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Research article

Complete convergence for weighted sums of negatively dependent random variables under sub-linear expectations

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Abstract: In this article, we study complete convergence and complete moment convergence for negatively dependent random variables under sub-linear expectations. The results obtained in sub-linear expectation spaces extend the corresponding ones in probability space.

Keywords: negatively dependent random variables; complete convergence; complete moment convergence; sub-linear expectations
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

Peng [1, 2] firstly introduced the important concepts of the sub-linear expectations space to study the uncertainty in probability. Inspired by the important works of Peng [1, 2], many scholars try to investigate the results under sub-linear expectations space, extending the corresponding ones in classic probability space. Zhang [3–5] established Donsker's invariance principle, exponential inequalities and Rosenthal's inequality under sub-linear expectations. Wu [6] obtained precise asymptotics for complete integral convergence under sub-linear expectations. Under sub-linear expectations, Xu and Cheng [7] investigated how small the increments of G-Brownian motion are. For more limit theorems under sub-linear expectations, the interested readers could refer to Xu and Zhang [8, 9], Wu and Jiang [10], Zhang and Lin [11], Zhong and Wu [12], Hu and Yang [13], Chen [14], Chen and Wu [15], Zhang [16], Hu, Chen and Zhang [17], Gao and Xu [18], Kuczmaszewska [19], Xu and Cheng [7, 20–23] and references therein.

In classic probability space, Hsu and Robbins [24] introduced concept of complete convergence, Chow [25] investigated complete moment convergence for independent random variables, Zhang and Ding [26] proved the complete moment convergence of the partial sums of moving average processes under some proper assumptions, Meng et al. [27] established complete convergence and complete moment convergence for weighted sums of extended negatively dependent random variables. For references on complete moment convergence in linear expectation space, the interested reader could refer to Ko [28], Meng et al. [29], Hosseini and Nezakati [30] and references therein. Encouraged by the work of Meng et al. [27], since the fact that X is independent to Y under sub-linear expectations implies that X is negatively dependent to Y under sub-linear expectations, we try to study the complete convergence and complete moment convergence for weighted sums of identically distributed, negatively dependent random variables under sub-linear expectations, which extends the corresponding results in Meng et al. [27].

We organize the remainders of this paper as follows. We give necessary basic notions, concepts and relevant properties, and present necessary lemmas under sub-linear expectations in the next section. In Section 3, we give our main results, Theorems 3.1 and 3.2, the proofs of which are presented in Section 4.

2. Preliminaries

As in Xu and Cheng [22], we use similar notations as in the work by Peng [2], Chen [14], Zhang [5]. Suppose that $(\Omega, \mathcal{F})$ is a given measurable space. Assume that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $I_A \in \mathcal{H}$ (cf. Chen [14]), where $I(A)$ or I_A represent the indicator function of A throughout this paper, $A \in \mathcal{F}$, and $X_1, \cdots, X_n \in \mathcal{H}$ implies $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ represents the linear space of (local lipschitz) function φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ both depending on φ.

Definition 2.1. *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) *Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;*

(b) *Constant preserving: $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;*

(c) *Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$;*

(d) *Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $\infty - \infty$ or $-\infty + \infty$.*

Remark 2.1. *In* (c) *of Definition 2.1, positive homogeneity could be understood by Theorem 1.2.1 of Peng [2], which says that a sub-linear expectation could be represented as a supremum of linear expectations. In Theorem 3.1, $\mathbb{E}[X] = \mathbb{E}[-X] = 0$ could imply that $\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X]$ for all $\alpha \in \mathbb{R}$, but $\mathbb{E}[X] = \mathbb{E}[-X] = 0$ could not imply that $\mathbb{E}[\alpha X^\beta] = \alpha \mathbb{E}[X^\beta]$ for all $\alpha \in \mathbb{R}$ and $\beta \neq 1$. By Lemma 2.1, in order to justify $\mathbb{E}[X] = \mathbb{E}[-X] = 0$ in Theorem 3.1, we should have $\mathbb{E}[Z + X] = \mathbb{E}[Z - X]$, for all $Z \in \mathcal{H}$.*

A set function $V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called sub-additive if $V(A+B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this article, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, write $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\} = \mathbb{E}[I_A]$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [4]). $\mathbb{V}$ is a sub-additive capacity. Define

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

Suppose that $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is called to be negatively dependent to $\mathbf{X}$, if for each function $\psi_1 \in C_{l,Lip}(\mathbb{R}^m)$, $\psi_2 \in C_{l,Lip}(\mathbb{R}^n)$, we have $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \leq \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \geq 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \geq 0$, $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [4], Definition 1.5 of Zhang [5], Definition 2.5 in Chen [14]). $\{X_n\}_{n=1}^\infty$ is named a sequence of negatively dependent random variables, if X_{n+1} is negatively dependent to $(X_1, \cdots, X_n)$ for each $n \geq 1$.

Suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are named identically distributed if for every Borel-measurable function ψ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1, \psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)],$$

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^\infty$ is named to be identically distributed if for each $i \geq 1$, X_i and X_1 are identically distributed.

In this sequel we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^\infty \mathbb{E}(X_n)$, whenever $X \leq \sum_{n=1}^\infty X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Let C stand for a positive constant which may differ from place to place.

As discussed in Zhang [5], by the definition of negative dependence, if $X_1, X_2, \ldots, X_n$ are negatively dependent random variables and $f_1, f_2, \ldots, f_n$ are all non increasing (or non decreasing) functions, then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are still negatively dependent random variables.

We cite the following lemmas under sub-linear expectations.

Lemma 2.1. *(See Proposition 1.3.7 of Peng [2]) Under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, if $X, Y \in \mathcal{H}$, $\mathbb{E}[Y] = \mathbb{E}[-Y] = 0$, then $\mathbb{E}[X + \alpha Y] = \mathbb{E}[X]$, for any $\alpha \in \mathbb{R}$.*

Lemma 2.2. *(See Lemma 4.5 (iii) of Zhang [4]) If $\mathbb{E}$ is countably sub-additive under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, then for $X \in \mathcal{H}$,*

$$\mathbb{E}|X| \leq C_{\mathbb{V}}(|X|).$$

Lemma 2.3. *(See Theorem 2.1 and its proof of Zhang [5]) Assume that $p > 1$ and $\{X_n; n \geq 1\}$ is a sequence of negatively dependent random varables under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then for each $n \geq 1$, there exists a positive constant $C = C(p)$ depending on p such that for $1 < p \leq 2$,*

$$\mathbb{E}\left|\sum_{i=1}^n X_i\right|^p \leq C\left[\sum_{i=1}^n \mathbb{E}|X_i|^p + \left(\sum_{i=1}^n [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^p\right], \tag{2.1}$$

and for $p > 2$,

$$\mathbb{E}\left|\sum_{i=1}^n X_i\right|^p \leq C\left\{\sum_{i=1}^n \mathbb{E}|X_i|^p + \left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{p/2} + \left(\sum_{i=1}^n [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^p\right\}. \tag{2.2}$$

Proof. For reader's convenience, here we give the detailed proof. We first prove (2.1). Set $T_k = \max\{X_k, X_k + X_{k-1}, \ldots, X_k + \cdots + X_1\}$, $\breve{T}_n = \max\{|X_n|, |X_n + X_{n-1}|, \ldots, |X_n + \cdots + X_1|\}$. Since $T_k^+ + X_{k+1} + \ldots + X_n \le T_n$, $T_k^+ \le 2\breve{T}_n$. Substituting $x = X_k$ and $y = T_{k-1}^+$, $k = n, \ldots, 2$ to the following elementary inequality

$$|x+y|^p \le 2^{2-p}|x|^p + |y|^p + px|y|^{p-1}\mathrm{sgn}(y),\ 1 < p \le 2$$

results in

$$\begin{aligned}|T_n|^p &\le 2^{2-p}|X_n|^p + (T_{n-1}^+)^p + pX_n(T_{n-1}^+)^{p-1}\\ &\le 2^{2-p}|X_n|^p + |T_{n-1}|^p + pX_n(T_{n-1}^+)^{p-1}\\ &\le \ldots\\ &\le 2^{2-p}\sum_{i=1}^n |X_i|^p + p\sum_{i=2}^n X_i(T_{i-1}^+)^{p-1},\end{aligned}$$

which by the definition of negative dependence and Hölder inequality under sub-linear expectations (see Proposition 1.4.2 of Peng [2]), implies that

$$\begin{aligned}&\mathbb{E}|T_n|^p \le 2^{2-p}\mathbb{E}\left[\textstyle\sum_{i=1}^n |X_n|^p\right] + p\textstyle\sum_{i=2}^n \mathbb{E}\left[X_i(T_{i-1}^+)^{p-1}\right]\\ &\le 2^{2-p}\mathbb{E}\left[\textstyle\sum_{i=1}^n |X_n|^p\right] + p2^{p-1}\textstyle\sum_{i=2}^n(\mathbb{E}[X_i])^+(\mathbb{E}[\breve{T}_n^p])^{1-1/p}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\,|\max\{-X_n, -X_n - X_{n-1}, \ldots, -X_n - \cdots - X_1\}|^p\\ &\le 2^{2-p}\mathbb{E}\left[\textstyle\sum_{i=1}^n |X_n|^p\right] + p2^{p-1}\textstyle\sum_{i=2}^n(-\mathbb{E}[-X_i])^-(\mathbb{E}[\breve{T}_n^p])^{1-1/p}.\end{aligned}$$

Therefore

$$\mathbb{E}|\breve{T}_n^p| \le 2^{3-p}\mathbb{E}\left[\sum_{i=1}^n |X_n|^p\right] + p2^p\sum_{i=1}^n\left[(\mathbb{E}[X_i])^+ + (-\mathbb{E}[-X_i])^-\right](\mathbb{E}[\breve{T}_n^p])^{1-1/p}$$

which implies that (2.1) holds.

Next, by (2.4) of Zhang [5] and its proof, we see that for $p > 2$

$$\mathbb{E}\left[\max_{1\le k\le n}|S_k|^p\right] \le C_p\left\{\sum_{i=1}^n \mathbb{E}\left[|X_n|^p\right] + \left(\sum_{i=1}^n \mathbb{E}[|X_i|^2]\right)^{p/2} + \left(\sum_{i=1}^n[(-\mathbb{E}[-X_i])^- + (\mathbb{E}[X_i])^+]\right)^p\right\}, \tag{2.3}$$

which implies that (2.2) holds. □

By Lemma 2.3 and the similar argument as in Theorem 2.3.1 in Stout [31], we could obtain the following lemma.

Lemma 2.4. *Assume that $q > 1$ and $\{X_n; n \ge 1\}$ is a sequence of negatively dependent random varables under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then for each $n \ge 1$, there exists a positive constant $C = C(q)$ depending only on q such that* $1 < q \le 2$,

$$\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^j X_i\right|^q\right) \le C(\log n)^q\left\{\sum_{i=1}^n \mathbb{E}\,|X_i|^q + \left(\sum_{i=1}^n[|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q\right\}, \tag{2.4}$$

and for $q > 2$,

$$\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^q\right) \le C(\log n)^q \left\{\sum_{i=1}^{n}\mathbb{E}|X_i|^q + \left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + \left(\sum_{i=1}^{n}[|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q\right\}. \tag{2.5}$$

Proof. For reader's convenience, here we also give detailed proof. We only prove (2.4), since (2.5) is obvious from (2.3). We first prove (2.4) for $n = 2^k$, k being an any positive integer. To avoid confusing the main idea, we just give the proof for $k = 6$. Let $X_{r,s} = \sum_{i=r+1}^{s} X_i$ for $0 \le r < s \le 2^6$. We consider the following collections of $X_{r,s}$:

$$\begin{array}{ll}
\{X_{0,64}\} & \\
\{X_{0,32}, X_{32,64}\} & \\
\{X_{0,16}, X_{16,32}, X_{32,48}, X_{48,64}\} & \\
\{X_{0,8}, \dots, & X_{56,64}\} \\
\{X_{0,4}, \dots, & X_{60,64}\} \\
\{X_{0,2}, \dots, & X_{62,64}\} \\
\{X_{0,1}, \dots, & X_{63,64}\}.
\end{array}$$

There are $k + 1 = 7$ collections. We choose $1 \le i \le 2^6$, and expand S_i, by using the terms of this expansion from the collections above and using the minimal possible number of terms in the expansion. Clearly at most one term is needed from each collections. As an example,

$$X_{0,62} = X_{0,32} + X_{32,48} + X_{48,56} + X_{56,60} + X_{60,62}.$$

Hence each expansion has at most $k + 1 = 7$ terms in it. Denote the expansion of S_i by

$$S_i = \sum_{j=1}^{h} X_{i_{j-1},i_j}, \ (h \le 7).$$

It follows from Hölder inequality that

$$|S_i|^q \le 7^{q-1}\sum_{j=1}^{h}(|X_{i_{j-1},i_j}|)^q.$$

Now

$$\begin{aligned}
\sum_{j=1}^{h}(|X_{i_{j-1},i_j}|)^q \quad \le \quad & |X_{0,64}|^q + (|X_{0,32}|^q + |X_{32,64}|^q) + (|X_{0,16}|^q + |X_{16,32}|^q + |X_{32,48}|^q + |X_{48,64}|^q) \\
& + \cdots + (|X_{0,1}|^q + |X_{1,2}|^q + \cdots + |X_{63,64}|^q).
\end{aligned}$$

Hence,

$$\begin{aligned}
\max_{1\le i\le 2^6}|S_i|^q \quad \le \quad & 7^{q-1}\left[|X_{0,64}|^q + (|X_{0,32}|^q + |X_{32,64}|^q) + (|X_{0,16}|^q + |X_{16,32}|^q + |X_{32,48}|^q + |X_{48,64}|^q)\right. \\
& \left. + \cdots + (|X_{0,1}|^q + |X_{1,2}|^q + \cdots + |X_{63,64}|^q)\right].
\end{aligned}$$

There are $k+1=7$ parenthetical expressions inside square brackets. By the C_r inequality, we see that

$$\sum_{i=1}^{m} |\xi_i|^q \leq \left(\sum_{i=1}^{m} |\xi_i|\right)^q, \ \forall \xi_i \in \mathbb{R}, m \geq 1,$$

which implies

$$\left(\sum_{i=1}^{32} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q + \left(\sum_{i=33}^{64} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q \leq \left(\sum_{i=1}^{2^6} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q,$$

$$\left(\sum_{i=1}^{16} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q + \cdots + \left(\sum_{i=49}^{64} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q \leq \left(\sum_{i=1}^{2^6} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q,$$

$$\cdots$$

$$[|\mathbb{E}(-X_1)| + |\mathbb{E}(X_1)|]^q + \cdots + [|\mathbb{E}(-X_{64})| + |\mathbb{E}(X_{64})|]^q \leq \left(\sum_{i=1}^{2^6} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q.$$

By (2.1) and the above discussion,

$$\mathbb{E}\left[\max_{1\leq i\leq 2^6} |S_i|^q\right] \leq 7^{q-1}\cdot 7C_q\left[\sum_{i=1}^{2^6} \mathbb{E}|X_i|^q + \left(\sum_{i=1}^{2^6} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q\right].$$

Using an appropriate notion, the above discussion extended to any $k \geq 1$ implies

$$\mathbb{E}\left[\max_{1\leq i\leq 2^k} |S_i|^q\right] \leq (k+1)^q C_q\left[\sum_{i=1}^{2^k} \mathbb{E}|X_i|^q + \left(\sum_{i=1}^{2^k} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q\right]. \tag{2.6}$$

Given an n such that $n \neq 2^k$ for any $k \geq 1$, choose k satisfying $2^{k-1} < n < 2^k$ and redefine $X_i = 0$ if $n < i \leq 2^k$. By (2.6), we see that

$$\mathbb{E}\left[\max_{1\leq i\leq n} |S_i|^q\right] \leq (k+1)^q C_q\left[\sum_{i=1}^{n} \mathbb{E}|X_i|^q + \left(\sum_{i=1}^{n} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^q\right].$$

Since $2^{k-1} < n$ implies $(k+1)^q \leq [\log(4n)/\log 2]^q$, (2.4) follows. □

3. Main results

Our main results are the following.

Theorem 3.1. *Suppose $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $\{X_n; n \geq 1\}$ is a sequence of negatively dependent random variables, identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that $\mathbb{E}(X) = \mathbb{E}(-X) = 0$ while $p > 1$. Suppose that $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers being all nonnegative or all non-positive such that*

$$\sum_{i=1}^{n} |a_{ni}|^p = O(n^{\delta}) \text{ for } 0 < \delta < 1. \tag{3.1}$$

Let $C_{\mathbb{V}}(|X|^p) < \infty$. Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right\} < \infty. \tag{3.2}$$

Theorem 3.2. *Suppose $p > 1$, $\alpha \ge \frac{1}{2}$, $\alpha p > 1$ and $\{X_n; n \ge 1\}$ is a sequence of negatively dependent random variables, identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that $\mathbb{E}(X) = \mathbb{E}(-X) = 0$. Suppose that $\{a_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of real numbers being all nonnegative or all non-positive such that (3.1) holds. Let $C_{\mathbb{V}}(|X|^p) < \infty$. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} C_{\mathbb{V}}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty. \tag{3.3}$$

Remark 3.1. *Under the assumptions of Theorem 3.2, we see that for all $\varepsilon > 0$,*

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} C_{\mathbb{V}}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\varepsilon n^{\alpha}} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t\right) \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{\varepsilon n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t\right) \mathrm{d}t \\
&\ge C \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > 2\varepsilon n^{\alpha}\right).
\end{aligned} \tag{3.4}$$

By (3.4), we can conclude that the complete moment convergence implies the complete convergence.

4. Proof of major results

4.1. Proof of Theorem 3.1

Proof. For all $1 \le i \le n$, $n \ge 1$, write

$$\begin{aligned}
Y_{ni} &= -n^{\alpha} I\left(a_{ni}X_i < -n^{\alpha}\right) + a_{ni}X_i I\left(|a_{ni}X_i| \le n^{\alpha}\right) + n^{\alpha} I\left(a_{ni}X_i > n^{\alpha}\right), \\
T_{nj} &= \sum_{i=1}^{j} \left(Y_{ni} - \mathbb{E}Y_{ni}\right), \qquad j = 1, 2, \ldots, n.
\end{aligned}$$

We easily observe that for all $\varepsilon > 0$,

$$\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right\} \subset \left\{\max_{1\le j\le n} |a_{nj}X_j| > n^{\alpha}\right\} \bigcup \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon n^{\alpha}\right\}, \tag{4.1}$$

which results in

$$
\begin{aligned}
&\mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right\}\\
&\le \mathbb{V}\left(\max_{1\le j\le n}|a_{nj}X_j| > n^{\alpha}\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon n^{\alpha}\right) \qquad (4.2)\\
&\le \sum_{j=1}^{n}\mathbb{V}\left(|a_{nj}X_j| > n^{\alpha}\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|T_{nj}\right| > \varepsilon n^{\alpha} - \max_{1\le j\le n}\left|\sum_{i=1}^{j}\mathbb{E}Y_{ni}\right|\right).
\end{aligned}
$$

Firstly, we will establish that

$$
n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j}\mathbb{E}Y_{ni}\right| \to 0,\ \text{as } n\to\infty. \qquad (4.3)
$$

We study the following three cases.

(i) If $\frac{1}{2} < \alpha \le 1$, then $p > 1$. By $\mathbb{E}X = \mathbb{E}(-X) = 0$, $|\mathbb{E}(X-Y)| \le \mathbb{E}|X-Y|$, $C_{\mathbb{V}}(|X|^p) < \infty$, Lemmas 2.1 and 2.2, we can see that

$$
\begin{aligned}
n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j}\mathbb{E}Y_{ni}\right| &\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}Y_{ni}|\\
&\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}[Y_{ni}-a_{ni}X_i]|\\
&\le n^{-\alpha}\sum_{i=1}^{n}\mathbb{E}\left|Y_{ni}-a_{ni}X_i\right| \le C\sum_{i=1}^{n}\frac{\mathbb{E}\left|a_{ni}X\right|^p}{n^{\alpha p}} \qquad (4.4)\\
&\le Cn^{-\alpha p}\sum_{i=1}^{n}|a_{ni}|^p\mathbb{E}|X|^p\\
&\le Cn^{\delta-\alpha p}C_{\mathbb{V}}\left(|X|^p\right) \to 0,\ \text{as } n\to\infty.
\end{aligned}
$$

(ii) If $\alpha > 1$, $p < 1$, then by $C_{\mathbb{V}}\left(|X|^p\right) < \infty$, and Lemma 2.2, we see that

$$
\begin{aligned}
n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j}\mathbb{E}Y_{ni}\right| &\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}Y_{ni}|\\
&\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}a_{ni}X_iI\left(|a_{ni}X_i| \le n^{\alpha}\right)| + C\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X_i| > n^{\alpha}\right)\\
&\le n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}a_{ni}XI\left(|a_{ni}X| \le n^{\alpha}\right)| + C\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X| > n^{\alpha}\right)\\
&\le C\sum_{i=1}^{n}\frac{\mathbb{E}\left(|a_{ni}X|^p\right)}{n^{\alpha p}} + \sum_{i=1}^{n}\frac{\mathbb{E}\left(|a_{ni}X|^p\right)}{n^{\alpha p}} \qquad (4.5)\\
&\le Cn^{-\alpha p}\sum_{i=1}^{n}|a_{ni}|^p\mathbb{E}|X|^p
\end{aligned}
$$

$$\leq \quad Cn^{\delta-\alpha p}C_{\mathbb{V}}\left(|X|^{p}\right) \to 0, \text{ as } n \to \infty.$$

(iii) If $\alpha > 1$, $p \geq 1$, then by $\mathbb{E}|X| \leq (\mathbb{E}|X|^p)^{1/p} \leq (C_{\mathbb{V}}(|X|^p))^{1/p} < \infty$, Markov inequality under sub-linear expectations, Hölder inequality, we see that

$$\begin{aligned} n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\mathbb{E}Y_{ni}\right| &\leq n^{-\alpha}\sum_{i=1}^{n}|\mathbb{E}Y_{ni}| \\ &\leq n^{-\alpha}\sum_{i=1}^{n}\mathbb{E}\left|a_{ni}X_{i}I\left(|a_{ni}X_{i}|\leq n^{\alpha}\right)\right| + \sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X_{i}| > n^{\alpha}\right) \\ &\leq Cn^{-\alpha}\sum_{i=1}^{n}|a_{ni}| + Cn^{-\alpha}\sum_{i=1}^{n}|a_{ni}| \qquad (4.6)\\ &\leq Cn^{-\alpha}\left(\sum_{i=1}^{n}|a_{ni}|^{p}\right)^{1/p}n^{1-1/p} \\ &\leq Cn^{1-\alpha-(1-\delta)/p} \to 0, \text{ as } n\to\infty. \end{aligned}$$

Combining (4.4)–(4.6) results in (4.3) immediately. Hence, for n sufficiently large,

$$\mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_{i}\right| > \varepsilon n^{\alpha}\right) \leq \sum_{j=1}^{n}\mathbb{V}\left(|a_{nj}X_{j}| > n^{\alpha}\right) + \mathbb{V}\left(\max_{1\leq j\leq n}|T_{nj}| > \frac{\varepsilon n^{\alpha}}{2}\right). \qquad (4.7)$$

To prove (3.2), we only need to establish that

$$I := \sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X_{i}| > n^{\alpha}\right) < \infty \qquad (4.8)$$

and

$$II := \sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\max_{1\leq j\leq n}|T_{nj}| > \frac{\varepsilon n^{\alpha}}{2}\right) < \infty. \qquad (4.9)$$

For I, by Markov inequality under sub-linear expectations, and Lemma 2.2, we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X| > n^{\alpha}\right) \\ &\leq C\sum_{n=1}^{\infty}n^{-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^{p} \qquad (4.10)\\ &\leq C\sum_{n=1}^{\infty}n^{\delta-2}C_{\mathbb{V}}\left(|X|^{p}\right) < \infty. \end{aligned}$$

As pointed before Lemma 2.2, we see that $\{Y_{ni}-\mathbb{E}Y_{ni}; 1\leq i\leq n, n\geq 1\}$ is also a sequence of negatively dependent random variables. By Lemma 2.4, Markov inequality under sub-linear expectations, and the C_r inequality, we conclude that for $q > 2$,

$$II \leq C\sum_{n=1}^{\infty}n^{\alpha p-2}n^{-\alpha q}\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni}-\mathbb{E}Y_{ni})\right|^{q}\right)$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\mathbb{E}|Y_{ni}-\mathbb{E}Y_{ni}|^q+\left(\sum_{i=1}^{n}\mathbb{E}|Y_{ni}-\mathbb{E}Y_{ni}|^2\right)^{q/2}\right. \qquad (4.11)\\
&\quad \left.+\left(\sum_{i=1}^{n}[|\mathbb{E}(-Y_{ni})|+|\mathbb{E}(Y_{ni})|]\right)^q\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\sum_{i=1}^{n}\mathbb{E}|Y_{ni}|^q+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\mathbb{E}|Y_{ni}|^2\right)^{q/2}\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}[|\mathbb{E}(-Y_{ni})|+|\mathbb{E}(Y_{ni})|]\right)^q\\
&=: II_1+II_2+II_3.
\end{aligned}
$$

Taking $q > \max\{2, p\}$, by the C_r inequality, Markov inequality under sub-linear expectations, and Lemma 2.2, we have

$$
\begin{aligned}
II_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\sum_{i=1}^{n}[\mathbb{E}|a_{ni}X_i|^q I(|a_{ni}X_i|\le n^{\alpha})+n^{\alpha q}\mathbb{V}(|a_{ni}X_i|>n^{\alpha})]\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^q I(|a_{ni}X|\le n^{\alpha})\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>n^{\alpha})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^q I(|a_{ni}X|\le n^{\alpha})}{n^{\alpha q}}\\
&\quad +C\sum_{n=1}^{\infty} n^{-2}(\log n)^q\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^p \qquad (4.12)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha p}}+C\sum_{n=1}^{\infty} n^{-2}(\log n)^q\sum_{i=1}^{n}|a_{ni}|^p C_{\mathbb{V}}(|X|^p)\\
&\le C\sum_{n=1}^{\infty} n^{-2}(\log n)^q\sum_{i=1}^{n}|a_{ni}|^p C_{\mathbb{V}}(|X|^p)+C\sum_{n=1}^{\infty} n^{\delta-2}(\log n)^q\\
&\le C\sum_{n=1}^{\infty} n^{\delta-2}(\log n)^q<\infty.
\end{aligned}
$$

For II_2, we study the following cases.

(i) If $p \ge 2$, observe that $\sum_{i=1}^{n} a_{ni}^2 \le \left(\sum_{i=1}^{n}|a_{ni}|^p\right)^{2/p} n^{1-2/p} \le n^{1-2(1-\delta)/p}$. Taking $q > \max\left\{2, \frac{2p(\alpha p-1)}{2\alpha p-p+2(1-\delta)}\right\}$, by the C_r inequality, $\mathbb{E}X^2 \le (\mathbb{E}(|X|^p))^{1/p} \le (C_{\mathbb{V}}(|X|^p))^{1/p} < \infty$, we see that

$$
\begin{aligned}
II_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\left[\mathbb{E}|a_{ni}X_i|^2 I(|a_{ni}X_i|\le n^{\alpha})+n^{2\alpha}\mathbb{V}(|a_{ni}X_i|>n^{\alpha})\right]\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I(|a_{ni}X|\le n^{\alpha})\right)^{q/2}
\end{aligned}
$$

$$+C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X|>n^{\alpha}\right)\right)^{q/2}$$
$$\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}a_{ni}^2\right)^{q/2}+C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}a_{ni}^2\right)^{q/2}$$
$$\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left(n^{1-2(1-\delta)/p}\right)^{q/2}$$
$$\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q+\frac{q}{2}-\frac{(1-\delta)q}{p}}(\log n)^q<\infty. \tag{4.13}$$

(ii) If $p<2$, we take $q>2(\alpha p-1)/(\alpha p-\delta)$. By the C_r inequality, Markov inequality under sub-linear expectations, and Lemma 2.2, we see that

$$\begin{aligned}
II_2 &\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\left[\mathbb{E}|a_{ni}X_i|^2I(|a_{ni}X_i|\leq n^{\alpha})+n^{2\alpha}\mathbb{V}\left(|a_{ni}X_i|>n^{\alpha}\right)\right]\right)^{q/2}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left(\sum_{i=1}^{n}\left[\mathbb{E}|a_{ni}X|^2I(|a_{ni}X|\leq n^{\alpha})+n^{2\alpha}\mathbb{V}\left(|a_{ni}X|>n^{\alpha}\right)\right]\right)^{q/2}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^2I(|a_{ni}X|\leq n^{\alpha})}{n^{2\alpha}}\right)^{q/2}+C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(n^{-\alpha p}\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^p\right)^{q/2}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha p}}\right)^{q/2}+C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(n^{-\alpha p}\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^p\right)^{q/2}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)^q\left(n^{-\alpha p}\sum_{i=1}^{n}|a_{ni}|^p\right)^{q/2}\left(C_{\mathbb{V}}\left(|X|^p\right)\right)^{q/2}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2+q(\delta-\alpha p)/2}(\log n)^q<\infty.
\end{aligned}\tag{4.14}$$

For II_3, we study the following cases.

(i) If $\frac{1}{2}<\alpha\leq 1$, then $p>1$. Taking $q>\frac{\alpha p-1}{\alpha p-\delta}$, by $\mathbb{E}(X)=\mathbb{E}(-X)=0$, Lemmas 2.1 and 2.2, we see that

$$\begin{aligned}
II_3 &\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left[\sum_{i=1}^{n}\left[|\mathbb{E}[-Y_{ni}+a_{ni}X_i]|+|\mathbb{E}[Y_{ni}-a_{ni}X_i]|\right]\right]^q\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left[\sum_{i=1}^{n}\left[\mathbb{E}[|-Y_{ni}+a_{ni}X_i|]+\mathbb{E}[|Y_{ni}-a_{ni}X_i|]\right]\right]^q\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q\left[\sum_{i=1}^{n}\left[\frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha(p-1)}}+\frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha(p-1)}}\right]\right]^q\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}(\log n)^q n^{-\alpha(p-1)q+\delta q}\left(C_{\mathbb{V}}\left(|X|^p\right)\right)^q
\end{aligned}$$

$$\leq \quad C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha pq+\delta q}(\log n)^q < \infty. \tag{4.15}$$

(ii) If $\alpha > 1$, $p < 1$, taking $q > \frac{\alpha p-1}{\alpha p-\delta}$, by Lemma 2.2, we obtain

$$\begin{aligned}
II_3 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q \left[\sum_{i=1}^{n} [\mathbb{E}|a_{ni}X|I\{|a_{ni}X| \leq n^\alpha\} + n^\alpha \mathbb{V}(|a_{ni}X| > n^\alpha)]\right]^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \left(\sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X|I\{|a_{ni}X| \leq n^\alpha\}}{n^\alpha}\right)^q \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \left(n^{-\alpha p}\sum_{i=1}^{n} \mathbb{E}|a_{ni}X|^p\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \left(\sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha p}}\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \left(n^{-\alpha p}\sum_{i=1}^{n} |a_{ni}|^p\right)^q (C_{\mathbb{V}}(|X|^p))^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2+q(\delta-\alpha p)}(\log n)^q < \infty.
\end{aligned}$$

(iii) If $\alpha > 1$, $p > 1$, then $\mathbb{E}|X| \leq (\mathbb{E}|X|^p)^{1/p} \leq (C_{\mathbb{V}}(|X|^p))^{1/p} < \infty$. We take $q > \frac{(\alpha p-1)p}{\alpha p-p+(1-\delta)}$. Hence by C_r inequality, Markov inequality under sub-linear expectations, and Hölder inequality, we see that

$$\begin{aligned}
II_3 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q \left[\sum_{i=1}^{n} [\mathbb{E}|a_{ni}X|I\{|a_{ni}X| \leq n^\alpha\} + n^\alpha \mathbb{V}(|a_{ni}X| > n^\alpha)]\right]^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q \left[\sum_{i=1}^{n} [|a_{ni}| + \mathbb{E}|a_{ni}X|]\right]^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q \left[\sum_{i=1}^{n} |a_{ni}|\right]^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}(\log n)^q \left[\left(\sum_{i=1}^{n} |a_{ni}|^p\right)^{1/p} n^{1-1/p}\right]^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q-\frac{(1-\delta)q}{p}}(\log n)^q < \infty.
\end{aligned}$$

Hence, the proof of Theorem 3.1 is finished. □

4.2. Proof of Theorem 3.2

Proof. For all $\varepsilon > 0$ and any $t > 0$, we see that

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} C_{\mathbb{V}}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^\alpha\right)^+$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t\right) \mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{n^{\alpha}} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha} + t\right) \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha} + t\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{n^{\alpha}} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \mathrm{d}t \qquad (4.16)\\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > t\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > t\right) \mathrm{d}t \\
&=: III_1 + III_2.
\end{aligned}
$$

By Theorem 3.1, we conclude that $III_1 < \infty$. Therefore, it is enough to establish $III_2 < \infty$. Without loss of restriction, assume that $a_{ni} \ge 0$. For all $1 \le i \le n$, $n \ge 1$, $t \ge n^{\alpha}$, write

$$
\begin{aligned}
Y'_{ni} &= -tI(a_{ni}X_i < -t) + a_{ni}X_i I(|a_{ni}X_i \le t|) + tI(a_{ni}X_i > t), \\
Z_{ni} &= a_{ni}X_i - Y'_{ni} = (a_{ni}X_i + t)I(a_{ni}X_i < -t) + (a_{ni}X_i - t)I(a_{ni}X_i > t), \\
T'_{nj} &= \sum_{i=1}^{j} (Y'_{ni} - \mathbb{E}Y'_{ni}), \ j = 1, 2, \ldots, n.
\end{aligned}
$$

We easily see that for all $\varepsilon > 0$,

$$
\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > t\right) \le \sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X_i| > t\right) + \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y'_{ni}\right| > t\right), \tag{4.17}
$$

which results in

$$
\begin{aligned}
III_2 :&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > t\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(|a_{ni}X_i| > t\right) \mathrm{d}t + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y'_{ni}\right| > t\right) \mathrm{d}t \\
&=: III_{21} + III_{22}. \qquad (4.18)
\end{aligned}
$$

For III_{21}, by $p > 1$, and Lemma 2.2, we obtain

$$
III_{21} := \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} \int_{n^{\alpha}}^{\infty} \mathbb{V}\left(|a_{ni}X_i| > t\right) \mathrm{d}t
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} \int_{n^{\alpha p}}^{\infty} \mathbb{V}\left(|a_{ni}X|^{p} > s\right) \frac{1}{p} s^{\frac{1}{p}-1} \mathrm{d}s \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} \int_{n^{\alpha p}}^{\infty} \mathbb{V}\left(|a_{ni}X|^{p} > s\right) \left(n^{\alpha p}\right)^{\frac{1}{p}-1} \mathrm{d}s \\
&\leq C \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} C_{\mathbb{V}}\left(|a_{ni}X|^{p}\right) \qquad (4.19) \\
&= C \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}|^{p} C_{\mathbb{V}}\left(|X|^{p}\right) \\
&\leq C \sum_{n=1}^{\infty} n^{\delta-2} < \infty.
\end{aligned}
$$

For III_{22}, we firstly establish that

$$
\sup_{t \geq n^{\alpha}} \frac{1}{t} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} Y'_{ni} \right| \to 0, \text{ as } n \to \infty. \qquad (4.20)
$$

For $1 \leq i \leq n$, $n \geq 1$ and $p > 1$, by $\mathbb{E}X_n = \mathbb{E}(-X_n) = 0$ and Lemma 2.1, we see that $\mathbb{E}Y'_{ni} = \mathbb{E}(-Z_{ni})$. If $a_{ni}X_i > t$, $0 < Z_{ni} = a_{ni}X_i - t < a_{ni}X_i$. If $a_{ni}X_i < -t$, $a_{ni}X_i < Z_{ni} = a_{ni}X_i + t \leq 0$. Hence $|Z_{ni}| \leq |a_{ni}X_i| I\left(|a_{ni}X_i| > t\right)$. Then, by Lemma 2.2, we have

$$
\begin{aligned}
\sup_{t \geq n^{\alpha}} \frac{1}{t} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} Y'_{ni} \right| &= \sup_{t \geq n^{\alpha}} \frac{1}{t} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E}(-Z_{ni}) \right| \\
&\leq C \sup_{t \geq n^{\alpha}} \frac{1}{t} \sum_{i=1}^{n} \mathbb{E}|Z_{ni}| \\
&\leq C \sup_{t \geq n^{\alpha}} \frac{1}{t} \sum_{i=1}^{n} \mathbb{E}|a_{ni}X_i| I\left(|a_{ni}X_i| > t\right) \\
&\leq C \sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X| I\left(|a_{ni}X| > n^{\alpha}\right)}{n^{\alpha}} \qquad (4.21) \\
&\leq C \sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X|^{p}}{n^{\alpha p}} \\
&\leq C n^{\delta-\alpha p} C_{\mathbb{V}}\left(|X|^{p}\right) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Hence, while n is large enough, for $t \geq n^{\alpha}$,

$$
\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} Y'_{ni} \right| \leq \frac{t}{2}, \qquad (4.22)
$$

which results in

$$
\mathbb{V}\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} Y'_{ni} \right| > t \right) \leq \mathbb{V}\left(\max_{1 \leq j \leq n} \left| T'_{nj} \right| > \frac{t}{2} \right). \qquad (4.23)
$$

In the following, we present $III_{22} < \infty$, for $1 < p \le 2$ and $p > 2$.

(i) If $1 < p \le 2$, by (4.23), Lemma 2.4, Markov inequality under sub-linear expectations, and the C_r inequality, we obtain

$$\begin{aligned}
III_{22} &\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} \mathbb{V}\left(\max_{1\le j\le n}\left|T'_{nj}\right| > \frac{t}{2}\right) \mathrm{d}t \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-2}\mathbb{E}\left(\max_{1\le j\le n}\left|T'_{nj}\right|^2\right) \mathrm{d}t \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_{n^\alpha}^{\infty} t^{-2}(\log n)^2\left(\sum_{i=1}^{n}\mathbb{E}\left|Y'_{ni}-\mathbb{E}Y'_{ni}\right|^2\right. \qquad (4.24)\\
&\quad \left.+\left(\sum_{i=1}^{n}\left[|\mathbb{E}(-Y'_{ni})|+|\mathbb{E}(Y'_{ni})|\right]\right)^2\right)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 \int_{n^\alpha}^{\infty} t^{-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}X_i|^2 I\left(|a_{ni}X_i|\le n^\alpha\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 \int_{n^\alpha}^{\infty} t^{-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}X_i|^2 I\left(n^\alpha<|a_{ni}X_i|\le t\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{i=1}^{n}\int_{n^\alpha}^{\infty}\mathbb{V}\left(|a_{ni}X_i|>t\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 \int_{n^\alpha}^{\infty} t^{-2}\left(\sum_{i=1}^{n}\left[|\mathbb{E}(-Y'_{ni})|+|\mathbb{E}(Y'_{ni})|\right]\right)^2\mathrm{d}t \\
&=: III_{221}+III_{222}+III_{223}+III_{224}.
\end{aligned}$$

For III_{221}, by $1 < p \le 2$, we see that

$$\begin{aligned}
III_{221} &\le C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^2\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X_i|^2 I\left(|a_{ni}X_i|\le n^\alpha\right)}{n^{2\alpha}} \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^2\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|\le n^\alpha\right)}{n^{2\alpha}} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^2\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^p I\left(|a_{ni}X|\le n^\alpha\right)}{n^{\alpha p}} \qquad (4.25)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^2 n^{-\alpha p+\delta}C_{\mathbb{V}}\left(|X|^p\right) \\
&\le C\sum_{n=1}^{\infty} n^{\delta-2}(\log n)^2 < \infty.
\end{aligned}$$

For III_{222}, by $1 < p \le 2$, by Markov inequality under sub-linear expectations, and Lemma 2.2, we see that

$$III_{222} = C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2\int_{n^\alpha}^{\infty} t^{-2}\mathbb{E}|a_{ni}X|^2 I\left(n^\alpha<|a_{ni}X|\le t\right)\mathrm{d}t$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} t^{-2}\mathbb{E}|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le t\right)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} t^{-2}\int_0^t \mathbb{V}\left(|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le t\right) > s^2\right)2s\mathrm{d}s\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1}\int_0^{(m+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le (m+1)^\alpha\right) > s^2\right)2s\mathrm{d}s \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1}\left[\int_0^{n^\alpha} \mathbb{V}\left(|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le (m+1)^\alpha\right) > s^2\right)2s\mathrm{d}s\right. \\
&\quad \left. + \int_{n^\alpha}^{(m+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le (m+1)^\alpha\right) > s^2\right)2s\mathrm{d}s\right] \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1} n^{2\alpha}\mathbb{V}\left(|a_{ni}X| > n^\alpha\right) \\
&\quad +C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1}\sum_{j=n}^{m}\int_{j^\alpha}^{(j+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^2 I\left(n^\alpha < |a_{ni}X| \le (m+1)^\alpha\right) > s^2\right)2s\mathrm{d}s \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1} n^{2\alpha}\frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha p}} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \\
&\quad \times\sum_{j=n}^{\infty}\sum_{m=j}^{\infty} m^{-\alpha-1}\int_{j^\alpha}^{(j+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^p I\left(n^\alpha < |a_{ni}X| \le (m+1)^\alpha\right) > s^p\right) s^{p-1}2ps^{2-p}\mathrm{d}s \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{m=n}^{\infty} m^{-\alpha-1} n^{2\alpha} n^{-\alpha p}|a_{ni}|^p C_{\mathbb{V}}\left(|X|^p\right) \\
&\quad +C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{j=n}^{\infty}\sum_{m=j}^{\infty} m^{-\alpha-1}\int_{j^\alpha}^{(j+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^p > s^p\right) s^{p-1}2pj^{\alpha(2-p)}\mathrm{d}s \\
&\le C\sum_{n=1}^{\infty} n^{-2+\delta}(\log n)^2 \\
&\quad +C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 \sum_{j=n}^{\infty}\int_{j^\alpha}^{(j+1)^\alpha} \mathbb{V}\left(|a_{ni}X|^p > s^p\right) s^{p-1}2pj^{\alpha-\alpha p}\mathrm{d}s \\
&\le C\sum_{n=1}^{\infty} n^{-2+\delta}(\log n)^2 + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{\alpha p-2-\alpha}(\log n)^2 C_{\mathbb{V}}\left(|a_{ni}X|^p\right) n^{\alpha-\alpha p} \\
&\le C\sum_{n=1}^{\infty} n^{-2+\delta}(\log n)^2 + C\sum_{n=1}^{\infty} n^{-2+\delta}(\log n)^2 C_{\mathbb{V}}\left(|X|^p\right) \\
&\le C\sum_{n=1}^{\infty} n^{-2+\delta}(\log n)^2 < \infty. \qquad (4.26)
\end{aligned}
$$

For $III_{223} < \infty$, by the proof of $III_{21} < \infty$, we can see that $III_{223} < \infty$. For III_{224}, by $1 < p \leq 2$, $\mathbb{E}(-X) = \mathbb{E}(X) = 0$, and Lemma 2.1, we obtain

$$\begin{aligned}
III_{224} &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 n^{-\alpha}\left(\sum_{i=1}^{n}\left[|\mathbb{E}[-Y'_{ni}+a_{ni}X_i]|+|\mathbb{E}[Y'_{ni}-a_{ni}X_i]|\right]\right)^2\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 n^{-\alpha}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|I\{|a_{ni}X|>n^{\alpha}\}\right)^2\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^2 n^{-\alpha}\left(\frac{\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^p}{n^{\alpha(p-1)}}\right)^2\\
&\leq C\sum_{n=1}^{\infty} n^{2\delta-\alpha p-2}(\log n)^2 (C_{\mathbb{V}}(|X|^p))^2 < \infty.
\end{aligned}$$

Therefore, we conclude that $III_{22} < \infty$ for $1 < p \leq 2$.

(ii) If $p > 2$, by (4.23), $\mathbb{E}(X) = \mathbb{E}(-X) = 0$, Markov inequality under sub-linear expectations, the C_r inequality, and Lemma 2.4 (for $q > 2$), we see that

$$\begin{aligned}
III_{22} &\leq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\int_{n^{\alpha}}^{\infty}\mathbb{V}\left(\max_{1\leq j\leq n}\left|T'_{nj}\right|>\frac{t}{2}\right)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\int_{n^{\alpha}}^{\infty}t^{-q}\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y'_{ni}-\mathbb{E}Y'_{ni})\right|^q\right)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\int_{n^{\alpha}}^{\infty}t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}|Y'_{ni}-\mathbb{E}Y'_{ni}|^q+\left(\sum_{i=1}^{n}\mathbb{E}(Y'_{ni}-\mathbb{E}Y'_{ni})^2\right)^{q/2}\right.\\
&\quad\left.+\left(\sum_{i=1}^{n}\left[|\mathbb{E}(Y'_{ni})|+|\mathbb{E}(-Y'_{ni})|\right]\right)^q\right)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\sum_{i=1}^{n}\int_{n^{\alpha}}^{\infty}t^{-q}\mathbb{E}|Y'_{ni}|^q\mathrm{d}t\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\int_{n^{\alpha}}^{\infty}t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}{Y'_{ni}}^2\right)^{q/2}\mathrm{d}t\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\int_{n^{\alpha}}^{\infty}t^{-q}\left(\sum_{i=1}^{n}\left[|\mathbb{E}(Y'_{ni})|+|\mathbb{E}(-Y'_{ni})|\right]\right)^q\mathrm{d}t\\
&=: IV_1+IV_2+IV_3.
\end{aligned}\tag{4.27}$$

For IV_1, we obtain

$$\begin{aligned}
IV_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\sum_{i=1}^{n}\int_{n^{\alpha}}^{\infty}t^{-q}\mathbb{E}|a_{ni}X|^qI(|a_{ni}X|\leq n^{\alpha})\,\mathrm{d}t\\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q\sum_{i=1}^{n}\int_{n^{\alpha}}^{\infty}t^{-q}\mathbb{E}|a_{ni}X|^qI(n^{\alpha}<|a_{ni}X|\leq t)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \sum_{i=1}^{n}\int_{n^\alpha}^{\infty} \mathbb{V}\left(|a_{ni}X|>t\right)\mathrm{d}t \qquad (4.28)\\
= &: IV_{11}+IV_{12}+IV_{13}.
\end{aligned}
$$

By the similar proofs of $III_{221}<\infty$ and $III_{222}<\infty$ (with q in place of the exponent 2), we can see that $IV_{11}<\infty$ and $IV_{12}<\infty$. Similarly, by the proof of $III_{21}<\infty$, we can see that $IV_{13}<\infty$.

For IV_2, we obtain

$$
\begin{aligned}
IV_2 \le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \int_{n^\alpha}^{\infty} t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|\le n^\alpha\right)\right)^{q/2}\mathrm{d}t\\
&+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \int_{n^\alpha}^{\infty} t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(n^\alpha<|a_{ni}X|\le t\right)\right)^{q/2}\mathrm{d}t \qquad (4.29)\\
&+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \int_{n^\alpha}^{\infty} \left(\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X|>t\right)\right)^{q/2}\mathrm{d}t\\
= &: IV_{21}+IV_{22}+IV_{23}.
\end{aligned}
$$

For IV_{21}, taking $q>\max\left\{2,\frac{2p(\alpha p-1)}{2\alpha p-p+2(1-\delta)}\right\}$, by the C_r inequality, Jensen inequality under sub-linear expectations, and Lemma 2.2, we obtain

$$
\begin{aligned}
IV_{21} =\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \int_{n^\alpha}^{\infty} t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|\le n^\alpha\right)\right)^{q/2}\mathrm{d}t\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q n^{\alpha-\alpha q}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|\le n^\alpha\right)\right)^{q/2}\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q n^{\alpha-\alpha q}\left(\sum_{i=1}^{n}a_{ni}^2\right)^{q/2}\left(\mathbb{E}|X|^p\right)^{q/p}\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q n^{\alpha-\alpha q}\left(n^{1-2(1-\delta)/p}\right)^{q/2}\left(C_{\mathbb{V}}\left(|X|^p\right)\right)^{q/p} \qquad (4.30)\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+\frac{q}{2}-\frac{(1-\delta)q}{p}}(\log n)^q<\infty.
\end{aligned}
$$

For IV_{22}, taking $q>\max\left\{2,\frac{2(\alpha p-1)}{(\alpha p-\delta)}\right\}$, by Jensen inequality under sub-linear expectations, and Lemma 2.2, we have

$$
\begin{aligned}
IV_{22} =\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \int_{n^\alpha}^{\infty} t^{-q}\left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(n^\alpha<|a_{ni}X|\le t\right)\right)^{q/2}\mathrm{d}t\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)^q \left(\sum_{i=1}^{n}\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|>n^\alpha\right)\right)^{q/2}\int_{n^\alpha}^{\infty} t^{-q}\mathrm{d}t\\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \left(\sum_{i=1}^{n}\frac{\mathbb{E}|a_{ni}X|^2 I\left(|a_{ni}X|>n^\alpha\right)}{n^{2\alpha}}\right)^{q/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} (\log n)^q \left(\sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X|^p I\left(|a_{ni}X| > n^{\alpha}\right)}{n^{\alpha p}} \right)^{q/2} \qquad (4.31) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} (\log n)^q n^{-\alpha p q/2} n^{\delta q/2} \left(C_{\mathbb{V}}\left(|X|^p\right)\right)^{q/2} \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha p q/2+\delta q/2} (\log n)^q < \infty.
\end{aligned}
$$

For IV_{23}, by Markov inequality under sub-linear expectations, and Lemma 2.2, we conclude that

$$
\begin{aligned}
\sup_{t \geq n^{\alpha}} \sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > t\right) &\leq \sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > n^{\alpha}\right) \\
&\leq \sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X|^p}{n^{\alpha p}} \qquad (4.32) \\
&\leq C n^{\delta-\alpha p} \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Since $t \geq n^{\alpha}$, for all n sufficiently large, we deduce that

$$
\sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > t\right) < 1. \qquad (4.33)
$$

By (4.29), we obtain

$$
\begin{aligned}
IV_{23} &= C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} (\log n)^q \int_{n^{\alpha}}^{\infty} \left(\sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > t\right) \right)^{q/2} \mathrm{d}t \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} (\log n)^q \int_{n^{\alpha}}^{\infty} \left(\sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > t\right) \right) \mathrm{d}t \qquad (4.34) \\
&\leq C \sum_{n=1}^{\infty} n^{\delta-2} (\log n)^q < \infty.
\end{aligned}
$$

For IV_3, taking $q > \max\left\{\frac{\alpha p-1}{\alpha p-\delta}, 2\right\}$, by $\mathbb{E}(X) = \mathbb{E}(-X) = 0$, Lemma 2.1, and Lemma 2.2, we see that

$$
\begin{aligned}
IV_3 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} (\log n)^q n^{-(q-1)\alpha} \left(\sum_{i=1}^{n} \left[|\mathbb{E}(Y'_{ni} - a_{ni}X_i)| + |\mathbb{E}(-Y'_{ni} + a_{ni}X_i)| \right] \right)^q \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha-(q-1)\alpha} (\log n)^q \left(\sum_{i=1}^{n} \frac{\mathbb{E}|a_{ni}X_i|^p}{n^{\alpha(p-1)}} \right)^q \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha-(q-1)\alpha} (\log n)^q n^{-\alpha q(p-1)+\delta q} \left(C_{\mathbb{V}}\left(|X|^p\right)\right)^q \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q p+\delta q} (\log n)^q < \infty.
\end{aligned}
$$

Hence, the proof of Theorem 3.2 is finished. □

5. Conclusions

We have established the new results of complete convergence and complete moment convergence for weighted sums of negatively dependent random variables under sub-linear expectations. Theorems of this article are the extensions of convergence properties for weighted sums of extended negatively dependent random variables under classical probability space.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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