Mathematics

Research article

Fractional view evaluation system of Schrödinger-KdV equation by a comparative analysis

Rasool Shah[1], Abd-Allah Hyder[2,3], Naveed Iqbal[4] and Thongchai Botmart[5,*]

[1] Department of Mathematics, Abdul Wali khan university Mardan 23200, Pakistan

[2] Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia

[3] Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo, Egypt

[4] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 81451, Saudi Arabia

[5] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: The time-fractional coupled Schrödinger-KdV equation is an interesting mathematical model because of its wide and significant application in mathematics and applied sciences. A fractional coupled Schrödinger-KdV equation in the sense of Caputo derivative is investigated in this article. Namely, we provide a comparative study of the considered model using the Adomian decomposition method and the homotopy perturbation method with Shehu transform. Approximate solutions obtained using the Adomian decomposition and homotopy perturbation methods were numerically evaluated and presented in graphs and tables. Then, these solutions were compared to the exact solutions, demonstrating the simplicity, effectiveness, and good accuracy of the applied method. To demonstrate the accuracy and efficiency of the suggested techniques, numerical problem are provided.

1. Introduction

Due to their broad use in research and engineering, fractional differential equations have been studied for many decades. The numerical solutions to fractional differential equations are of

tremendous interest. No approach provides an accurate solution for fractional differential equations. Approximate solutions can only be generated using series solution methods or linearization. Nonlinear phenomena can be observed in numerous engineering and scientific disciplines, such as chemical kinetics, nonlinear spectroscopy, solid state physics, fluid physics, computational biology, quantum mechanics, and thermodynamics, among others. Nonlinearity is defined by various nonlinear partial differential equations of greater order. For all physical systems, nonlinear models describe fundamental events [1, 2]. Most of the properties of science and nature described by employing non-linear partial differential equations are strongly connected with the basic properties of scientific fields. Recently, partial differential equations have been utilised to examine properties of several real-world problems likely to arise in plasma physics, population ecology, fluid mechanics, solid state physics, shallow-water wave propagation, quantum mechanics, heat, biology and optical fibers. Furthermore, their mathematical frameworks have also been described to literary works [3–5]. Theoretical partial differential equations analysis offers a wide range of applications. It's worth noting that there are certain really difficult equations that even supercomputers can't solve. All one can do in these situations is an attempt to get qualitative data about the answer. Moreover, the formulation of the equation, as well as its associated side conditions, is significant . A physical or technical issue model is frequently used to generate the equation. It is not immediately clear that the model is continuous in that it results in a solved partial differential equation [6–8]. Furthermore, in most circumstances, the answer should be one of a kind and stable under minor data disruptions. Theoretical comprehension of the equation aids in determining whether or not these requirements are satisfied. Several methods for solving classical partial differential equations have already been suggested, and several solutions have been discovered [9–15].

Because of its extensive applicability range, the nonlinear Schrodinger equation (NLSE) is one kind of PDE whose study has had a lot of success for decades. Different types of NLSE are used to represent specific phenomena in diverse areas, including nonlinear optics [16, 17], fluid dynamics [18], and Bose-Einstein Delicacy [19, 20] among others. The nonlinearity to create additional light on the disruption, that happens when electromagnetic pulses travel in the optical extreme [21, 22] is a distinguishing feature among the forms of the discovered NLSE. Furthermore, studies on the soliton and numerical solutions for NLSEs have been described in the literature [23] utilising various techniques. The current research varies from previous works on the Schrödinger-KdV problem in that it has identified an accurate numerical solution using a series scheme developed using a non singular kernel operator.

$$
\begin{cases}
\frac{\partial^\wp \varphi}{\partial \eta^\wp} = \frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta), \\
\frac{\partial^\wp \zeta}{\partial \eta^\wp} = \frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta), \\
\frac{\partial^\wp \phi}{\partial \eta^\wp} = -6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}, \quad 0 < \wp \le 1.
\end{cases} \tag{1.1}
$$

Transformations are intrinsically intriguing, because they facilitate the solution of mathematical problems such as ordinary, partial, and fractional-order differential equations by researchers. Laplace and Fourier transformations are two well-known transformations that were initially utilized to solve ordinary and partial differential equations. Later, in the field of fractional calculus, these transformations were applied to differential equations of fractional order. In recent years, scholars have proposed numerous more transformations to address a vast array of mathematical difficulties. In

this study, inspired by the aforementioned research, we present the Shehu transform, a Laplace-type integral transform for solving both ordinary and partial differential equations. The integral transform described is applied successfully to both ordinary and partial differential equations. Using the Laplace or Fourier integral transforms, it is simple to validate the results obtained in the applications section. The Shehu transform is represented throughout this work by the operator **S**.

In this paper, we enhance the Shehu transform decomposition method (STDM) and the homotopy perturbation Shehu transform technique (HPSTM) in order to discover an approximation to the aforementioned problem. Adomian decomposition methodology (ADM) is a well-known systematic method for solving deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations, integral equations, and integro-differential equations, among others. The ADM is an advanced method that provides quick algorithms for approximate analytic answers and numerical simulations in applied sciences and engineering. In 1998, however, he developed the homotopy perturbation technique [26, 27]. The applications of the homotopy perturbation method, which are thoroughly discussed in [28–32], have recently received considerable interest. This method views the solution as the sum of an infinite series, which typically converges to precise solutions rapidly. This method has been applied to a variety of mathematical problems.

2. Basic definitions

In order to perform our research, we require various terminologies and postulate outcomes from a literature.

Definition 2.1. Shehu transform (ST) for a mapping $\mathcal{U}(\mu)$ containing exponential order defined on the set of mappings is described as follows [33–35]:

$$\mathbf{S} = \left\{ \mathcal{U}(\mu) \mid \exists \mathcal{P}, p_1, p_2 > 0, \mid \mathcal{U}(\mu) \mid < \mathcal{P} \exp\left(|\mu|/p_j\right), \text{if } \mu \in (-1)' \times [0, \infty), J = 1, 2; (\mathcal{P}, p_1, p_2 > 0) \right\}, (2.1)$$

where $\mathcal{U}(\mu)$ is denoted by $\mathbf{S}[\mathcal{U}(\mu)] = \mathcal{S}(\varepsilon, \upsilon)$, is stated as

$$\mathbf{S}[\mathcal{U}(\mu)] = \int_0^\infty \mathcal{U}(\mu) \exp\left(-\frac{\varepsilon}{\upsilon}\mu\right) d\mu = \mathcal{S}(\varepsilon, \upsilon), \mu \le 0, \upsilon \in [\kappa_1, \kappa_2].$$

The following is an example of a supportive ST:

$$\mathbf{S}[\mu^\wp] = \int_0^\infty \exp\left(-\frac{\varepsilon}{\upsilon}\mu\right) \mu^\wp d\mu = \Gamma(\wp + 1)\left(\frac{\upsilon}{\varepsilon}\right)^{\wp+1}.$$

Definition 2.2. The inverse ST of a function $\mathcal{U}(\mu)$ is described as [33–35]

$$\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{m\wp+1}\right] = \frac{\mu^{m\wp}}{\Gamma(m\wp + 1)}, \Re(\wp) > 0, \text{ and } m > 0.$$

Lemma 2.3. *Consider ST of $\mathcal{U}_1(\mu)$ and $\mathcal{U}_2(\mu)$ as $\mathcal{P}(\varepsilon, \upsilon)$ and $Q(\varepsilon, \upsilon)$, respectively [33–35],*

$$\mathbf{S}[\gamma_1 \mathcal{U}_1(\mu) + \gamma_2 \mathcal{U}_2(\mu)] = \mathbf{S}[\gamma_1 \mathcal{U}_1(\mu)] + \mathbf{S}[\gamma_2 \mathcal{U}_2(\mu)]$$
$$= \gamma_1 \mathcal{P}(\varepsilon, \upsilon) + \gamma_2 Q(\varepsilon, \upsilon),$$

where γ_1 and γ_2 are unspecified terms.

Lemma 2.4. *The ST of the Caputo fractional derivative (CFD) is defined as [33–35]*

$$\mathbf{S}\left[\mathcal{D}_\mu^\wp \mathcal{U}(\mu)\right] = \left(\frac{\varepsilon}{\upsilon}\right)^\wp \mathbf{S}[\mathcal{U}(\phi,\mu)] - \sum_{k=0}^{m-1}\left(\frac{\varepsilon}{\upsilon}\right)^{\wp-\kappa-1}\mathcal{U}^{(k)}(\phi,0), m-1 \le \wp \le m, m \in \mathbb{N}.$$

3. Homotopy perturbation Shehu transform method [36]

In order to understand the basic idea, we give the general implementation of the given technique.

$$D_\eta^\wp \varphi(\mu,\eta) + \mathcal{P}_1\varphi(\mu,\eta) + \mathcal{Q}_1\varphi(\mu,\eta) = 0 \ \ 0 < \wp \le 1, \tag{3.1}$$

with initial conditions

$$\varphi(\mu,0) = h(\mu).$$

Here $D_\eta^\wp$ represents the fractional differential operator, $\mathcal{P}_1$ and $\mathcal{Q}_1$ represent the linear and nonlinear terms respectively.

Apply Shehu transform to (3.1),

$$\mathbf{S}[D_\eta^\wp \varphi(\mu,\eta)] + \mathbf{S}[\mathcal{P}_1\varphi(\mu,\eta) + \mathcal{Q}_1\varphi(\mu,\eta)] = 0, \tag{3.2}$$

$$\left(\frac{\varepsilon}{\upsilon}\right)^\wp\{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\varphi(0)\} + \mathbf{S}[\mathcal{P}_1\varphi(\mu,\eta) + \mathcal{Q}_1\varphi(\mu,\eta)] = 0. \tag{3.3}$$

From (3.3) we have

$$\mathcal{M}(\varphi) = \frac{\upsilon}{\varepsilon}h(\mu) - \left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}[\mathcal{P}_1\varphi(\mu,\eta) + \mathcal{Q}_1\varphi(\mu,\eta)]. \tag{3.4}$$

Using the inverse Shehu transform to Eq. (3.4), we get

$$\varphi(\mu,\eta) = H(\mu) - \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}\{\mathcal{P}_1\varphi(\mu,\eta) + \mathcal{Q}_1\varphi(\mu,\eta)\}\right]. \tag{3.5}$$

The term $H(\mu)$ comes in result of source term and the given initial sources. Now, by homotopy perturbation method

$$\varphi(\mu,\eta) = \sum_{k=0}^{\infty} p^k \varphi_k(\mu,\eta). \tag{3.6}$$

The nonlinear terms are derived as

$$\mathcal{Q}_1\varphi(\mu,\eta) = \sum_{k=0}^{\infty} p^k H_k(\varphi), \tag{3.7}$$

for He's polynomials H_k as

$$H_k(\varphi_0, \varphi_1, \cdots, \varphi_k) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left[\mathcal{Q}_1\left(\sum_{i=0}^{\infty} p^i \varphi_i\right)\right]_{p=0}.$$
$$k = 0, 1, 2, 3 \cdots \tag{3.8}$$

Putting (3.7) and (3.8) in equation (3.5)

$$\sum_{k=0}^{\infty} p^k \varphi_k(\mu,\eta) = H(\mu) - p \times \left(\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}\{\mathcal{P}_1 \sum_{k=0}^{\infty} p^k \varphi_k(\mu,\eta) + \sum_{k=0}^{\infty} p^k H_k(\varphi)\}\right]\right). \tag{3.9}$$

By rearranging based on powers of p-terms, we obtain

$$p^0 : \varphi_0(\mu, \eta) = H(\mu),$$

$$p^1 : \varphi_1(\mu, \eta) = -\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}(\mathcal{P}_1\varphi_0(\mu, \eta) + H_0(\varphi))\right],$$

$$p^2 : \varphi_2(\mu, \eta) = -\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}(\mathcal{P}_1\varphi_1(\mu, \eta) + H_1(\varphi))\right],$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$p^k : \varphi_k(\mu, \eta) = -\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}(\mathcal{P}_1\varphi_{k-1}(\mu, \eta) + H_{k-1}(\varphi))\right], \tag{3.10}$$

$$k > 0, k \in N.$$

Thus we easily calculate $\varphi_k(\mu, \eta)$, with the help of which we get convergent series. At $p \to 1$, we have

$$\varphi(\mu, \eta) = \lim_{M\to\infty} \sum_{k=1}^{M} \varphi_k(\mu, \eta). \tag{3.11}$$

The series form solution converges quickly towards the exact solution.

4. Shehu transform decomposition method [37]

In order to understand the basic idea, we give the general implementation of the given technique.

$$D_\eta^{\wp}\varphi(\mu, \eta) = \mathcal{P}_1(\mu, \eta) + Q_1(\mu, \eta) + \mathcal{R}_1(\mu, \eta), 0 < \wp \leq 1, \tag{4.1}$$

having initial sources

$$\varphi(\mu, 0) = \varphi(\mu).$$

Where $D_\eta^{\wp} = \frac{\partial^{\wp}}{\partial\eta^{\wp}}$ represents the fractional differential operator, $\mathcal{P}_1$ and Q_1 represent the linear and nonlinear terms and $\mathcal{R}_1$ is the source term.
Apply Shehu transform to (4.1),

$$\mathbf{S}[D_\eta^{\wp}\varphi(\mu, \eta)] = \mathbf{S}[\mathcal{P}_1(\mu, \eta) + Q_1(\mu, \eta) + \mathcal{R}_1(\mu, \eta)]. \tag{4.2}$$

Applying the differentiation property of Shehu transform, we get

$$\left(\frac{\varepsilon}{\upsilon}\right)^{\wp}\{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\varphi(0)\} = \mathbf{S}[\mathcal{P}_1(\mu, \eta) + Q_1(\mu, \eta) + \mathcal{R}_1(\mu, \eta)]. \tag{4.3}$$

From (4.3) we have

$$\mathcal{M}(\varphi) = \frac{\upsilon}{\varepsilon}\varphi(0) + \left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}[\mathcal{P}_1(\mu, \eta) + Q_1(\mu, \eta) + \mathcal{R}_1(\mu, \eta)], \tag{4.4}$$

Using the inverse Shehu transform

$$\varphi(\mu, \eta) = \varphi(0) + \mathbf{S}^{-1}[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}[\mathcal{P}_1(\mu, \eta) + Q_1(\mu, \eta) + \mathcal{R}_1(\mu, \eta)]. \tag{4.5}$$

STDM series form solution of $\varphi(\mu, \eta)$ is defined as

$$\varphi(\mu, \eta) = \sum_{m=0}^{\infty} \varphi_m(\mu, \eta). \tag{4.6}$$

The nonlinear terms Q_1 is defined as

$$Q_1(\mu, \eta) = \sum_{m=0}^{\infty} \mathcal{A}_m. \tag{4.7}$$

The nonlinear terms by means of Adomian polynomials is determined as

$$\mathcal{A}_m = \frac{1}{m!} \left[\frac{\partial^m}{\partial \wp^m} \left\{ Q_1 \left(\sum_{k=0}^{\infty} \wp^k \mu_k, \sum_{k=0}^{\infty} \wp^k \eta_k \right) \right\} \right]_{\wp=0}, \tag{4.8}$$

By putting Eq (4.6) and Eq (4.8) into (4.5) , we obtain

$$\sum_{m=0}^{\infty} \varphi_m(\mu, \eta) = \varphi(0) + \mathbf{S}^{-1}[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp} \mathbf{S}\{\mathcal{R}_1(\mu, \eta)\}] + \mathbf{S}^{-1}\left(\frac{\upsilon}{\varepsilon}\right)^{\wp} \left[\mathbf{S} \left\{ \mathcal{P}_1\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \eta_m \right) + \sum_{m=0}^{\infty} \mathcal{A}_m \right\} \right]. \tag{4.9}$$

we define the following terms,

$$\varphi_0(\mu, \eta) = \varphi(0) + \mathbf{S}^{-1}[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp} \mathbf{S}\{\mathcal{R}_1(\mu, \eta)\}], \tag{4.10}$$

$$\varphi_1(\mu, \eta) = \mathbf{S}^{-1} \left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp} \mathbf{S}\{\mathcal{P}_1(\mu_0, \eta_0) + \mathcal{A}_0\} \right],$$

the general for $m \geq 1$, is defined as

$$\varphi_{m+1}(\mu, \eta) = \mathbf{S}^{-1} \left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp} \mathbf{S}\{\mathcal{P}_1(\mu_m, \eta_m) + \mathcal{A}_m\} \right].$$

5. Applications

In this section, the solution to time-fractional Schrödinger-KdV equation is derived by implementing STDM and HPSTM.

Example

Consider the time-fractional Schrödinger-KdV equation of the following form

$$\frac{\partial^{\wp} \varphi}{\partial \eta^{\wp}} = \frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta), \quad 0 < \wp \leq 1,$$

$$\frac{\partial^{\wp} \zeta}{\partial \eta^{\wp}} = \frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta), \tag{5.1}$$

$$\frac{\partial^{\wp} \phi}{\partial \eta^{\wp}} = -6\phi \frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi \frac{\partial \phi}{\partial \mu} + 2\zeta \frac{\partial \zeta}{\partial \mu},$$

having initial conditions

$$\begin{cases} \varphi(\mu, 0) = \cos(\mu), \\ \zeta(\mu, 0) = \sin(\mu), \\ \phi(\mu, 0) = \frac{3}{4}. \end{cases}$$

At $\wp = 1$, the exact solution of Eq (5.1) is

$$\begin{aligned} \varphi(\mu, \eta) &= \cos(\mu + \frac{\eta}{4}), \\ \zeta(\mu, \eta) &= \sin(\mu + \frac{\eta}{4}), \\ \phi(\mu, \eta) &= \frac{3}{4}. \end{aligned} \tag{5.2}$$

Apply Shehu transform to (5.1),

$$\begin{aligned} \mathbf{S}\left\{\frac{\partial^\wp \varphi}{\partial \eta^\wp}\right\} &= \mathbf{S}\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right], \\ \mathbf{S}\left\{\frac{\partial^\wp \zeta}{\partial \eta^\wp}\right\} &= \mathbf{S}\left[\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right], \\ \mathbf{S}\left\{\frac{\partial^\wp \phi}{\partial \eta^\wp}\right\} &= \mathbf{S}\left[-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right], \end{aligned}$$

Applying the Shehu transform differentiation property, we have

$$\begin{aligned} \left(\frac{\varepsilon}{\upsilon}\right)^\wp \{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\varphi(0)\} &= \mathbf{S}\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right], \\ \left(\frac{\varepsilon}{\upsilon}\right)^\wp \{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\zeta(0)\} &= \mathbf{S}\left[\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right], \\ \left(\frac{\varepsilon}{\upsilon}\right)^\wp \{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\phi(0)\} &= \mathbf{S}\left[-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right], \end{aligned} \tag{5.3}$$

$$\begin{aligned} \mathcal{M}(u) &= \frac{\upsilon}{\varepsilon}\varphi(0) + \left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right], \\ \mathcal{M}(u) &= \frac{\upsilon}{\varepsilon}\zeta(0) + \left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}\left[\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right], \\ \mathcal{M}(u) &= \frac{\upsilon}{\varepsilon}\phi(0) + \left(\frac{\upsilon}{\varepsilon}\right)^\wp \mathbf{S}\left[-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right]. \end{aligned} \tag{5.4}$$

Using the Shehu inverse transform

$$\begin{aligned} \varphi(\mu, \eta) &= \varphi(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp \left\{\mathbf{S}\left(\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right], \\ \zeta(\mu, \eta) &= \zeta(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp \left\{\mathbf{S}\left(\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right], \\ \phi(\mu, \eta) &= \zeta(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp \left\{\mathbf{S}\left(-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right)\right\}\right], \end{aligned} \tag{5.5}$$

$$\varphi(\mu, \eta) = \cos(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \left\{\mathbf{S}\left(\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right],$$

$$\zeta(\mu, \eta) = \sin(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \left\{\mathbf{S}\left(\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right], \qquad (5.6)$$

$$\phi(\mu, \eta) = \frac{3}{4} + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \left\{\mathbf{S}\left(-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right)\right\}\right],$$

Now, by homotopy perturbation method

$$\varphi(\mu, \eta) = \varphi_0 + \varphi_1 p + \varphi_2 p^2 + \cdots,$$
$$\zeta(\mu, \eta) = \zeta_0 + \zeta_1 p + \zeta_2 p^2 + \cdots, \qquad (5.7)$$
$$\phi(\mu, \eta) = \phi_0 + \phi_1 p + \phi_2 p^2 + \cdots$$

$$\sum_{k=0}^{\infty} p^k \varphi_k(\mu, \eta) = \cos(\mu) + p\left(\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \mathbf{S}\left[(\sum_{k=0}^{\infty} p^k \varphi_k(\mu, \eta))_{\mu\mu} + \sum_{k=0}^{\infty} p^k H_k^1(\mu, \eta)\right]\right]\right),$$

$$\sum_{k=0}^{\infty} p^k \zeta_k(\mu, \eta) = \sin(\mu) + p\left(\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \mathbf{S}\left[(\sum_{k=0}^{\infty} p^k \zeta_k(\mu, \eta))_{\mu\mu} - \sum_{k=0}^{\infty} p^k H_k^2(\mu, \eta)\right]\right]\right), \qquad (5.8)$$

$$\sum_{k=0}^{\infty} p^k \phi_k(\mu, \eta) = \frac{3}{4} + p\left(\mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\varphi \mathbf{S}\left[-(\sum_{k=0}^{\infty} p^k \phi_k(\mu, \eta))_{\mu\mu\mu} - \sum_{k=0}^{\infty} p^k H_k^3(\mu, \eta)\right]\right]\right),$$

where $H_k(\mu)$ He's polynomials representing the nonlinear terms. The He's polynomials first few terms are given by

$$H_0^1(\mu) = \zeta_0 \phi_0$$
$$H_1^1(\mu) = \zeta_1 \phi_0 + \phi_1 \zeta_0,$$
$$H_2^1(\mu) = \zeta_1 \phi_1 + \phi_0 \zeta_2 + \zeta_0 \phi_2,$$
$$\vdots$$
$$H_0^2(\mu) = \varphi_0 \phi_0,$$
$$H_1^2(\mu) = \varphi_1 \phi_0 + \phi_1 \varphi_0,$$
$$H_2^2(\mu) = \varphi_1 \phi_1 + \phi_0 \varphi_2 + \varphi_0 \phi_2,$$
$$\vdots$$
$$H_0^3(\mu) = -6\phi_0 \phi_{0\mu} + 2\varphi_0 \phi_{0\mu} + 2\zeta_0 \zeta_{0\mu},$$
$$H_1^3(\mu) = -6\phi_1 \phi_{0\mu} - 6\phi_{1\mu} \phi_0 + 2\varphi_1 \phi_{0\mu} + 2\phi_{1\mu} \varphi_0 + 2\zeta_1 \zeta_{0\mu} + 2\zeta_{1\mu} \zeta_0,$$
$$H_2^3(\mu) = -6\phi_2 \phi_{0\mu} - 6\phi_{1\mu} \phi_1 - 6\phi_2 \phi_{0\mu} + 2\varphi_2 \phi_{0\mu} + 2\phi_{1\mu} \varphi_1 + 2\varphi_0 \phi_{2\mu} + 2\zeta_2 \zeta_{0\mu} + 2\zeta_{1\mu} \zeta_1 + 2\zeta_0 \zeta_{2\mu}.$$

By rearranging based on powers of p-terms, we obtain

$$p^0 : \varphi_0(\mu, \eta) = \cos(\mu),$$

$$\zeta_0(\mu, \eta) = \sin(\mu),$$

$$\phi_0(\mu, \eta) = \frac{3}{4},$$

$$p^1 : \varphi_1(\mu, \eta) = -\frac{sin(\mu)}{4\Gamma(\wp + 1)},$$

$$\zeta_1(\mu, \eta) = -\frac{7cos(\mu)}{4\Gamma(\wp + 1)},$$

$$\phi_1(\mu, \eta) = \frac{2sin(x)cos(\mu)}{\Gamma(\wp + 1)},$$

$$p^2 : \varphi_2(\mu, \eta) = -\frac{cos(\mu)(-39 + 32cos^2(\mu))}{16\Gamma(2\wp + 1)},$$

$$\zeta_2(\mu, \eta) = -\frac{sin(\mu)(-7 + 32cos^2(\mu))}{16\Gamma(2\wp + 1)},$$

$$\phi_2(\mu, \eta) = \frac{(2cos^2(\mu) - 1)(8cos(\mu) - 9)}{2\Gamma(2\wp + 1)},$$

$$p^3 : \varphi_3(\mu, \eta) = -\frac{sin(\mu)(-25(\Gamma(\wp + 1))^2 - 480(\Gamma(\wp + 1))^2 cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$-\frac{512(\Gamma(\wp + 1))^2 cos^3(\mu) + 256(\Gamma(\wp + 1))^2 cos(\mu) + 224\Gamma(2\wp + 1)cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} - \cdots$$

$$\zeta_3(\mu, \eta) = -\frac{cos(\mu)(1329(\Gamma(\wp + 1))^2 - 1824(\Gamma(\wp + 1))^2 cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$+\frac{(512(\Gamma(\wp + 1))^2 cos^3(\mu) - 256(\Gamma(\wp + 1))^2 cos(\mu) - 32\Gamma(2\wp + 1) + 32\Gamma(2\wp + 1)cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2},$$

$$\phi_3(\mu, \eta) = -\frac{sin(\mu)(-272(\Gamma(\wp + 1))^2 + 576(\Gamma(\wp + 1))^2 cos^2(\mu) - 70(\Gamma(\wp + 1))^2 cos(\mu))}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$-\frac{512(\Gamma(\wp + 1))^2 cos^3(\mu) + 384\Gamma(2\wp + 1)cos^3(\mu) - 143\Gamma(2\wp + 1)cos(\mu) + 16\Gamma(2\wp + 1)cos^2(\mu) - 8\Gamma(2\wp + 1)}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2},$$

$$\vdots$$

At $p \to 1$ we have

$$\varphi(\mu, \eta) = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots$$

$$= \cos(\mu) - \frac{sin(\mu)}{4\Gamma(\wp + 1)} - \frac{cos(\mu)(-39 + 32cos^2(\mu))}{16\Gamma(2\wp + 1)} - \frac{sin(\mu)(-25(\Gamma(\wp + 1))^2 - 480(\Gamma(\wp + 1))^2 cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$- \frac{512(\Gamma(\wp + 1))^2 cos^3(\mu) + 256(\Gamma(\wp + 1))^2 cos(\mu) + 224\Gamma(2\wp + 1)cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} - \cdots$$

$$\zeta(\mu, \eta) = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \cdots$$

$$= \sin(\mu) - \frac{7\cos(\mu)}{4\Gamma(\wp + 1)} - \frac{\sin(\mu)(-7 + 32\cos^2(\mu))}{16\Gamma(2\wp + 1)} - \frac{\cos(\mu)(1329(\Gamma(\wp + 1))^2 - 1824(\Gamma(\wp + 1))^2\cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$+ \frac{(512(\Gamma(\wp + 1))^2\cos^3(\mu) - 256(\Gamma(\wp + 1))^2\cos(\mu) - 32\Gamma(2\wp + 1) + 32\Gamma(2\wp + 1)\cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} - \cdots$$

$$\phi(\mu, \eta) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$

$$= \frac{3}{4} + \frac{2\sin(x)\cos(\mu)}{\Gamma(\wp + 1)} + \frac{(2\cos^2(\mu) - 1)(8\cos(\mu) - 9)}{2\Gamma(2\wp + 1)}$$

$$- \frac{\sin(\mu)(-272(\Gamma(\wp + 1))^2 + 576(\Gamma(\wp + 1))^2\cos^2(\mu) - 70(\Gamma(\wp + 1))^2\cos(\mu))}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$

$$- \frac{512(\Gamma(\wp + 1))^2\cos^3(\mu) + 384\Gamma(2\wp + 1)\cos^3(\mu) - 143\Gamma(2\wp + 1)\cos(\mu) + 16\Gamma(2\wp + 1)\cos^2(\mu) - 8\Gamma(2\wp + 1)}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} + \cdots$$

STDM Solution

Apply Shehu transform to (5.1),

$$\mathbf{S}\left\{\frac{\partial^\wp \varphi}{\partial\eta^\wp}\right\} = \mathbf{S}\left[\frac{\partial^2\varphi}{\partial\mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right],$$

$$\mathbf{S}\left\{\frac{\partial^\wp \zeta}{\partial\eta^\wp}\right\} = \mathbf{S}\left[\frac{\partial^2\zeta}{\partial\mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right],$$

$$\mathbf{S}\left\{\frac{\partial^\wp \phi}{\partial\eta^\wp}\right\} = \mathbf{S}\left[-6\phi\frac{\partial\phi}{\partial\mu} - \frac{\partial^3\phi}{\partial\mu^3} + 2\varphi\frac{\partial\phi}{\partial\mu} + 2\zeta\frac{\partial\zeta}{\partial\mu}\right],$$

Applying the Shehu transform differentiation property, we have

$$\left(\frac{\varepsilon}{\upsilon}\right)^\wp\{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\varphi(0)\} = \mathbf{S}\left[\frac{\partial^2\varphi}{\partial\mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right],$$

$$\left(\frac{\varepsilon}{\upsilon}\right)^\wp\{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\zeta(0)\} = \mathbf{S}\left[\frac{\partial^2\zeta}{\partial\mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right], \tag{5.9}$$

$$\left(\frac{\varepsilon}{\upsilon}\right)^\wp\{\mathcal{M}(u) - \frac{\upsilon}{\varepsilon}\phi(0)\} = \mathbf{S}\left[-6\phi\frac{\partial\phi}{\partial\mu} - \frac{\partial^3\phi}{\partial\mu^3} + 2\varphi\frac{\partial\phi}{\partial\mu} + 2\zeta\frac{\partial\zeta}{\partial\mu}\right].$$

Using the Shehu inverse transform

$$\varphi(\mu, \eta) = \varphi(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp\left\{\mathbf{S}\left(\frac{\partial^2\varphi}{\partial\mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right],$$

$$\zeta(\mu, \eta) = \zeta(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp\left\{\mathbf{S}\left(\frac{\partial^2\zeta}{\partial\mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right], \tag{5.10}$$

$$\phi(\mu, \eta) = \phi(0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^\wp\left\{\mathbf{S}\left(-6\phi\frac{\partial\phi}{\partial\mu} - \frac{\partial^3\phi}{\partial\mu^3} + 2\varphi\frac{\partial\phi}{\partial\mu} + 2\zeta\frac{\partial\zeta}{\partial\mu}\right)\right\}\right],$$

$$\varphi(\mu, \eta) = \cos(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\left\{\mathbf{S}\left(\frac{\partial^2 \varphi}{\partial \mu^2} + \zeta(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right],$$

$$\zeta(\mu, \eta) = \cos(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\left\{\mathbf{S}\left(\frac{\partial^2 \zeta}{\partial \mu^2} - \varphi(\mu, \eta)\phi(\mu, \eta)\right)\right\}\right], \tag{5.11}$$

$$\phi(\mu, \eta) = \frac{3}{4} + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\left\{\mathbf{S}\left(-6\phi\frac{\partial \phi}{\partial \mu} - \frac{\partial^3 \phi}{\partial \mu^3} + 2\varphi\frac{\partial \phi}{\partial \mu} + 2\zeta\frac{\partial \zeta}{\partial \mu}\right)\right\}\right].$$

STDM series form solution of $\varphi(\mu, \eta), \zeta(\mu, \eta)$ and $\phi(\mu, \eta)$ are defined as:

$$\varphi(\mu, \eta) = \sum_{m=0}^{\infty} \varphi_m(\mu, \eta),$$

$$\zeta(\mu, \eta) = \sum_{m=0}^{\infty} \zeta_m(\mu, \eta), \quad and$$

$$\phi(\mu, \eta) = \sum_{m=0}^{\infty} \phi_m(\mu, \eta)$$

Where the Adomian polynomials $\zeta\phi = \sum_{m=0}^{\infty} \mathcal{A}_m$, $\varphi\phi = \sum_{m=0}^{\infty} \mathcal{B}_m$ and $-6\phi\phi_\mu + 2\varphi\phi_\mu + 2\zeta\zeta_\mu = \sum_{m=0}^{\infty} C_m$ and the nonlinear terms have been characterised. Thus, Eq. (5.11) can be rewritten as

$$\sum_{m=0}^{\infty} \varphi_m(\mu, \eta) = \varphi(\mu, 0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \sum_{m=0}^{\infty} \mathcal{A}_m\right]\right],$$

$$\sum_{m=0}^{\infty} \zeta_m(\mu, \eta) = \zeta(\mu, 0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[\frac{\partial^2 \zeta}{\partial \mu^2} - \sum_{m=0}^{\infty} \mathcal{B}_m\right]\right],$$

$$\sum_{m=0}^{\infty} \phi_m(\mu, \eta) = \phi(\mu, 0) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[-\frac{\partial^3 \phi}{\partial \mu^3} + \sum_{m=0}^{\infty} C_m\right]\right],$$

$$\sum_{m=0}^{\infty} \varphi_m(\mu, \eta) = \cos(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \sum_{m=0}^{\infty} \mathcal{A}_m\right]\right],$$

$$\sum_{m=0}^{\infty} \zeta_m(\mu, \eta) = \sin(\mu) + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[\frac{\partial^2 \zeta}{\partial \mu^2} - \sum_{m=0}^{\infty} \mathcal{B}_m\right]\right], \tag{5.12}$$

$$\sum_{m=0}^{\infty} \phi_m(\mu, \eta) = \frac{3}{4} + \mathbf{S}^{-1}\left[\left(\frac{\upsilon}{\varepsilon}\right)^{\wp}\mathbf{S}\left[-\frac{\partial^3 \phi}{\partial \mu^3} + \sum_{m=0}^{\infty} C_m\right]\right].$$

The nonlinear terms by means of Adomian polynomials is determined as,

$\mathcal{A}_0 = \zeta_0\phi_0, \quad \mathcal{A}_1 = \zeta_1\phi_0 + \phi_1\zeta_0, \quad \mathcal{A}_2 = \zeta_1\phi_1 + \phi_0\zeta_2 + \zeta_0\phi_2,$

$\mathcal{B}_0 = \varphi_0\phi_0, \quad \mathcal{B}_1 = \varphi_1\phi_0 + \phi_1\varphi_0, \quad \mathcal{B}_2 = \varphi_1\phi_1 + \phi_0\varphi_2 + \varphi_0\phi_2,$

$C_0 = -6\phi_0\phi_{0\mu} + 2\varphi_0\phi_{0\mu} + 2\zeta_0\zeta_{0\mu}, \quad C_1 = -6\phi_1\phi_{0\mu} - 6\phi_{1\mu}\phi_0 + 2\varphi_1\phi_{0\mu} + 2\phi_{1\mu}\varphi_0 + 2\zeta_1\zeta_{0\mu} + 2\zeta_{1\mu}\zeta_0,$

$C_2 = -6\phi_2\phi_{0\mu} - 6\phi_{1\mu}\phi_1 - 6\phi_2\phi_{0\mu} + 2\varphi_2\phi_{0\mu} + 2\phi_{1\mu}\varphi_1 + 2\varphi_0\phi_{2\mu} + 2\zeta_2\zeta_{0\mu} + 2\zeta_{1\mu}\zeta_1 + 2\zeta_0\zeta_{2\mu},$

On comparing both sides of Eq (5.12), we get

$$\varphi_0(\mu, \eta) = \cos(\mu),$$
$$\zeta_0(\mu, \eta) = \sin(\mu),$$
$$\phi_0(\mu, \eta) = \frac{3}{4}.$$

For $m = 0$

$$\varphi_1(\mu, \eta) = -\frac{sin(\mu)}{4\Gamma(\wp + 1)},$$
$$\zeta_1(\mu, \eta) = -\frac{7cos(\mu)}{4\Gamma(\wp + 1)},$$
$$\phi_1(\mu, \eta) = \frac{2sin(x)cos(\mu)}{\Gamma(\wp + 1)}.$$

For $m = 1$

$$\varphi_2(\mu, \eta) = -\frac{cos(\mu)(-39 + 32cos^2(\mu))}{16\Gamma(2\wp + 1)},$$
$$\zeta_2(\mu, \eta) = -\frac{sin(\mu)(-7 + 32cos^2(\mu))}{16\Gamma(2\wp + 1)},$$
$$\phi_2(\mu, \eta) = \frac{(2cos^2(\mu) - 1)(8cos(\mu) - 9)}{2\Gamma(2\wp + 1)}.$$

For $m = 2$

$$p^3 : \varphi_3(\mu, \eta) = -\frac{sin(\mu)(-25(\Gamma(\wp + 1))^2 - 480(\Gamma(\wp + 1))^2cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$- \frac{512(\Gamma(\wp + 1))^2cos^3(\mu) + 256(\Gamma(\wp + 1))^2cos(\mu) + 224\Gamma(2\wp + 1)cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$\zeta_3(\mu, \eta) = -\frac{cos(\mu)(1329(\Gamma(\wp + 1))^2 - 1824(\Gamma(\wp + 1))^2cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$+ \frac{(512(\Gamma(\wp + 1))^2cos^3(\mu) - 256(\Gamma(\wp + 1))^2cos(\mu) - 32\Gamma(2\wp + 1) + 32\Gamma(2\wp + 1)cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2},$$
$$\phi_3(\mu, \eta) = -\frac{sin(\mu)(-272(\Gamma(\wp + 1))^2 + 576(\Gamma(\wp + 1))^2cos^2(\mu) - 70(\Gamma(\wp + 1))^2cos(\mu))}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$- \frac{512(\Gamma(\wp + 1))^2cos^3(\mu) + 384\Gamma(2\wp + 1)cos^3(\mu) - 143\Gamma(2\wp + 1)cos(\mu) + 16\Gamma(2\wp + 1)cos^2(\mu) - 8\Gamma(2\wp + 1)}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2},$$

$$\vdots$$

The STDM remaining solution elements like ξ_m ζ_m and ϕ_m for $(m \geq 3)$ are simply to calculate. Hence, we define the series form solution as follows:

$$\varphi(\mu, \eta) = \sum_{m=0}^{\infty} \varphi_m(\mu, \eta) = \varphi_0(\mu, \eta) + \varphi_1(\mu, \eta) + \varphi_2(\mu, \eta) + \varphi_3(\mu, \eta) + \cdots$$

$$\zeta(\mu, \eta) = \sum_{m=0}^{\infty} \zeta_m(\mu, \eta) = \zeta_0(\mu, \eta) + \zeta_1(\mu, \eta) + \zeta(\mu, \eta) + \zeta_3(\mu, \eta) + \cdots$$

$$\phi(\mu, \eta) = \sum_{m=0}^{\infty} \phi_m(\mu, \eta) = \phi_0(\mu, \eta) + \phi_1(\mu, \eta) + \phi(\mu, \eta) + \phi_3(\mu, \eta) + \cdots$$

$$\varphi(\mu, \eta) = \cos(\mu) - \frac{\sin(\mu)}{4\Gamma(\wp + 1)} - \frac{\cos(\mu)(-39 + 32\cos^2(\mu))}{16\Gamma(2\wp + 1)} - \frac{\sin(\mu)(-25(\Gamma(\wp + 1))^2 - 480(\Gamma(\wp + 1))^2\cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$- \frac{512(\Gamma(\wp + 1))^2\cos^3(\mu) + 256(\Gamma(\wp + 1))^2\cos(\mu) + 224\Gamma(2\wp + 1)\cos^2(\mu)}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} - \cdots$$

$$\zeta(\mu, \eta) = \sin(\mu) - \frac{7\cos(\mu)}{4\Gamma(\wp + 1)} - \frac{\sin(\mu)(-7 + 32\cos^2(\mu))}{16\Gamma(2\wp + 1)} - \frac{\cos(\mu)(1329(\Gamma(\wp + 1))^2 - 1824(\Gamma(\wp + 1))^2\cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$+ \frac{(512(\Gamma(\wp + 1))^2\cos^3(\mu) - 256(\Gamma(\wp + 1))^2\cos(\mu) - 32\Gamma(2\wp + 1) + 32\Gamma(2\wp + 1)\cos^2(\mu))}{64\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} - \cdots$$

$$\phi(\mu, \eta) = \frac{3}{4} + \frac{2\sin(x)\cos(\mu)}{\Gamma(\wp + 1)} + \frac{(2\cos^2(\mu) - 1)(8\cos(\mu) - 9)}{2\Gamma(2\wp + 1)}$$
$$- \frac{\sin(\mu)(-272(\Gamma(\wp + 1))^2 + 576(\Gamma(\wp + 1))^2\cos^2(\mu) - 70(\Gamma(\wp + 1))^2\cos(\mu))}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2}$$
$$- \frac{512(\Gamma(\wp + 1))^2\cos^3(\mu) + 384\Gamma(2\wp + 1)\cos^3(\mu) - 143\Gamma(2\wp + 1)\cos(\mu) + 16\Gamma(2\wp + 1)\cos^2(\mu) - 8\Gamma(2\wp + 1)}{8\Gamma(3\wp + 1)(\Gamma(\wp + 1))^2} + \cdots$$

6. Result and discussion

This section explains the numerical solution to the time-fractional Schrödinger-KdV equation. Example 5.1 is graphical simulation is shown in Figure 1. Figure 1 is a graphical depiction of the solution acquired by the offered methods and the exact outcome of Example 5.1. Figure 2 depicts the results of proposed techniques with $\alpha = 0.90$ and $\alpha = 0.80$, respectively. Example 5.1 depicts the analysis to several fractional decimal places. Figure 3 depicts the reaction to the solution for various α values. The solution of Example 5.1 is depicted graphically in Figures 4 and 5. Figure 4 depicts the graphical simulations of the suggested methods' results for $\alpha = 1$ and the precise answer, while Figure 5 depicts the graphs for the solution of Example 5.1 for $\alpha = 0.9$ and $\alpha = 0.8$. Figure 6 depicts the graphical comparison for distinct α values, respectively.
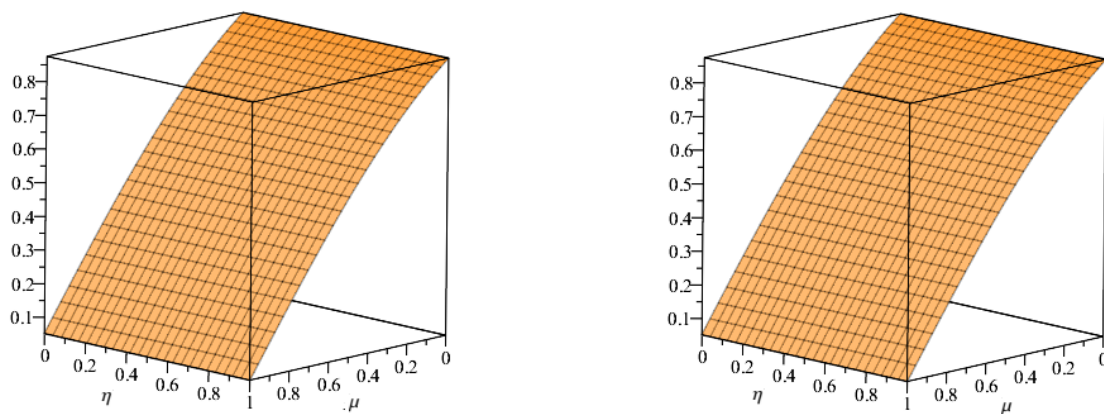
Figure 1. The exact solution and analytical solution for $\varphi(\mu, \eta)$.
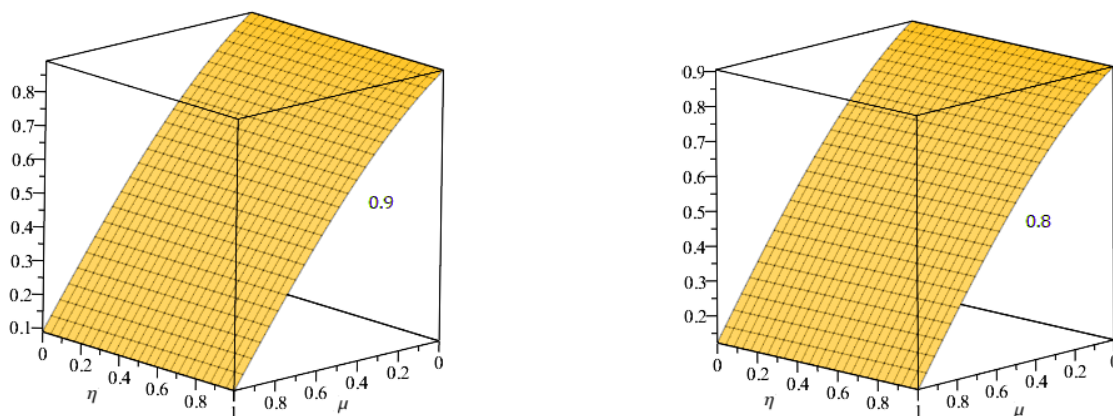

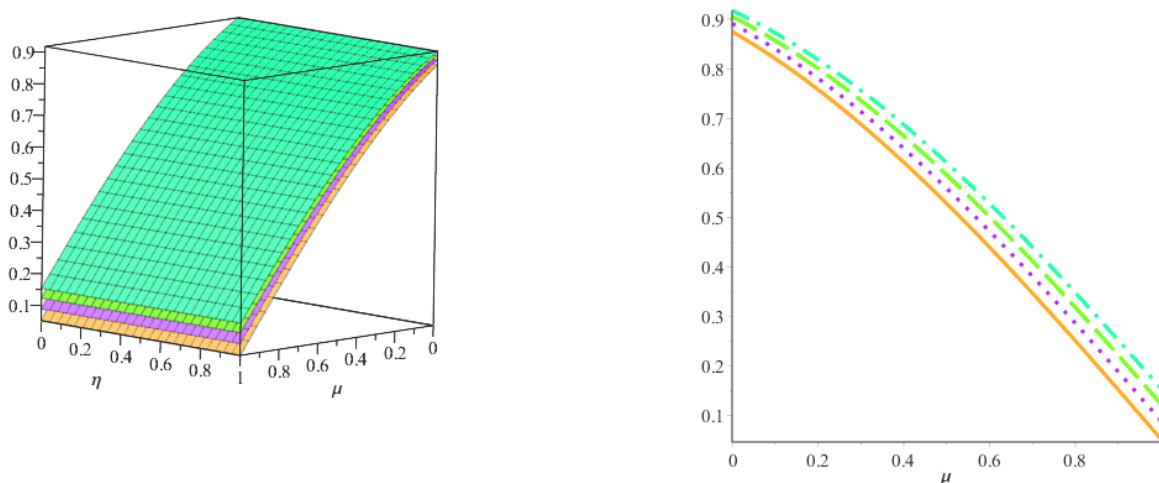**Figure 2.** Proposed methods solution at $\wp = 0.9, 0.8$ for $\varphi(\mu, \eta)$.

Figure 3. 3D and 2D behaviour of proposed methods solution at different fractional-orders for $\varphi(\mu, \eta)$.
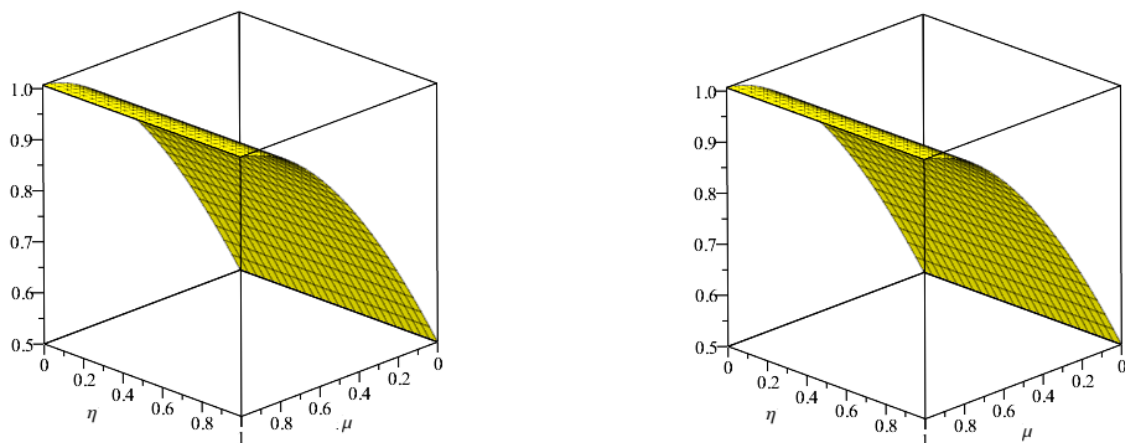
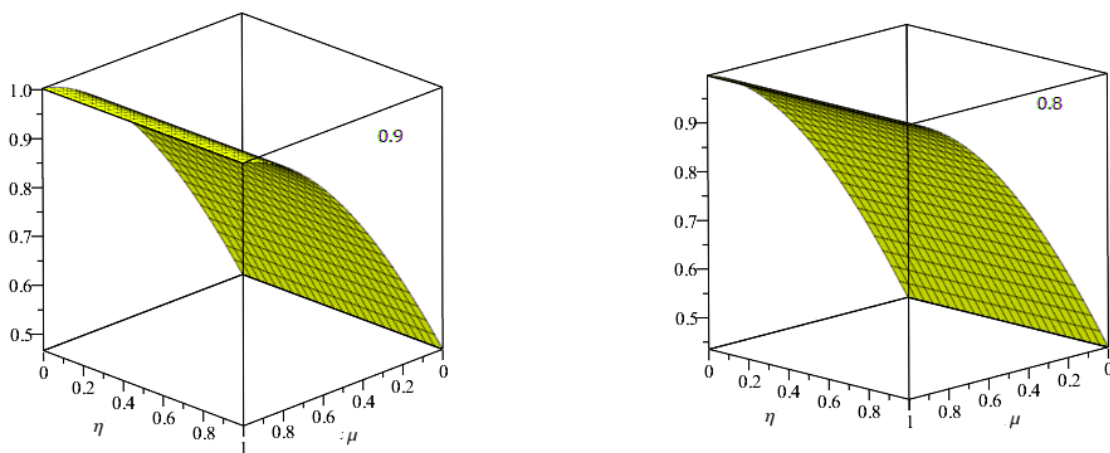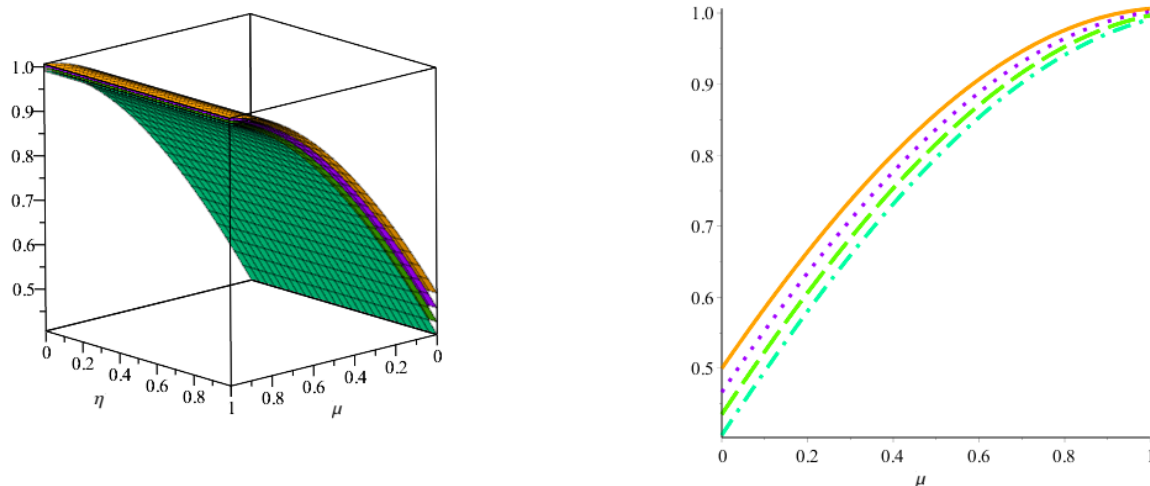**Figure 4.** The exact solution and analytical solution for $\zeta(\mu, \eta)$.

Figure 5. Proposed methods solution at $\wp = 0.9, 0.8$ for $\zeta(\mu, \eta)$.

Figure 6. 3D and 2D behaviour of proposed methods solution at different fractional-orders for $\zeta(\mu, \eta)$.

7. Conclusions

Using two novel methodologies, the solutions to several non-linear fractional Schrödinger-KdV equations are shown in this study. HPSTM and STDM are the most basic and uncomplicated approaches that can be utilized to solve FPDEs and associated systems successfully. The obtained results show that the suggested methods are reliable and that it may be applied to nonlinear coupled fractional evolution problems. They also show that the proposed methods are very efficient and powerful methodologies for obtaining nonlinear coupled time fractional differential equations solutions. The proposed methods key advantage is that it have less computing than other numerical and analytical approximate methods, resulting in a significant improvement in solving coupled fractional nonlinear equations over existing methods in the open literature. The suggested method's applications for solving the time fractional coupled Schrödinger-KdV equation demonstrate its simplicity, validity, and efficiency. As future work, we plan to extend the apply of HPSTM and STDM system to investigate higher dimensional physical applications. So, this indicates that the aforesaid approach is a remarkably and proper potent approach in achieving the analytical and approximate results of many linear and nonlinear fractional problems arising in engineering and applied physics.

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgment

References

1. V. Prajapati, R. Meher, A robust analytical approach to the generalized Burgers-Fisher equation with fractional derivatives including singular and non-singular kernels, *J. Ocean Eng. Sci.*, 2022. https://doi.org/10.1016/j.joes.2022.06.035

2. L. Verma, R. Meher, Z. Avazzadeh, O. Nikan, Solution for generalized fuzzy fractional Kortewege-de Varies equation using a robust fuzzy double parametric approach, *J. Ocean Eng. Sci.*, 2022. https://doi.org/ 10.1016/j.joes.2022.04.026

3. M. Alqhtani, K. Saad, W. Weera, W. Hamanah, Analysis of the fractional-order local Poisson equation in fractal porous media, *Symmetry*, **14** (2022), 1323. https://doi.org/ 10.3390/sym14071323

4. M. Al-Sawalha, R. Agarwal, R. Shah, O. Ababneh, W. Weera, A reliable way to deal with fractional-order equations that describe the unsteady flow of a polytropic gas, *Mathematics*, **10** (2022), 2293. https://doi.org/ 10.3390/math10132293

5. S. Mukhtar, S. Noor, The numerical investigation of a fractional-order multi-dimensional model of Navier-Stokes equation via novel techniques, *Symmetry*, **14** (2022), 1102. https://doi.org/ 10.3390/sym14061102

6. N. Shah, Y. Hamed, K. Abualnaja, J. Chung, A. Khan, A comparative analysis of fractional-order Kaup-Kupershmidt equation within different operators, *Symmetry*, **14** (2022), 986. https://doi.org/ 10.3390/sym14050986

7. N. Shah, H. Alyousef, S. El-Tantawy, J. Chung, Analytical investigation of fractional-order Korteweg-De-Vries-Type equations under Atangana-Baleanu-Caputo operator: Modeling nonlinear waves in a plasma and fluid, *Symmetry*, **14** (2022), 739. https://doi.org/ 10.3390/sym14040739

8. N. Aljahdaly, A. Akgul, R. Shah, I. Mahariq, J. Kafle, A comparative analysis of the fractional-order coupled Korteweg-De Vries equations with the Mittag-Leffler law, *J. Math.*, **2022** (2022), 1–30. https://doi.org/ 10.1155/2022/8876149

9. H. Ismael, H. Bulut, H. Baskonus, W-shaped surfaces to the nematic liquid crystals with three nonlinearity laws, *Soft Comput.*, **25** (2020), 4513–4524. https://doi.org/ 10.1007/s00500-020-05459-6

10. M. Yavuz, T. Sulaiman, A. Yusuf, T. Abdeljawad, The Schrodinger-KdV equation of fractional order with Mittag-Leffler nonsingular kernel, *Alex. Eng. J.*, **60** (2021), 2715–2724. https://doi.org/ 10.1016/j.aej.2021.01.009

11. K. Safare, V. Betageri, D. Prakasha, P. Veeresha, S. Kumar, A mathematical analysis of ongoing outbreak COVID-19 in India through nonsingular derivative, *Numer. Meth. Part. D. E.*, **37** (2020), 1282–1298. https://doi.org/ 10.1002/num.22579

12. H. Ismael, H. Baskonus, H. Bulut, Abundant novel solutions of the conformable Lakshmanan-Porsezian-Daniel model, *Discrete Cont. Dyn.-S.*, **14** (2021), 2311. https://doi.org/ 10.3934/dcdss.2020398 .

13. H. Yasmin, N. Iqbal, A comparative study of the fractional coupled Burgers and Hirota-Satsuma KdV equations via analytical techniques, *Symmetry*, **14** (2022), 1364. https://doi.org/ 10.3390/sym14071364

14. R. Alyusof, S. Alyusof, N. Iqbal, M. Arefin, Novel evaluation of the fractional acoustic wave model with the exponential-decay kernel, *Complexity*, **2022** (2022), 1–14. https://doi.org/ 10.1155/2022/9712388

15. M. Yavuz, T. Sulaiman, A. Yusuf, T. Abdeljawad, The Schrodinger-KdV equation of fractional order with Mittag-Leffler nonsingular kernel, *Alex. Eng. J.*, **60** (2021), 2715–2724. https://doi.org/ 10.1016/j.aej.2021.01.009

16. H. Triki, A. Biswas, Dark solitons for a generalized nonlinear Schrodinger equation with parabolic law and dual-power law nonlinearities, *Math. Meth. Appl. Sci.*, **34** (2011), https://doi.org/ 10.1002/mma.1414

17. L. Zhang, J. Si, New soliton and periodic solutions of (1+2)-dimensional nonlinear Schrodinger equation with dual-power law nonlinearity, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 2747–2754. https://doi.org/ 10.1016/j.cnsns.2009.10.028

18. Q. Xu, S. Songhe, Numerical analysis of two local conservative methods for two-dimensional nonlinear Schrodinger equation, *SCIENTIA SINICA Math.*, **48** (2017), 345. https://doi.org/ 10.1360/scm-2016-0308

19. J. C. Bronski, L. D. Carr, B. Deconinck, J. N. Kutz, Bose-Einstein condensates in standing waves: The cubic nonlinear Schrödinger equation with a periodic potential, *Phys. Rev. Lett.*, **86** (2001), 1402–1405. https://doi.org/10.1103/PhysRevLett.86.1402

20. A. Trombettoni, A. Smerzi, Discrete solitons and breathers with dilute Bose-Einstein condensates, *Phys. Rev. Lett.*, **86** (2001), 2353–2356. https://doi.org/ 10.1103/physrevlett.86.2353

21. A. Biswas, Quasi-stationary non-Kerr law optical solitons, *Opt. Fiber Technol.*, **9** (2003), 224–259. https://doi.org/ 10.1016/s1068-5200(03)00044-0

22. M. Eslami, M. Mirzazadeh, Topological 1-soliton solution of nonlinear Schrödinger equation with dual-power law nonlinearity in nonlinear optical fibers, *Eur. Phys. J. Plus*, **128** (2013). https://doi.org/ 10.1140/epjp/i2013-13140-y

23. A. Hyder, The influence of the differential conformable operators through modern exact solutions of the double Schrödinger-Boussinesq system, *Phys. Scripta*, 2021. https://doi.org/ 10.1088/1402-4896/ac169f

24. G. Adomian, R. Rach, Inversion of nonlinear stochastic operators, *J. Math. Anal. Appl.*, **91** (1983), 39–46. https://doi.org/ 10.1016/0022-247x(83)90090-2

25. S. Alinhac, M. Baouendi, A counterexample to strong uniqueness for partial differential equations of Schrodinger's type, *Commun. Part. Diff. Eq.*, **19** (1994), 1727–1733. https://doi.org/ 10.1080/03605309408821069

26. J. H. He, Homotopy perturbation technique, *Comput. Meth. Appl. Mech. Eng.,* **178** (1999), 257–262. https://doi.org/ 10.1016/s0045-7825(99)00018-3

27. J. H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, *Int. J. Non-Linear Mech.,* **35** (2000), 743. https://doi.org/ 10.1016/s0020-7462(98)00085-7

28. D. D. Ganji, M. Rafei, Solitary wave solutions for a generalized Hirota Satsuma coupled KdV equation by homotopy perturbation method, *Phys. Lett. A,* **356** (2006), 131–137. https://doi.org/ 10.1016/j.physleta.2006.03.039

29. A. M. Siddiqui, R. Mahmood, Q. K. Ghori, Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder, *Phys. Lett. A*, **352** (2006), 404–410. https://doi.org/ 10.1016/j.physleta.2005.12.033

30. Y. Qin, A. Khan, I. Ali, M. Al Qurashi, H. Khan, R. Shah, D. Baleanu, An efficient analytical approach for the solution of certain fractional-order dynamical systems, *Energies*, **13** (2020), 2725. https://doi.org/ 10.3390/en13112725

31. M. Alaoui, R. Fayyaz, A. Khan, R. Shah, M. Abdo, Analytical investigation of Noyes-Field model for time-fractional Belousov-Zhabotinsky reaction, *Complexity*, **2021** (2021), 1–21. https://doi.org/ 10.1155/2021/3248376

32. J. H. He, Y. O. El-Dib, Homotopy perturbation method for Fangzhu oscillator, *J. Math. Chem.*, **58** (2020), 2245–2253. https://doi.org/ 10.1007/s10910-020-01167-6

33. J. Honggang, Z. Yanmin, Conformable double Laplace-Sumudu transform decomposition method for fractional partial differential equations, *Complexity*, **2022** (2022), 1–8. https://doi.org/ 10.1155/2022/7602254

34. L. Akinyemi, O. S. Iyiola, Exact and approximate solutions of time-fractional models arising from physics via Shehu transform, *Math. Meth. Appl. Sci.*, **43** (2020), 7442–7464. https://doi.org/ 10.1002/mma.6484

35. L. Ali, R. Shah, W. Weera, Fractional view analysis of Cahn-Allen equations by new iterative transform method, *Fractal Fract.*, **6** (2022), 293. https://doi.org/ 10.3390/fractalfract6060293

36. S. Maitama, W. Zhao, Homotopy perturbation Shehu transform method for solving fractional models arising in applied sciences, *J. Appl. Math. Comput. Mech.*, **20** (2021), 71–82. https://doi.org/ 10.17512/jamcm.2021.1.07

37. M. Liaqat, A. Khan, M. Alam, M. Pandit, S. Etemad, S. Rezapour, Approximate and Closed-Form solutions of Newell-Whitehead-Segel eEquations via modified conformable Shehu transform decomposition method, *Math. Probl. Eng.*, **2022** (2022), 1–14. https://doi.org/ 10.1155/2022/6752455