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Research article

Study of weak solutions of variational inequality systems with degenerate parabolic operators and quasilinear terms arising American option pricing problems

Jia Li and Changchun Bi*

School of Management, Shenyang Urban Construction University, Shenyang, Liaoning, 110167, China

* **Correspondence:** Email: bichangchun_1983@163.com.

Abstract: In this paper, we study variational inequality systems with quasilinear degenerate parabolic operators in a bounded domain. As a series of penalty problems, the existence of the solutions in the weak sense is proved by a limit process. The uniqueness of the solution is also proved.

Keywords: parabolic variational inequality; weak solution; penalty problem; existence; uniqueness; Americian option pricing
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1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded simple connected domain, $0 < T < \infty$ and $Q_T = \Omega \times [0, T]$. We study the following parabolic systems

$$\begin{cases} \min\{L_i u_i - f_i(x, t, u_1, u_2), u_i - u_{i,0}\} = 0, & (x, t) \in Q_T, \\ u_i(0, x) = u_{i0}(x), & x \in \Omega, \\ u_i(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T), \end{cases} \tag{1.1}$$

with quasilinear degenerate parabolic operators, where

$$L_i u_i = \frac{\partial u_i}{\partial t} - \operatorname{div}(|\nabla u_i|^{p_i - 2} \nabla u_i), i = 1, 2.$$

The initial boundary value problem of variational inequalities arise in many application in pricing American options and their derivatives . Through the risk neutral strategy, the intrinsic value of many American options can ultimately be attributed to the solution of a variational inequality in the Black-Scholes models. The author refers to [1–3] and the references their in.

The nonexistence, existence and uniqueness theory for weak solutions of parabolic systems were studied by many existing works, see e.g., [4–6]. In particular, Hassnaoui and Idrissi [6] studied the existence and uniqueness of weak solutions for a nonlinear parabolic system with non-degenerate case of (1.1). Escher, Laurencot and Matioc in [7] proved the global existence of nonnegative weak solutions to a degenerate parabolic system without quasilinear terms in (1.1). Furthermore, the authors showed that these weak solutions converge at an exponential rate.

In recent years, there has been tremendous interest in developing existence and uniqueness theory for weak solutions of parabolic variational inequality (see, for example, [3, 8–14] and the references therein). In 2014, the authors in [9] discussed the problem

$$\begin{cases} \min\{u_t - Lu - F(u,x,t), u(x,t) - u_0(x)\} = 0 \text{ in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \\ u(x,t) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

with second order degenerate elliptic operator

$$Lu = -u\mathrm{div}(a(u)|\nabla u|^{p(x,t)-2}\nabla u) - \gamma|\nabla u|^{p(x,t)}.$$

Under the assumptions about u_0 and F, they proved the existence and uniqueness of the weak solution. When $a(u) = 1$, and $p(x,t) = 2$, the authors in [10, 11] discussed the existence and numerical algorithm of solution. In [12], a new property of variable exponent Lebesgue and Sobolev spaces was examined. Using these properties, the authors proved the existence of the solution of some parabolic variational inequality.

To the best of our knowledge, the existence and uniqueness for multi-variable problem of parabolic variational inequalities (called variational inequality systems) were less studied. We cannot easily put the method in [10, 11] to the multi-variable case since the systems are coupled with quasilinear terms.

The aim of this paper is to study the existence and uniqueness of solution for parabolic systems with quasilinear degenerate inequalities in a bounded domain. We mainly use comparison theorem and penalty method to construct a sequence of approximation solutions with the help of monotone iteration technique. Then we obtain the existence of solutions to the system (1.1) by a standard limiting process.

The paper is organized as follows. In Section 2, we present our main theorems. Section 3 gives some estimates about penalty problems to prove our main results. Section 4 analyses the existence and uniqueness of solutions to variational inequality system (1.1).

2. The main results of weak solutions

In spirit of [3] and [9], we introduce the following maximal monotone graph

$$G(x) = \begin{cases} 0, & x > 0, \\ 1, & x = 0. \end{cases} \tag{2.1}$$

The purpose of the paper is to obtain the existence and uniqueness of weak solutions of (1.1), and the weak solution is defined as follows.

Definition 2.1. Function $\{(u_1, \xi_1), (u_2, \xi_2)\}$ is called a generalized solution of the systems (1.1) if $u_i \in L^\infty(Q_T) \cap L(0, T, W_0^{1,p_i}(\Omega))$, $\partial_t u_i \in L^2(\Omega_T)$, $\xi_i \in L^\infty(0, T; L^\infty(\Omega))$, $i = 1, 2$, and satisfies (a) $u_i(x,t) \geq$

$u_{i0}(x)$, (b) $u_i(x,0) = u_{i0}(x)$, (c) $\xi_i \in G(u_i - u_{i0})$, (d) for every test-function $\varphi_i \in C_0^1(\bar{Q}_T)$

$$\begin{aligned}&\int\int_{Q_T} (-u_i \cdot \partial_t\varphi_i + |\nabla u_i|^{p_i-2}\nabla u_i \nabla\varphi_i)\mathrm{d}x\mathrm{d}t - \int_{Q_T} u_{i0} \cdot \varphi_i(x,0)\mathrm{d}x\\ &= \int\int_{Q_T} f_i(x,t,u_1,u_2) \cdot \varphi_i \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega \xi_i \cdot \varphi_i \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.2}$$

Here $C_0^1(\bar{\Omega}_T)$ is the space of all continuous and differentiable functions satisfying

$$\varphi_i(x,T) = 0 \text{ for } (x,t) \in \partial\Omega \times (0,T), i = 1,2.$$

Condition (d) of Definition 2.1 implies that

$$\begin{aligned}&\int_0^t \int_\Omega (-u_i \cdot \partial_t\varphi_i + |\nabla u_i|^{p_i-2}\nabla u_i \nabla\varphi_i)\mathrm{d}x\mathrm{d}t + \int_{Q_T} u_i(x,t) \cdot \varphi_i(x,t)\mathrm{d}x - \int_{\Omega_T} u_{i0} \cdot \varphi_i(x,0)\mathrm{d}x\\ &= \int_0^t \int_\Omega f_i(x,t,u_1,u_2) \cdot \varphi_i \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega \xi_i \cdot \varphi_i \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.3}$$

We introduce the constrains to the nonlinear functions $f_i, i = 1,2$ involved in this paper as follows.

Definition 2.2. A function $f = f(u_1,u_2)$ is quasimonotone nondecreasing (resp., nonincreasing) if for fixed u_1 (or u_2), f is nondecreasing (resp., nonincreasing) in u_2 (or u_1).

To study the problem (1.1), we make the following assumptions:

(H1) $f_i(x,t,u_1,u_2)$ is quasimonotonically nondecreasing for u_1, u_2, $i = 1,2$.

(H2) $f_i(x,t,u_1,u_2) \in C(\Omega \times [0,T] \times R^2)$, and there exists a nonnegative function $g(s) \in C^1(\mathrm{R})$ such that

$$|f_i(x,t,u_1,u_2)| \le \min\{g(u_1), g(u_2)\}.$$

Our main results are present as follows:

Theorem 2.1. Let $(H1)$ and $(H2)$ be satisfied, and $u_{i0} \in L^\infty(\Omega_T) \cap W_0^{1,p_i}(\Omega)$,$i = 1,2$. Then problem (1.1) has a solution $u = (u_1,u_2)$ in the sense of Definition 2.2.

Theorem 2.2. Assume that $f = (f_1,f_2)$ is Lipschitz continuous in (u_1,u_2), then the solution of problem (1.1) is unique.

3. A penalty problem

To prove the theorem, we consider the following penalty problem

$$\begin{cases} L_{i\varepsilon}u_{i\varepsilon} = f_i(x,t,u_{1\varepsilon},u_{2\varepsilon}) - \beta_\varepsilon(u_{i\varepsilon} - u_{i0}), & (x,t) \in Q_T,\\ u_{i\varepsilon}(x,0) = u_{i0\varepsilon}(x) = u_{i0} + \varepsilon, & x \in \Omega,\\ u_{i\varepsilon}(x,t) = \varepsilon, & (x,t) \in \partial Q_T,\end{cases} \tag{3.1}$$

where

$$L_{i\varepsilon}u_{i\varepsilon} = \frac{\partial u_{i\varepsilon}}{\partial t} - \mathrm{div}\Big(\big(|\nabla u_{i\varepsilon}|^2 + \varepsilon\big)^{\frac{p_i-2}{2}} \nabla u_{i\varepsilon}\Big).$$

Here, $\beta_\varepsilon(\cdot)$ is the penalty function satisfying

$$\begin{aligned}&\varepsilon \in (0,1), \beta_\varepsilon(\cdot) \in C^2(R), \beta_\varepsilon(x) \le 0, \beta_\varepsilon(0) = -1,\\ &\beta_\varepsilon'(x) \ge 0, \beta_\varepsilon''(x) \le 0, \ \lim_{\varepsilon\to 0+} \beta_\varepsilon(x) = \begin{cases} 0, & x > 0,\\ -1, & x = 0.\end{cases}\end{aligned} \tag{3.2}$$

It is worth noting that when $u_i > u_{i,0}$, $L_i u_i - f_i(x,t,u_1,u_2) = 0$, and when $u_i = u_{i,0}$, one gets $L_i u_i \geq f_i(x,t,u_1,u_2)$ in (1.1). In (3.1), $\beta_\varepsilon(u_{i\varepsilon} - u_{i0})$ plays a similar role. When $u_{i\varepsilon} > u_{i0} + \varepsilon$,

$$L_{i\varepsilon} u_{i\varepsilon} - f_i(x,t,u_{1\varepsilon},u_{2\varepsilon}) = -\beta_\varepsilon(u_{i\varepsilon} - u_{i0}) = 0,$$

and when $u_{i0} \leq u_{i\varepsilon} \leq u_{i0} + \varepsilon$, we have

$$L_{i\varepsilon} u_{i\varepsilon} - f_i(x,t,u_{1\varepsilon},u_{2\varepsilon}) = -\beta_\varepsilon(u_{i\varepsilon} - u_{i0}) \geq 0.$$

With a similar method as in [8], we can prove that regularized problem has a unique weak solution

$$u_i \in L^\infty(Q_T) \cap L(0,T,W_0^{1,p_i}(\Omega)), \partial_t u_i \in L^2(Q_T), i = 1,2,$$

satisfying the following integral identities

$$\begin{aligned}&\int_\Omega \partial_t u_i \cdot \varphi_i \mathrm{d}x + \int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i \nabla\varphi_i \mathrm{d}x + \int_\Omega \beta_\varepsilon(u_{i\varepsilon} - u_{i0})\varphi_i \mathrm{d}x\\ &= \int_{Q_T} f_i(x,t,u_1,u_2)\cdot\varphi_i \mathrm{d}x\end{aligned} \tag{3.3}$$

with $\varphi_i \in C^1(\bar{\Omega}_T)$ and $t \in (0,T)$.

We start with two preliminary results that will be used several times henceforth.

Lemma 3.1. [[15], Lemma2.1.] Let $M(s) = |s|^{p(x,t)-2}s$, then $\forall \xi,\eta \in \mathrm{R}^N$

$$(M(\xi) - M(\eta))\cdot(\xi-\eta) \geq \begin{cases} 2^{-p}|\xi-\eta|^p, & 2 \leq p < \infty,\\ (p-1)|\xi-\eta|^2(|\xi|^p + |\eta|^p)^{\frac{p-2}{p}}, & 1 \leq p < 2.\end{cases}$$

Lemma 3.2. (Comparison principle) Assume u_i and v_i are in $L^{p_i}(0,T;W^{1,p_i}(\Omega))$. If $L_{i\varepsilon}u_i \geq L_{i\varepsilon}v_i$ in Q_T and $u_i(x,t) \leq v_i(x,t)$ on ∂Q_T, then $u_i(x,t) \leq v_i(x,t)$ in Q_T, $i = 1,2$.

Proof. We argue by contradiction. Suppose $u_i(x,t)$ and $v_i(x,t)$ satisfy $L_{i\varepsilon}u_i \geq L_{i\varepsilon}v_i$ in Q_T, and there is a $\delta > 0$ such that for some $0 < \tau \leq T$, $w_i = u_i - v_i > \delta$ on the set

$$\Omega_\delta = \Omega \cap \{x : w_i(x,t) > \delta\}$$

and $|\Omega_\delta| > 0$, $i = 1,2$. Let

$$F_\varepsilon(\xi) = \begin{cases} 2\cdot\varepsilon^{-\frac{1}{2}} - 2\cdot\xi^{-\frac{1}{2}}, & \text{if } \xi > \varepsilon,\\ 0, & \text{if } \xi \leq \varepsilon,\end{cases}$$

where$\delta > \varepsilon > 0$. Since $F_\varepsilon(w_i) \leq 0$, we multiply $L_{i\varepsilon}u_i \geq L_{i\varepsilon}v_i$ by $F_\varepsilon(w_i)$ and integrate in Q_τ to have

$$\int\int_{Q_\tau}\frac{\partial}{\partial t}w_i\cdot F_\varepsilon(w_i)\mathrm{d}x\mathrm{d}t + \int\int_{Q_\tau}[\left(|\nabla u_{i\varepsilon}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}\nabla u_{i\varepsilon} - \left(|\nabla v_{i\varepsilon}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}\nabla v_{i\varepsilon}]\nabla F_\varepsilon(w_i)\mathrm{d}x\mathrm{d}t \leq 0$$

or equivalently

$$J_1 + J_2 \leq 0, \tag{3.4}$$

where $Q_{\tau,\varepsilon} = \{(x,t) \in Q_T | w > \varepsilon\}$,

$$J_1 = \int\int_{Q_{\tau,\varepsilon}}\frac{\partial}{\partial t}w_i\cdot F_\varepsilon(w_i)\mathrm{d}x\mathrm{d}t,$$

$$J_2=\frac{1}{2}\int\int_{Q_{\tau,\varepsilon}} w_i^{-\frac{3}{2}}[\left((|\nabla u_{i\varepsilon}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}\nabla u_{i\varepsilon}-\left(|\nabla v_{i\varepsilon}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}\nabla v_{i\varepsilon}]\nabla w_i \mathrm{d}x\mathrm{d}t.$$

Now let $t_0=\inf\{t\in(0,\tau]: w>\varepsilon\}$, then we estimate J_1 as follows

$$\begin{aligned} J_1 &= \int\int_{Q_{\tau,\theta}} \tfrac{\partial}{\partial t} w_i F_\varepsilon(w_i)\mathrm{d}x\mathrm{d}t = \int_\Omega (\int_0^{t_0} \tfrac{\partial}{\partial t} w_i F_\varepsilon(w_i)\mathrm{d}t + \int_0^{t_0} \tfrac{\partial}{\partial t} w_i F_\varepsilon(w_i)\mathrm{d}t)\mathrm{d}x \\ &\geq \int_\Omega \int_\varepsilon^{w_i} F_\varepsilon(s)\mathrm{d}s\mathrm{d}x \geq \int_{\Omega_\delta}\int_\varepsilon^{w_i} F_\varepsilon(\mathrm{s})\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.5}$$

Using $\delta>\varepsilon>0$ and the function $F_\varepsilon(\cdot)$ to (3.5), we have

$$J_1 \geq \int_{\Omega_\theta} (w_i-\varepsilon)F_\varepsilon(\mathrm{s})\mathrm{d}x \geq (w_i-\varepsilon)F_\varepsilon(\varepsilon)|\Omega_\delta|. \tag{3.6}$$

By the virtue of the first inequality of Lemma 3.1, we use $w_i=u_i-v_i>\delta$ to arrive at

$$J_2 \geq 2^{-(p_i+1)}\int\int_{Q_{\tau,\theta}} w_i^{-\frac{3}{2}}|\nabla w_i|^{p_i}\mathrm{d}x\mathrm{d}t \geq 0. \tag{3.7}$$

Since $v_i\in L^{p_i}(0,T;W^{1,p_i}(\Omega))$, and we plug the above estimates (3.6) and (3.7) into (3.4) and drop the nonnegative terms, we arrive at

$$(\delta-\varepsilon)\varepsilon^{-\frac{1}{2}}|\Omega_\delta|<\tilde{C}.$$

Note that $\lim\limits_{\varepsilon\to0}(\delta-\varepsilon)\varepsilon^{-\frac{1}{2}}|\Omega_\delta|=+\infty$, we obtain a contradiction. This means that $|\Omega_\delta|=0$ and $w_i\leq 0$ a.e. in Q_τ, $i=1,2$.□

Lemma 3.3. Let be weak solutions of (3.1). Then

$$u_{i0\varepsilon}\leq u_{i\varepsilon}\leq |u_{i0}|_\infty+\varepsilon, i=1,2, \tag{3.8}$$

$$u_{i\varepsilon_1}\leq u_{i\varepsilon_2} \text{ for } \varepsilon_1\leq\varepsilon_2, i=1,2, \tag{3.9}$$

where $|u_0|=\sup\limits_{x\in\Omega}|u_0(x)|$, for details, see [16, 17].

Proof. First, we prove $u_{i\varepsilon}\geq u_{i0\varepsilon}$ by contradiction. Assume $u_{i\varepsilon}\leq u_{i0\varepsilon}$ in Q_T^0, $i=1,2$, $Q_T^0\subset Q_T$. Noting $u_{i\varepsilon}\geq u_{i0\varepsilon}$ on ∂Q_T, we may assume that $u_{i\varepsilon}=u_{i0\varepsilon}$ on $\partial \mathrm{Q}_T^0$. With (3.1) and letting $t=0$, it is easy to see that

$$Lu_{i0,\varepsilon}=-\beta_\varepsilon(u_{i0,\varepsilon}-u_{i0,\varepsilon})=1, i=1,2, \tag{3.10}$$

$$Lu_{i\varepsilon}=-\beta_\varepsilon(u_{i\varepsilon}-u_{i0,\varepsilon})\leq 1, i=1,2. \tag{3.11}$$

From Lemma 3.2, we have that

$$u_{i\varepsilon}(x,t)\geq u_{i0,\varepsilon}(x) \text{ for any } (x,t)\in Q_T, i=1,2. \tag{3.12}$$

Therefore, we obtain a contradiction.

Second, we pay attention to $u_{i\varepsilon}(t,x)\leq|u_{i0}|_\infty+\varepsilon$. Applying the definition of $\beta_\varepsilon(\cdot)$ gives

$$L(|u_{i0}|_\infty+\varepsilon)=0,\ Lu_{i\varepsilon}=-\beta_\varepsilon(u_{i\varepsilon}-u_{i0,\varepsilon})\geq 0,\ i=1,2. \tag{3.13}$$

Using Lemma 3.2, (3.13) leads to

$$u_{i\varepsilon}(t,x)\leq|u_{i0}|_\infty+\varepsilon \text{ on } \partial\Omega\times(0,T),$$

$$u_{i\varepsilon}(t,x) \leq |u_{i0}|_\infty + \varepsilon \text{ in } \Omega,\ i=1,2. \tag{3.14}$$

Thus, combining (3.13) and (3.14) and repeating Lemma 3.2, we have

$$u_{i\varepsilon}(t,x) \leq |u_{i0}|_\infty + \varepsilon \text{ in } Q_T,\ i=1,2. \tag{3.15}$$

Third, we aim to prove (3.9). From (3.1), it is easy to see that

$$Lu_{i\varepsilon_1} = \beta_{\varepsilon_1}(u_{i\varepsilon_1} - u_{i0,\varepsilon_1}),\ i=1,2. \tag{3.16}$$

$$Lu_{i\varepsilon_2} = \beta_{\varepsilon_2}(u_{i\varepsilon_2} - u_{i0,\varepsilon_2}),\ i=1,2. \tag{3.17}$$

It follows by $\varepsilon_1 \leq \varepsilon_2$ and the definition of $\beta_\varepsilon(\cdot)$ that

$$\begin{aligned} &Lu_{i0,\varepsilon_2} + \beta_{\varepsilon_1}(u_{i\varepsilon_2} - u_{i0,\varepsilon}) \\ &= \beta_{\varepsilon_2}(u_{i\varepsilon_2} - u_{i0,\varepsilon}) - \beta_{\varepsilon_1}(u_{i\varepsilon_1} - u_{i0,\varepsilon}) = \beta_{\varepsilon_2}(u_{i\varepsilon_2} - u_{i0,\varepsilon}) - \beta_{\varepsilon_1}(u_{i\varepsilon_2} - u_{i0,\varepsilon}) \geq 0, \end{aligned} \tag{3.18}$$

$i = 1,2$. Combining initial and boundary condition in (3.1), we obtain that the inequality (3.9) holds by Lemma 3.2. □

Lemma 3.4. For any $(x,t) \in \Omega_T$, the solution of problem (3.1) satisfies the estimate

$$|\nabla u_{i\varepsilon}|_{L^{p_i}(Q_T)} \leq \int\int_{Q_T} \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \left|\nabla u_{i\varepsilon}\right|^2 \mathrm{d}x\mathrm{d}t \leq C, \tag{3.19}$$

where C is a constant independent of ε.

Proof. Choosing $\varphi_i = u_{i\varepsilon}$ in (3.3), we have

$$\begin{aligned} &\int\int_{Q_T} \partial_t u_{i\varepsilon} \cdot u_{i\varepsilon} - \mathrm{div}(\left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \nabla u_{i\varepsilon}) \cdot u_{i\varepsilon} \mathrm{d}x\mathrm{d}t \\ &= \int\int_{Q_T} f_i(x,t,u_{1\varepsilon},u_{2\varepsilon})u_{i\varepsilon} - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})u_{i\varepsilon}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.20}$$

It is easy to see that

$$\int\int_{Q_T} \partial_t u_{i\varepsilon} \cdot u_{i\varepsilon}\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int\int_{Q_T} \partial_t (u_{i\varepsilon})^2 \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_{Q_T} u_{i\varepsilon}(\cdot,T) - u_{i\varepsilon}(\cdot,0)\mathrm{d}x. \tag{3.21}$$

Then we substitute (3.21) into (3.20) to arrive at

$$\begin{aligned} &\int\int_{Q_T} \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \left|\nabla u_{i\varepsilon}\right|^2 \mathrm{d}x\mathrm{d}t \\ &= \int\int_{Q_T} f_i(x,t,u_{1\varepsilon},u_{2\varepsilon})u_{i\varepsilon} - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})u_{i\varepsilon}\mathrm{d}x\mathrm{d}t - \tfrac{1}{2}\int_{Q_T} u_{i\varepsilon}(\cdot,T) - u_{i\varepsilon}(\cdot,0)\mathrm{d}x. \end{aligned}$$

By (3.8) and the property of f_i,

$$\left|\int\int_{Q_T} f_i(x,t,u_{1\varepsilon},u_{2\varepsilon})u_{i\varepsilon}\mathrm{d}x\mathrm{d}t\right| \leq C. \tag{3.22}$$

Applying (3.2) and (3.8) obtains

$$\int\int_{Q_T} -\beta_\varepsilon(u_{i\varepsilon} - u_{i0})u_{i\varepsilon}\mathrm{d}x\mathrm{d}t \leq \varepsilon\,|\Omega|\,T \leq |\Omega|\,T. \tag{3.23}$$

Then Lemma 3.4 is proved by combining (3.21)–(3.23).□

Lemma 3.5. The solution of problem (3.1) satisfies the estimate

$$\left\|\partial_t u_{i\varepsilon}\right\|_{L^2(Q_T)} \leq C(p_i, T, |\Omega|), i = 1, 2. \tag{3.24}$$

Proof. From (3.3), we have that

$$\begin{aligned}
&\int\int_{Q_T} (\partial_t u_{i\varepsilon})^2 \mathrm{d}x\mathrm{d}t \\
&= -\int\int_{Q_T} \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \nabla u_{i\varepsilon} \nabla \partial_t u_{i\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\quad + \int\int_{Q_T} [f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon}) - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})] \cdot \partial_t u_{i\varepsilon} \mathrm{d}x\mathrm{d}t \\
&= -A_1 + A_2,
\end{aligned} \tag{3.25}$$

where

$$A_1 = \int\int_{Q_T} \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \nabla u_{i\varepsilon} \nabla \partial_t u_{i\varepsilon} \mathrm{d}x\mathrm{d}t,$$

$$A_2 = \int\int_{Q_T} [f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon}) - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})] \cdot \partial_t u_{i\varepsilon} \mathrm{d}x\mathrm{d}t.$$

First, we pay attention to A_1. Using some differential transform technique obtains

$$A_1 = -\frac{1}{2}\int\int_{Q_T} \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i-2}{2}} \partial_t \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right) \mathrm{d}x\mathrm{d}t = -\frac{1}{p_i}\int\int_{Q_T} \partial_t \left(|\nabla u_{i\varepsilon}|^2 + \varepsilon\right)^{\frac{p_i}{2}} \mathrm{d}x\mathrm{d}t.$$

Since $u_{i0\varepsilon}(x) = u_{i0} + \varepsilon$, then

$$A_1 \leq -\frac{1}{p_i}\int\int_{Q_T} \partial_t \left(|\nabla u_{i\varepsilon}|^{p_i}\right) \mathrm{d}x\mathrm{d}t \leq \int_\Omega |\nabla u_{i0}(\cdot, 0)|^{p_i} \mathrm{d}x. \tag{3.26}$$

Applying Holder inequalities again, we have that

$$A_2 \leq \frac{1}{2}\int\int_{Q_T} [f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon}) - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})]^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int\int_{Q_T} (\partial_t u_{i\varepsilon})^2 \mathrm{d}x\mathrm{d}t. \tag{3.27}$$

Using $(\mathrm{a} + \mathrm{b})^2 \leq 2(a^2 + b^2)$, the property of f_i and (3.2), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int\int_{Q_T} [f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon}) - \beta_\varepsilon(u_{i\varepsilon} - u_{i0})]^2 \mathrm{d}x\mathrm{d}t \\
&\leq \int\int_{Q_T} f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon})^2 \mathrm{d}x\mathrm{d}t + \int\int_{Q_T} \beta_\varepsilon(u_{i\varepsilon} - u_{i0})^2 \mathrm{d}x\mathrm{d}t \\
&\leq \int\int_{Q_T} f_i(x, t, u_{1\varepsilon}, u_{2\varepsilon})^2 \mathrm{d}x\mathrm{d}t + T|\Omega| \leq C.
\end{aligned} \tag{3.28}$$

Then, we obtain Lemma 3.5 by submitting (3.26)–(3.28) into (3.25).□

4. Proof of the main result

In this section, we are ready to prove that the system (1.1) has a unique generalized solution. By (3.18), (3.19) and (3.24) and the uniqueness of the weak limits, we know that there are functions

$$u_i \in L^\infty(Q_T) \cap L(0, T, W_0^{1,p_i}(\Omega)), \text{ as } \varepsilon \to 0,$$

such that for some subsequence of $(u_{1\varepsilon}, u_{2\varepsilon})$, denoted again by $(u_{1\varepsilon}, u_{2\varepsilon})$,

$$u_{i\varepsilon} \to u_i, f(x,t,u_{1\varepsilon},u_{2\varepsilon}) \to f(x,t,u_1,u_2) \text{ a.e. in } Q_T, \tag{4.1}$$

$$\nabla u_{i\varepsilon} \xrightarrow{w} \nabla u_i \text{ in } L^{p_i}(Q_T), \tag{4.2}$$

$$\left|\nabla u_{i\varepsilon}\right|^{p_i-2}\nabla u_{i\varepsilon} \xrightarrow{w} w_i \text{ in } L^{\frac{p_i}{p_i-1}}(Q_T), \text{ forsome } w_i, \tag{4.3}$$

$$\partial_t u_{i\varepsilon} \xrightarrow{w} \partial_t u_i \text{ in } L^2(Q_T), \tag{4.4}$$

where $\xrightarrow{w}$ stands for weak convergence, $i = 1, 2$.

Lemma 4.1. For any $(x,t) \in \Omega_T$, $w_i = |\nabla u_i|^{p_i-2}\nabla u_i$,$i = 1, 2$.

Proof. Applying triangle inequality $|a+b| \le |a| + |b|$, $(a, b \in \mathrm{R})$, it is easy to see that

$$\begin{aligned}
&\int\int_{Q_T} \left||\nabla u_{i\varepsilon}|^{p_i-2}\nabla u_{i\varepsilon} - |\nabla u_i|^{p_i-2}\nabla u_i\right| \mathrm{d}x\mathrm{d}t \\
&\le \int\int_{Q_T} \left||\nabla u_{i\varepsilon}|^{p_i-2} - |\nabla u_i|^{p_i-2}\right| \cdot |\nabla u_{i\varepsilon}|\mathrm{d}x\mathrm{d}t + \int\int_{Q_T} |\nabla u_i|^{p_i-2} \cdot \left|\nabla u_{i\varepsilon} - \nabla u_i\right| \mathrm{d}x\mathrm{d}t \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \int\int_{Q_T} \left||\nabla u_{i\varepsilon}|^{p_i-2} - |\nabla u_i|^{p_i-2}\right| \cdot |\nabla u_{i\varepsilon}|\mathrm{d}x\mathrm{d}t, \ I_2 = \int\int_{Q_T} |\nabla u_i|^{p_i-2} \cdot \left|\nabla u_{i\varepsilon} - \nabla u_i\right| \mathrm{d}x\mathrm{d}t.$$

By mean of the inequality $|a^r - b^r| \le |a-b|^r$, $(r \in [0,1], a, b > 0)$, we have

$$I_1 = \int\int_{Q_T} \left|(|\nabla u_{i\varepsilon}|^{p_i})^{\frac{p_i-2}{p_i}} - (|\nabla u_i|^{p_i})^{\frac{p_i-2}{p_i}}\right| \cdot |\nabla u_{i\varepsilon}|\mathrm{d}x\mathrm{d}t \le \int\int_{Q_T} \left||\nabla u_{i\varepsilon}|^{p_i} - |\nabla u_i|^{p_i}\right|^{\frac{p_i-2}{p_i}} \cdot |\nabla u_{i\varepsilon}|\mathrm{d}x\mathrm{d}t.$$

Applying Holder inequality and (4.2), we have

$$I_1 \le \left(\int\int_{Q_T} \left||\nabla u_{i\varepsilon}|^{p_i} - |\nabla u_i|^{p_i}\right| \mathrm{d}x\mathrm{d}t\right)^{\frac{p_i-2}{p_i}} \cdot \left(\int\int_{Q_T} |\nabla u_{i\varepsilon}|^{\frac{p_i}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p_i}} \to 0 \ \ (\varepsilon \to 0).$$

Now we pay our attention to I_2. From (4.2), we know that $\nabla u_i \in L^{p_i}(Q_{T_1})$. By (4.1), we may conclude that

$$\nabla u_{i\varepsilon} \to \nabla u_i \quad \text{a.e. in } Q_T.$$

If not, there exists a measurable domain O_T satisfying

$$\int\int_{\mathrm{O}_T} \left|\nabla u_{i\varepsilon} \to \nabla u_i\right| \mathrm{d}x\mathrm{d}t > 0.$$

Then, we obtain a contradiction with (4.2). Applying Holder inequality, we have

$$I_2 = \left(\int\int_{Q_T} |\nabla u_i|^{p_i}\mathrm{d}x\mathrm{d}t\right)^{\frac{p_i-2}{p_i}} \cdot \left(\int\int_{Q_T} \left|\nabla u_{i\varepsilon} - \nabla u_i\right|^{\frac{p_i}{2}}\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p_i}} \to 0 \ \ (\varepsilon \to 0).$$

Hence Lemma 4.1 is proved.□

This proves that any weak convergence subsequence of $|\nabla u_{i\varepsilon}|^{p_i-2}\partial_{x_l}u_{i\varepsilon}$ will have $\partial_{x_l}w_i$ as its weak limit and hence by a standard argument, and we have that as $k \to \infty$,

$$\left|\nabla u_{i\varepsilon}\right| \partial_{x_l}u_{i\varepsilon} \xrightarrow{w} |\nabla u_i|^{p_i-2}\partial_{x_l}u_i \text{ in } L^{\frac{p_i}{p_i-1}}(Q_T). \tag{4.5}$$

Combining the above results, we have, in fact, proved that $u = (u_1, u_2)$ is a generalized solution of (1.1).

Lemma 4.2. For any $(x,t) \in \Omega_T$, it hold that

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \in G(u - u_0) \text{ as } \varepsilon \to 0. \tag{4.6}$$

Proof. Using (3.8) and the definition of β_ε, we have

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi as\varepsilon \to 0.$$

Now, we prove $\xi \in G(u - u_0)$. According to the definition of $G(\cdot)$, we only need to prove that if $u(x_0,t_0) > u_0(x_0)$,

$$\xi(x_0,t_0) = 0.$$

In fact, if $u(x_0,t_0) > u_0(x_0)$, there are a constant $\lambda > 0$ and a δ-neighborhood $B_\delta(x_0,t_0)$ such that if ε is small enough, we have

$$u_\varepsilon(x,t) \geq u_0(x) + \lambda, \forall (x,t) \in B_\delta(x_0,t_0).$$

Thus, if ε is small enough, we have

$$0 \geq \beta_\varepsilon(u_\varepsilon - u_0) \geq \beta_\varepsilon(\lambda) = 0, \forall (x,t) \in B_\delta(x_0,t_0).$$

Furthermore, it follows by $\varepsilon \to 0$ that

$$\xi(x,t) = 0, \forall (x,t) \in B_\delta(x_0,t_0).$$

Hence, (4.6) holds, and the proof of Lemma 4.3 completes.□

The proof of Theorem 2.1. Applying (3.8), (3.9), and Lemma 4.3, it is clear that

$$u(x,t) \leq u_0(x), \text{in } \Omega_T,\ u(x,0) = u_0(x), \text{in } \Omega,\ \xi \in G(u - u_0),$$

thus (a), (b), and (c) of Definition 1.1 hold. The rest arguments of existence part are the same as those of Theorem 2.1 in [8] by a standard limiting process. Thus, we omit the details.

The proof of Theorem 2.2. The following is the uniqueness result to the solution of the system. Assume that $\{(u_1,\xi_1),(u_2,\xi_2)\}$ and $\{(v_1,\zeta_1),(v_2,\zeta_2)\}$ are two solutions of (1.1). Let $\varphi_i = u_i - v_i$ in Definition 2.1, $i = 1, 2$, then by (2.3),

$$\begin{aligned}&\int_0^t \int_\Omega -u_i\partial_t\varphi_i + |\nabla u_i|^{p_i-2}\nabla u_i\nabla\varphi_i \mathrm{d}x\mathrm{d}t + \int_\Omega u_i(x,t)\varphi_i(x,t)\mathrm{d}x - \int_\Omega u_i(x,0)\varphi_i(x,0)\mathrm{d}x\\ &= \int_0^t \int_\Omega f_i(x,t,u_1,u_2)\varphi_i \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega \xi_i \cdot \varphi_i \mathrm{d}x\mathrm{d}t, \text{a.e. } t \in (0,T),\end{aligned}$$

$$\begin{aligned}&\int_0^t \int_\Omega -v_i\partial_t\varphi_i + |\nabla v_i|^{p_i-2}\nabla v_i\nabla\varphi_i \mathrm{d}x\mathrm{d}t + \int_\Omega v_i(x,t)\varphi_i(x,t)\mathrm{d}x - \int_\Omega v_i(x,0)\varphi_i(x,0)\mathrm{d}x\\ &= \int_0^t \int_\Omega f_i(x,t,v_1,v_2)\varphi_i \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega \zeta_i \cdot \varphi_i \mathrm{d}x\mathrm{d}t, \text{a.e. } t \in (0,T),\end{aligned}$$

$i = 1, 2$. Subtracting the 2 equations, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega {\varphi_i}^2 \mathrm{d}x = &\int_0^{\mathrm{t}}\int_\Omega (|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla v_i|^{p_i-2}\nabla v_i)\cdot\nabla\varphi_i \mathrm{d}x\mathrm{d}t\\ &+\int_0^{\mathrm{t}}\int_\Omega (f_i(x,t,u_1,u_2) - f_i(x,t,v_1,v_2))\cdot\varphi_i \mathrm{d}x\mathrm{d}t\\ &-\int_0^{\mathrm{t}}\int_\Omega (\xi_i - \zeta_i)\cdot\varphi_i \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now we prove

$$(\xi_i - \zeta_i)\cdot\varphi_i \le 0,\quad |(\xi_i - \zeta_i)\cdot\varphi_i| \le \varphi_i. \tag{4.7}$$

On one hand, if $u_i(x,t) > v_i(x,t)$, then using Lemma2.1 yields

$$u_i(x,t) > u_{i0}(x).$$

From (2.1) and above inequality, it is easy to see that

$$\xi_i = 0 \le \zeta_i. \tag{4.8}$$

Combining (2.1) and (4.8),

$$(\xi_i - \zeta_i)\cdot\varphi_i = (\xi_i - \zeta_i)\cdot(u_i - v_i) \le 0,\ (\xi_i - \zeta_i)\cdot\varphi_i \ge -\varphi_i. \tag{4.9}$$

On the other hand, if $u_i(x,t) < v_i(x,t)$, it is easy to have that$\xi_i \ge 0 = \zeta_i$.In this case,

$$(\xi_i - \zeta_i)\cdot\varphi_i = (\xi_i - \zeta_i)\cdot(u_i - v_i) \le 0,\ (\xi_i - \zeta_i)\cdot\varphi_i \ge \varphi_i. \tag{4.10}$$

Combining (4.9) and (4.10), (4.7) still holds.

Using the previous inequality and the Lipschitz condition, a simple calculation shows that

$$\begin{aligned}&\int_\Omega |u_1 - v_1|^2 + |u_2 - v_2|^2 \mathrm{d}x\\ &\le 2\mathrm{K}\int_0^{\mathrm{t}}\int_\Omega (|u_1 - v_1| + |u_2 - v_2|)^2 \mathrm{d}x\mathrm{d}t + \int_0^{\mathrm{t}}\int_\Omega |u_1 - v_1| + |u_2 - v_2| \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Furthermore, it follows by $(\mathrm{a}+\mathrm{b})^2 \le 2(a^2 + b^2)$ and Holder inequalities that

$$\int_\Omega |u_1 - v_1|^2 + |u_2 - v_2|^2 \mathrm{d}x \le (2K + \frac{1}{2}T|\Omega|)\int_0^{\mathrm{t}}\int_\Omega |u_1 - v_1|^2 + |u_2 - v_2|^2 \mathrm{d}x\mathrm{d}t.$$

Setting $F(t) = \int_0^{\mathrm{t}}\int_\Omega |u_1 - v_1|^2 + |u_2 - v_2|^2 \mathrm{d}x\mathrm{d}t$, then the above inequality can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) \le (2K + \frac{1}{2}T|\Omega|)\cdot F(t).$$

A standard argument show that $F(t) = 0$ since $F(0) = 0$, and hence $u_i = v_i, i = 1, 2$. The proof is complete.

5. Conclusions

In this paper, we study variational inequality systems with quasilinear degenerate parabolic operators in a bounded domain

$$\begin{cases} \min\{L_i u_i - f_i(x,t,u_1,u_2), u_i - u_{i,0}\} = 0, & (x,t) \in Q_T, \\ u_i(0,x) = u_{i0}(x), & x \in \Omega, \\ u_i(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

with quasilinear degenerate parabolic inequalities, where

$$L_i u_i = \frac{\partial u_i}{\partial t} - \mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i), i = 1,2.$$

The existence and uniqueness of the solutions in the weak sense are proved by using the penalty method and the reduction method with assumptions that p_1 and p_2 are constants satisfying $p_i > 2$. However, there are some problems that have not been solved: when $1 < p_i < 2$, $p_i > 2$ or p_i is x-functions, $i = 1,2$, we cannot use Lemmas 3.1 and 3.2 to prove Lemmas 3.3–3.5. We will continue to study this problem in future.

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Conflict of interest

The authors declare no conflict of interest.

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