = f(x),$$

we deduce the following equality

$$\Big[1 + h \exp\Big(-\lambda_j \int_0^T r^{\beta(r)-1} dr \Big) \Big] \langle y_0, \psi_j \rangle$$

$$+ h\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr$$
$$= \langle f,\psi_j\rangle. \tag{4.6}$$

This implies that the following equality is satisfied

$$\langle y_0,\psi_j\rangle = \frac{\langle f,\psi_j\rangle - h\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}. \tag{4.7}$$

Combining (4.4) and (4.7), we derive that

$$\begin{aligned}\langle y_h(.,t),\psi_j\rangle &= \frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\\ &+\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\\ &-\frac{-h\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}.\end{aligned} \tag{4.8}$$

By the theory of Fourier series, the mild solution is given by

$$\begin{aligned}y_h(x,t) &= \sum_{j=1}^\infty \frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\psi_j(x)\\ &-h\sum_{j=1}^\infty \frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\psi_j(x)\\ &+\sum_{j=1}^\infty\Big[\int_0^t r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^t z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\Big]\psi_j(x)\\ &= \mathbb{K}_1(x,t)+\mathbb{K}_2(x,t)+\mathbb{K}_3(x,t).\end{aligned} \tag{4.9}$$

Let us consider the first term $\mathbb{K}_1$. By Parseval's equality, using (3.30), (3.35) and noting that $1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big) > 1$, we derive

$$\begin{aligned}\Big\|\mathbb{K}_1\Big\|^2_{\mathbb{Z}^s(\Omega)} &= \sum_j \lambda_j^{2s}\left(\frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\right)^2\langle y_0,\psi_j\rangle^2\\ &\le \sum_j \lambda_j^{2s}\exp\Big(-2\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle^2\\ &\le C_1^2\Big(T^{b\varepsilon}+1\Big)^2 t^{-2b\varepsilon}\sum_j \lambda_j^{2s-2\varepsilon}\langle y_0,\psi_j\rangle^2\end{aligned}$$

$$= C_1^2 \left(T^{b\varepsilon} + 1\right)^2 t^{-2b\varepsilon} \left\|y_0\right\|^2_{\mathbb{Z}^{s-\varepsilon}(\Omega)}, \tag{4.10}$$

where we have used (3.31). Therefore, we obtain that the following estimate

$$\left\|\mathbb{K}_1\right\|_{\mathbb{Z}^s(\Omega)} \le C_1 \left(T^{b\varepsilon} + 1\right) t^{-b\varepsilon} \left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}. \tag{4.11}$$

Proof of i). Suppose $F \in L^\infty(0, T; \mathbb{Z}^s(\Omega))$.
We deal with the second term $\mathbb{K}_2$. We first obtain

$$\begin{aligned}
&\left\|\mathbb{K}_2\right\|^2_{\mathbb{Z}^s(\Omega)} \\
&= h^2 \sum_j \lambda_j^{2s} \left(\frac{\exp\left(-\lambda_j \int_0^t r^{\beta(r)-1} dr\right) \int_0^T r^{\beta(r)-1} \exp\left(-\lambda_j \int_r^T z^{\beta(z)-1} dz\right) \langle F(.,r), \psi_j\rangle dr}{1 + h \exp\left(-\lambda_j \int_0^T r^{\beta(r)-1} dr\right)} \right)^2 \\
&\le h^2 \sum_j \lambda_j^{2s} \exp\left(-2\lambda_j \int_0^t r^{\beta(r)-1} dr\right) \left(\int_0^T r^{\beta(r)-1} \exp\left(-\lambda_j \int_r^T z^{\beta(z)-1} dz\right) \langle F(.,r), \psi_j\rangle dr\right)^2 \\
&\le h^2 C_1^2 t^{-2b\varepsilon} \sum_j \lambda_j^{2s-2\varepsilon} \left(\int_0^T r^{\beta(r)-1} \exp\left(-\lambda_j \int_r^T z^{\beta(z)-1} dz\right) \langle F(.,r), \psi_j\rangle dr\right)^2.
\end{aligned} \tag{4.12}$$

From (3.43), we can easily to verify that

$$\begin{aligned}
\left\|J_2(.,T)\right\|^2_{\mathbb{Z}^{s-\varepsilon}(\Omega)} &= \sum_j \lambda_j^{2s-2\varepsilon} \left(\int_0^T r^{\beta(r)-1} \exp\left(-\lambda_j \int_r^T z^{\beta(z)-1} dz\right) \langle F(.,r), \psi_j\rangle dr\right)^2 \\
&\le \left(\frac{T^b + T^m}{m} \right)^2 \left\|F\right\|^2_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}.
\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we derive the following bound

$$\left\|\mathbb{K}_2\right\|_{\mathbb{Z}^s(\Omega)} \le C_1 h \left(\frac{T^b + T^m}{m} \right) t^{-b\varepsilon} \left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}. \tag{4.14}$$

Let us now treat the third term $\mathbb{K}_3$. In view of (3.43), we infer that

$$\left\|\mathbb{K}_3(.,t)\right\|_{\mathbb{Z}^s(\Omega)} = \left\|J_2(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \le \frac{t^b + t^m}{m} \left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{4.15}$$

Combining (4.9), (4.11), (4.14) and (4.15) yields

$$\begin{aligned}
\left\|y_h(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \le \sum_{j=1}^3 \left\|\mathbb{K}_j(.,t)\right\|_{\mathbb{Z}^s(\Omega)} &\le C_1 \left(T^{b\varepsilon} + 1\right) t^{-b\varepsilon} \left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} \\
&+ C_1 h \left(\frac{T^b + T^m}{m} \right) t^{-b\varepsilon} \left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))} + \frac{t^b + t^m}{m} \left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}.
\end{aligned} \tag{4.16}$$

Proof of ii). Suppose that $F \in L^\infty(0, T; \mathbb{Z}^{s-\delta}(\Omega))$.
From (3.70), we obtain the following bound

$$\left\|J_2(.,T)\right\|_{\mathbb{Z}^s(\Omega)} \le \overline{C}_{10} T^{m-b\delta} \left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}, \quad t > 0, \tag{4.17}$$

where we note that $t^{m-b\delta} \leq T^{m-b\delta}$. This estimate together with (4.12) yield

$$\left\|\mathbb{K}_2\right\|_{\mathbb{Z}^s(\Omega)} \leq C_1\overline{C}_{10}T^{m-b\delta}ht^{-b\varepsilon}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}. \tag{4.18}$$

In view of (3.70), we infer that

$$\left\|\mathbb{K}_3(.,t)\right\|_{\mathbb{Z}^s(\Omega)} = \left\|J_2(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \leq \overline{C}_{10}t^{m-b\delta}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}. \tag{4.19}$$

Combining (4.9), (4.11), (4.18) and (4.19), we deduce that

$$\begin{aligned}\left\|y_h(.,t)\right\|_{\mathbb{Z}^s(\Omega)} &\leq \sum_{j=1}^{3}\left\|\mathbb{K}_j(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \leq C_1\left(T^{b\varepsilon}+1\right)t^{-b\varepsilon}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}\\ &\quad + C_1\overline{C}_{10}T^{m-b\delta}ht^{-b\varepsilon}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))} + \overline{C}_{10}t^{m-b\delta}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\delta}(\Omega))}.\end{aligned} \tag{4.20}$$

The proof is completed. □

The following theorem shows the convergence of the mild solution to (3.1)-(1.3) when $h \to 0^-$.

Theorem 4.2. *i) Let $y_0 \in \mathbb{Z}^{s-\varepsilon}(\Omega)$ and $F \in L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))$ for any $0 < \varepsilon < \frac{1}{b}$. Then, $h \in (0,1)$ and $k \in (1,2)$ we get*

$$\left\|y_h(.,t) - y(.,t)\right\|_{L^p(0,T;\mathbb{Z}^s(\Omega))} \leq C\Big(h^{\frac{2-k}{2}}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} + h\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}\Big), \tag{4.21}$$

where C depends on T, b, ε, p.
ii) Let $y_0 \in \mathbb{Z}^{s-\varepsilon}(\Omega)$ for $\varepsilon > 0$ and $F \in L^\infty(0,T;\mathbb{Z}^s(\Omega))$ for any $\varepsilon > 0$. Let us assume that $2m > b$. Then we get

$$\left\|y_h(.,t) - y(.,t)\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))} \leq C\Big(h^{\frac{2-k}{2}}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} + h\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}\Big),\ \ 0 < h < 1, \tag{4.22}$$

where the hidden constant depends on T, b, ε, m.

Proof. First, we focus on the formulas of solutions (4.9) and (3.28). Taking the difference, we get

$$\begin{aligned}&y_h(x,t) - y(x,t)\\ &= \sum_{j=1}^{\infty}\frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\psi_j(x) - \sum_j \exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle\psi_j(x)\\ &- h\sum_{j=1}^{\infty}\frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\psi_j(x)\\ &= \mathbb{K}_0(x,t) + \mathbb{K}_2(x,t).\end{aligned} \tag{4.23}$$

By a simple transformation, it is easy to verify that

$$\mathbb{K}_0(x,t) = h\sum_{j=1}^{\infty}\frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\langle y_0,\psi_j\rangle\psi_j(x). \tag{4.24}$$

We need to consider the term

$$\mathbb{K}_4(t) = \frac{h\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}$$
$$= \frac{h\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)}{\Big(1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\Big)^{\frac{\int_0^t r^{\beta(r)-1}dr}{\int_0^T r^{\beta(r)-1}dr}}\Big(1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\Big)^{\frac{\int_t^T r^{\beta(r)-1}dr}{\int_0^T r^{\beta(r)-1}dr}}}. \tag{4.25}$$

We consider the denominator component of the above fraction. In view of the inequality

$$1+z > z^{\frac{k}{2}}, \quad 1<k<2, \quad z>0,$$

we get the following inequality

$$\Big(1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\Big)^{\frac{\int_0^t r^{\beta(r)-1}dr}{\int_0^T r^{\beta(r)-1}dr}} > h^{\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}\exp\Big(-\frac{\lambda_j k}{2}\int_0^t r^{\beta(r)-1}dr\Big)$$

and

$$\Big(1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\Big)^{\frac{\int_t^T r^{\beta(r)-1}dr}{\int_0^T r^{\beta(r)-1}dr}} > 1.$$

Since $k<2$, the latter three observations infer that

$$\mathbb{K}_4(t) \le h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}\exp\Big(\lambda_j\frac{k-2}{2}\int_0^t r^{\beta(r)-1}dr\Big) \le h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}. \tag{4.26}$$

From this result and (4.26), we derive that

$$\Big\|\mathbb{K}_0(.,t)\Big\|^2_{\mathbb{Z}^s(\Omega)} = \sum_j \lambda_j^{2s}\Big|\mathbb{K}_4(t)\Big|^2\exp\Big(-2\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle^2$$
$$\le h^{2-\frac{k\int_0^t r^{\beta(r)-1}dr}{\int_0^T r^{\beta(r)-1}dr}}\sum_j \lambda_j^{2s}\exp\Big(-2\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)\langle y_0,\psi_j\rangle^2. \tag{4.27}$$

Using (3.35), we obtain

$$\Big\|\mathbb{K}_0(.,t)\Big\|_{\mathbb{Z}^s(\Omega)} \le h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}\Big\|J_1(.,T)\Big\|_{\mathbb{Z}^s(\Omega)}$$
$$\le C_1\Big(T^{b\varepsilon}+1\Big)h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}T^{-b\varepsilon}\Big\|y_0\Big\|_{\mathbb{Z}^{s-\varepsilon}}$$
$$= C_1\Big(T^{-b\varepsilon}+1\Big)h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}}\Big\|y_0\Big\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}. \tag{4.28}$$

It is obvious to check that the following estimate holds

$$1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr} > \frac{2-k}{2}. \tag{4.29}$$

Since $0 < h < 1$, we get

$$h^{1-\frac{k\int_0^t r^{\beta(r)-1}dr}{2\int_0^T r^{\beta(r)-1}dr}} \le h^{\frac{2-k}{2}}. \tag{4.30}$$

Combining (4.28) and (4.30), we deduce that

$$\left\|\mathbb{K}_0(.,t)\right\|_{\mathbb{Z}^s(\Omega)} \le C_1\left(T^{-b\varepsilon}+1\right)h^{\frac{2-k}{2}}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} \tag{4.31}$$

Next, we consider $\|\mathbb{K}_2(.,t)\|_{\mathbb{Z}^s(\Omega)}$ in two cases corresponding to part i) and part ii). We use the results in the proof of Theorem (4.1).
Case 1. Proof of (4.21).
Since F is in the space $L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))$, we follows from (4.14) that

$$\left\|\mathbb{K}_2\right\|_{\mathbb{Z}^s(\Omega)} \le C_1 h\left(\frac{T^b+T^m}{m}\right)t^{-b\varepsilon}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}. \tag{4.32}$$

Combining (4.23), (4.31) and (4.32), we find that

$$\begin{aligned}\left\|y_h(.,t)-y(.,t)\right\|_{\mathbb{Z}^s(\Omega)} &\le \left\|\mathbb{K}_0\right\|_{\mathbb{Z}^s(\Omega)}+\left\|\mathbb{K}_2\right\|_{\mathbb{Z}^s(\Omega)}\\ &\le C_1\left(T^{-b\varepsilon}+1\right)h^{\frac{2-k}{2}}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}+C_1 h\left(\frac{T^b+T^m}{m}\right)t^{-b\varepsilon}\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}. \end{aligned} \tag{4.33}$$

Let us choose ε such that $0 < \varepsilon < \frac{1}{b}$. Since $1 < p < \frac{1}{b\varepsilon}$, we know that the proper integral $\int_0^T t^{-b\varepsilon p}dt$ is convergent. By a simple computation, we deduce that

$$\left\|y_h(.,t)-y(.,t)\right\|_{L^p(0,T;\mathbb{Z}^s(\Omega))} \le C\Big(h^{\frac{2-k}{2}}\left\|y_0\right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)}+h\left\|F\right\|_{L^\infty(0,T;\mathbb{Z}^{s-\varepsilon}(\Omega))}\Big), \tag{4.34}$$

where C depends on T, b, ε, p.
Case 2. Proof of (4.22).
From the definition of $\mathbb{K}_2$ as in (4.23), we derive that

$$\begin{aligned}&\left\|\mathbb{K}_2\right\|^2_{\mathbb{Z}^s(\Omega)}\\ &= h^2\sum_j \lambda_j^{2s}\left(\frac{\exp\Big(-\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr}{1+h\exp\Big(-\lambda_j\int_0^T r^{\beta(r)-1}dr\Big)}\right)^2\\ &\le h^2\sum_j \lambda_j^{2s}\exp\Big(-2\lambda_j\int_0^t r^{\beta(r)-1}dr\Big)\left(\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\right)^2\\ &\le h^2\sum_j \lambda_j^{2s}\left(\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\right)^2. \end{aligned} \tag{4.35}$$

We have the following observation

$$\sum_j \lambda_j^{2s}\left(\int_0^T r^{\beta(r)-1}\exp\Big(-\lambda_j\int_r^T z^{\beta(z)-1}dz\Big)\langle F(.,r),\psi_j\rangle dr\right)^2$$

$$= \left\| J_2(.,T) \right\|^2_{\mathbb{Z}^s(\Omega)} \leq \frac{T^{2m}}{m^2} \left\| F \right\|^2_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \tag{4.36}$$

Combining (4.23), (4.31) and (4.36), we find that

$$\begin{aligned} &\left\| y_h(.,t) - y(.,t) \right\|_{\mathbb{Z}^s(\Omega)} \leq \left\| \mathbb{K}_0 \right\|_{\mathbb{Z}^s(\Omega)} + \left\| \mathbb{K}_2 \right\|_{\mathbb{Z}^s(\Omega)} \\ &\leq C_1 \left(T^{-b\varepsilon} + 1 \right) h^{\frac{2-k}{2}} \left\| y_0 \right\|_{\mathbb{Z}^{s-\varepsilon}(\Omega)} + h \frac{T^m}{m} \left\| F \right\|_{L^\infty(0,T;\mathbb{Z}^s(\Omega))}. \end{aligned} \tag{4.37}$$

From the right-hand side of the above estimate, we deduce the desired result (4.22). The proof of our theorem is completed. □

5. Conclusions

This work considers a time-fractional parabolic equation with conformable variable derivative. We derive the well-posedness for mild solutions in Hilbert spaces for linear initial problem and linear nonlocal problem. We also shows the convergence of non-local solutions to local solutions. The techniques obtained in this study can be further extended to complicated nonlinear problem.

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Conflict of interest

The authors declare no conflict of interest.

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