Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(11): 19727–19738.
DOI: 10.3934/math.20221081
Received: 24 May 2022
Revised: 27 July 2022
Accepted: 16 August 2022
Published: 06 September 2022

Research article

Majorization results for non vanishing analytic functions in different domains

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Abstract: In recent years, many authors have studied and investigated majorization results for different subclasses of analytic functions. In this paper, we give some majorization results for certain non vanishing analytic functions, whose ratios are subordinated to different domains in the open unit disk.

Keywords: univalent function; starlike function; majorization problem
Mathematics Subject Classification: Primary 05A30, 30C45, Secondary 11B65, 47B38

1. Introduction

For better understanding of the main results, here in this part of the paper we give some basic and important concepts. We start from the very basic definition, which we denote by symbol $\mathfrak{A}$. The class $\mathfrak{A}$ consists of all analytic functions in the open disk

$$\mathfrak{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

and if $f(z)$ is contained in $\mathfrak{A}$, the relations

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1$$

are satisfied. In addition, the family $\mathfrak{S} \subset \mathfrak{A}$ includes all univalent functions. The coefficient conjecture stated by Biberbach [1] in 1916 contributed to the field's emergence as a viable area of future research, despite the fact that function theory was formed in 1851. De-Branges [2] proved this conjecture in 1985. Between 1916 and 1985, many of the world's leading scholars attempted to confirm or reject the Bieberbach conjecture. As a result, they found a number of subclasses of the $\mathfrak{S}$ family of normalised univalent functions that are linked to different image domains. The $\mathfrak{S}^*$ and $\mathcal{K}$, classes of starlike and convex functions, respectively, are the most fundamental and important subclasses of the functions class $\mathfrak{S}$, which are described as

$$\mathfrak{S}^* =: \left\{ f \in \mathfrak{S} : \Re\mathrm{e} \frac{zf'(z)}{f(z)} > 0 \qquad (z \in \mathfrak{D}) \right\}$$

and

$$\mathcal{K} =: \left\{ f \in \mathfrak{S} : \Re\mathrm{e} \frac{(zf'(z))'}{f'(z)} > 0 \qquad (z \in \mathfrak{D}) \right\}.$$

Robertson [3] established the concept of quasi-subordination between holomorphic functions in 1970. Two functions $\mathcal{F}_1(z), \mathcal{F}_1(z) \in \mathfrak{A}$ are related to the relationship of quasi-subordination, indicated mathematically by $\mathcal{F}_1(z) \prec_q \mathcal{F}_2(z)$, if there exist functions $\phi(z), u(z) \in \mathfrak{A}$ so that $\frac{zf'(z)}{\phi(z)}$ is regular in $\mathfrak{D}$ with

$$|\phi(z)| \leqq 1, \quad u(0) = 0 \quad \text{and} \quad |u(z)| \leqq |z|,$$

obeying the relationship

$$\mathcal{F}_1(z) = \phi(z)\mathcal{F}_2(u(z)) \qquad (z \in \mathfrak{D}). \tag{1.1}$$

Furthermore, by selecting

$$u(z) = z \quad \text{and} \quad \phi(z) = 1,$$

we gain one of the most useful geometric function theory ideas known as subordination between regular functions. In fact, if $\mathcal{F}_2(z) \in \mathfrak{S}$, then, for $\mathcal{F}_1(z), \mathcal{F}_2(z) \in \mathfrak{A}$, the subordination relationship has

$$\mathcal{F}_1(z) \prec \mathcal{F}_2(z) \iff [\mathcal{F}_1(\mathfrak{D}) \subset \mathcal{F}_2(\mathfrak{D}) \text{ with } \mathcal{F}_1(0) = \mathcal{F}_2(0)].$$

By taking $u(z) = z$, the above definition of quasi-subordination becomes the majorization between holomorphic functions and is written mathematically by

$$\mathcal{F}_1(z) \ll \mathcal{F}_2(z) \qquad (\mathcal{F}_1(z), \mathcal{F}_2(z) \in \mathfrak{A}).$$

That is

$$\mathcal{F}_1(z) \ll \mathcal{F}_2(z),$$

if the function $\phi(z) \in \mathfrak{A}$ having the condition $|\phi(z)| \leqq 1$ such that

$$\phi(z)\mathcal{F}_2(z) = \mathcal{F}_1(z) \qquad (z \in \mathfrak{D}). \tag{1.2}$$

MacGregor [4] developed this concept in 1967. Several papers have been written in which this concept has been utilized. The work of Srivastava and Altintas [5], Goyal and Goswami [6, 7], Cho et al. [8], Li et al. [9], Aouf and Prajapat [10], Goswami and Aouf [11], El-Ashwah and Panigraht [12] and Tang et al. [13, 14] are worth noting on this subject. For some recent study on this topic, we refer the readers to see [15–19].

Ma and Minda [20] examined the general form of the family $\mathfrak{S}^*$ in 1992, and it was given by

$$\mathfrak{S}^*(\Lambda) =: \left\{ f \in \mathfrak{S} : \frac{zf'(z)}{f(z)} \prec \Lambda(z), \ (z \in \mathfrak{D}) \right\}, \tag{1.3}$$

where $\Lambda(z)$ is analytic function having the condition $\Lambda'(0) > 0$ and in $\mathfrak{D}$ its real part is greater than 0. In addition, with regard to $\Lambda(0) = 1$, the function $\Lambda(z)$ maps $\mathfrak{D}$ onto a star-like shaped area. Ma and Minda [20] investigated on growth, distortion, and covering theorems, along with other aspects. Various sub-families of the normalized holomorphic class $\mathfrak{A}$ have been explored as a particular example of class $\mathfrak{S}^*(\Lambda)$ in recent years. Some of them are listed below:

(i) Choosing $\Lambda(z)$ as
$$\Lambda(z) = \frac{1 + Mz}{1 + Nz} \qquad (-1 \leq N < M \leq 1),$$
then we obtain the class given by
$$\mathfrak{S}^*[M, N] \equiv \mathfrak{S}^*\left(\frac{1 + Mz}{1 + Nz}\right),$$
where $\mathfrak{S}^*[M, N]$ is the functions class define in [21], see also [22, 23]. Furthermore, the class $\mathfrak{S}^*(\zeta)$ given by
$$\mathfrak{S}^*(\zeta) := \mathfrak{S}^*[1 - 2\zeta, -1] \qquad (0 \leq \zeta \leq 1),$$
where $\mathfrak{S}^*(\zeta)$ is the class of starlike function of order ζ.

(ii) The following class:
$$\mathfrak{S}^*_L \equiv \mathfrak{S}^*(\Lambda(z)) \left(\Lambda(z) = \sqrt{1 + z}\right),$$
was studied in [24] by Stankiewicz and Sokól.

(iii) By taking $\Lambda(z) = 1 + \sin z$, the family $\mathfrak{S}^*(\Lambda(z))$ leads to the class $\mathfrak{S}^*_{\sin}$, which was investigated by Cho et al. [25]. On the other hand, the function class given by
$$\mathfrak{S}^*_e \equiv \mathfrak{S}^*(e^z)$$
was studied in [26] (see also [27]).

(iv) The class $\mathfrak{S}^*_R \equiv \mathfrak{S}^*(\Lambda(z))$ with $\Lambda(z) = 1 + \frac{z}{J}\frac{J+z}{J-z}$, $J = 1 + \sqrt{2}$ is studied in [28]. While the following families:
$$\mathfrak{S}^*_{\cos} =: \mathfrak{S}^*(\cos(z))$$
and
$$\mathfrak{S}^*_{\cosh} := \mathfrak{S}^*(\cosh(z))$$
were considered, respectively, by Abdullah et.al [29] and Bano and Raza [30].

For some more recent and interesting investigations on some subclasses of analytic and bi-univalent functions, we may refer the readers to see [31–34].

Now, we choose the nonvanishing holomorphic functions $h_1(z)$ and $h_2(z)$ in $\mathfrak{D}$ with

$$h_1(0) = h_2(0) = 1.$$

Then, for the classes which we described in this article, contain such function $f(z) \in \mathfrak{A}$ whose ratios $\frac{f(z)}{zq(z)}$ and $q(z)$ are subordinated to $h_1(z)$ and $h_2(z)$, respectively, for certain holomorphic function $q(z)$ with $q(0) = 1$ as

$$\frac{f(z)}{zq(z)} \prec h_1(z) \quad \text{and} \quad q(z) \prec h_2(z).$$

Instead of $h_1(z)$ and $h_2(z)$, we will now select certain specific functions. These choices are

$$h_1(z) = \cos z,$$

or

$$h_1(z) = \sqrt{1+z},$$

or

$$h_1(z) = e^z,$$

or

$$h_1(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2,$$

and

$$h_2(z) = 1 + \tanh z.$$

We now investigate the following new subfamilies by using the above-mentioned concepts:

$$\mathcal{F}_{\cos} = \left\{ f \in \mathfrak{A} : \frac{f(z)}{zq(z)} \prec \cos z \ \&\ q(z) \prec h_2(z),\ z \in \mathfrak{D} \right\}, \tag{1.4}$$

$$\mathcal{F}_{\mathfrak{S}\mathcal{L}} = \left\{ f \in \mathfrak{A} : \frac{f(z)}{zq(z)} \prec \sqrt{1+z} \ \&\ q(z) \prec h_2(z),\ z \in \mathfrak{D} \right\}, \tag{1.5}$$

$$\mathcal{F}_{\exp} = \left\{ f \in \mathfrak{A} : \frac{f(z)}{zq(z)} \prec e^z \ \&\ q(z) \prec h_2(z),\ z \in \mathfrak{D} \right\}, \tag{1.6}$$

$$\mathcal{F}_{car} = \left\{ f \in \mathfrak{A} : \frac{f(z)}{zq(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \ \&\ q(z) \prec h_2(z),\ z \in \mathfrak{D} \right\}. \tag{1.7}$$

We will examine majorization problems for each of the above-mentined families in this article: $\mathcal{F}_{\cos}, \mathcal{F}_{\mathfrak{S}\mathcal{L}}, \mathcal{F}_{\exp}$ and $\mathcal{F}_{car}$.

2. Main results

To proved majorization results for the families $\mathcal{F}_{\cos}, \mathcal{F}_{\mathfrak{S}\mathcal{L}}, \mathcal{F}_{\exp}$ and $\mathcal{F}_{car}$, we need the following lemmas.

Lemma 2.1. *Let $q(z) \prec 1 + \tanh z$ and $|z| \leq r$. Then, $q(z)$ satisfies the following conditions:*

$$1 - \tanh r \sec^2 r \leq |q(z)| \leq 1 + \tanh r \sec^2 r \tag{2.1}$$

and

$$\left| \frac{zq'(z)}{q(z)} \right| \leq \frac{r \sec^2 r}{(1-r^2)(1-\tan r)}. \tag{2.2}$$

Proof. If $q(z) \prec 1 + \tanh z$, then

$$q(z) = 1 + \tanh u(z),$$

for certain Schwartz function $u(z)$. After a very simple computations, we now get

$$\frac{zq'(z)}{q(z)} = \frac{zu'(z)\sec h^2 u(z)}{1 + \tanh u(z)}. \tag{2.3}$$

Let $u(z) = R(e^{it})$, with $|z| = r \leq R$, $-\pi \leq \theta \leq \pi$. Upon certain simple computation, we get

$$\Re(\sec h^2 \left(Re^{i\theta}\right)) = 1 - \frac{\tanh^2(Rx)\sec^4(Ry) - \tan^2(Ry)\sec h^4(Rx)}{\tanh^4(Rx)\tan^4(Ry) + 2\tanh^2(Rx)\tan^2(Ry) + 1},$$

$$(R := |u(z)|;\ r := |z|),$$

where

$$y = \sin\theta,\ x = \cos\theta,\ y, x \in [-1, 1].$$

Now, we can write

$$1 \leq \sec^2(Ry) \leq \sec^2 R \leq \sec^2 r.$$

So, we have

$$\Re\left(\sec h^2 u(z)\right) \geq \sec h^2 R \geq \sec h^2 r. \tag{2.4}$$

Let us suppose

$$\left|\tanh(Re^{i\theta})\right|^2 = \frac{\tanh^2(R\cos t)\sec^4(R\sin t) + \tan^2(R\sin t)\sec h^4(R\cos t)}{1 + \tan^2(R\sin t)\tanh^2(R\cos t)} = \Psi(\theta). \tag{2.5}$$

A simple computation give us that 0, $\pm\frac{\pi}{2}$ and $\pm\pi$ are the zeros of $\Psi'(\theta)$ in $[-\pi, \pi]$. We observe that

$$\Psi(\frac{\pi}{2}) = \tan^2(R) \quad \text{and} \quad \Psi(0) = \Psi(\pi) = \tanh^2(R).$$

Furthermore, we see that

$$\max\{\Psi(0), \Psi(\frac{\pi}{2}), \Psi(\pi)\} = \tan^2(R). \tag{2.6}$$

Hence,

$$\left|\tanh R\left(e^{i\theta}\right)\right| \leq \tan(R) \leq \tan r. \tag{2.7}$$

Similarly, we demonstrate that

$$\sec h^2 r \leq \left|\sec h^2 u(z)\right| \leq \sec^2 r. \tag{2.8}$$

Now, from well-known inequality for schwarz function $u(z)$, we attain

$$|u'(z)| \leq \frac{1 - |u(z)|^2}{1 - |z|^2} = \frac{1 - R^2}{1 - |z|^2} \leq \frac{1}{1 - r^2}. \tag{2.9}$$

Now using (2.7)–(2.9) in (2.3), we get (2.2). □

Lemma 2.2. *Suppose that $q(z) \prec 1+\sin z$ $(|z| \le r)$. Then q satisfies the following conditions:*

$$1-\cosh r\sin r \le |q(z)| \le 1+\cosh r\sin r \tag{2.10}$$

and

$$\left|\frac{zq'(z)}{q(z)}\right| \le \frac{r\cosh r}{(1-r^2)(1-\sinh r)}. \tag{2.11}$$

Proof. For proof see [35]. □

Theorem 2.1. *Let $f(z) \in \mathfrak{A}$, $g \in \mathcal{F}_{\cos}$ and also assume that $f(z) \ll g(z)$ in $\mathfrak{D}$. Then, for $|z| \le r_1$,*

$$|f'(z)| \le |g'(z)|,$$

where r_1 is the smallest positive root of the equation

$$\left(\left(1-r^2-2r\right)\cos r - r\sinh r\right)(1-\tan r) - r\cos r\sec^2 r = 0. \tag{2.12}$$

Proof. If $g \in \mathcal{F}_{\cos}$, then by the subordination relationship, we get

$$\frac{g(z)}{zq(z)} = \cos u(z).$$

Now, by some simple computations, we obtain

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} - \frac{zu'(z)\sin u(z)}{\cos u(z)}. \tag{2.13}$$

Now by using (2.7)–(2.9) and (2.11) along with Lemma 2.1, we have

$$\begin{aligned}\left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{\left|1+\frac{zq'(z)}{q(z)} - \frac{zu'(z)\sin u(z)}{\cos u(z)}\right|} \\ &\le \frac{|z|}{1-\left|\frac{zq'(z)}{q(z)}\right| - \left|\frac{zu'(z)\sin u(z)}{\cos u(z)}\right|} \\ &\le \frac{r\left(1-r^2\right)\cos r(1-\tan r)}{(1-r^2)(1-\tan r)\cos r - r\cos r\sec^2 r - r\sinh r(1-\tan r)}.\end{aligned} \tag{2.14}$$

From (1.2), we can write

$$f(z) = \phi(z)\,g(z). \tag{2.15}$$

Differentiating (2.15) on both sides, we obtain

$$\begin{aligned}f'(z) &= \phi'(z)\,g(z) + \phi(z)\,g'(z) \\ &= \left(\phi(z) + \phi'(z)\frac{g(z)}{g'(z)}\right)g'(z).\end{aligned} \tag{2.16}$$

Also, the Schwarz function $\phi(z)$ satisfies the following inequality:

$$|\phi'(z)| \le \frac{1-|\phi'(z)|^2}{1-|z|^2} = \frac{1-|\phi'(z)|^2}{1-r^2} \quad (z \in \mathfrak{D}). \tag{2.17}$$

Now applying (2.14) and (2.17) in (2.16), we attain

$$|f'(z)| \leq |g'(z)| \left[\phi(z) + \frac{r\left(1 - |\phi'(z)|^2\right)\cos r\,(1 - \tan r)}{(1 - r^2)\cos r\,(1 - \tan r) - r\cos r\sec^2 r - r\sinh r\,(1 - \tan r)} \right],$$

which by putting

$$|\phi'(z)| = \eta \ \ (0 \leq \eta \leq 1) \tag{2.18}$$

becomes the inequality

$$|f'(z)| \leq \Xi_1(r, \eta)\,|g'(z)|,$$

where

$$\Xi_1(r, \rho) = \phi(z) + \frac{r\left(1 - |\phi'(z)|^2\right)\cos r\,(1 - \tan r)}{((1 - r^2)\cos r - r\sinh r)(1 - \tan r) - r\sec^2 r\cos r}.$$

To determine r_1, it is sufficient to choose

$$r_1 = \max\left(r \in [0, 1) : \Xi_1(r, \eta) \leq 1, \forall \eta \in [0, 1]\right),$$

or, equivalently,

$$r_1 = \max\left(r \in [0, 1) : \Phi_1(r, \eta) \geq 0, \forall \eta \in [0, 1]\right),$$

where

$$\Phi_1(r, \eta) = \left(\left(1 - r^2 - r(1 + \eta)\right)\cos r - r\sinh r\right)(1 - \tan r) - r\cos r\sec^2 r.$$

Obviously, if we choose $\eta = 1$, then we can see that the function $\Phi_1(r, \eta)$ gets its minimum value, namely,

$$\min\left(\Phi_1(r, \eta), \eta \in [0, 1]\right) = \Phi_1(r, 1) = \Phi_1(r),$$

where

$$\Phi_1(r) = \left(\left(1 - r^2 - 2r\right)\cos r - r\sinh r\right)(1 - \tan r) - r\cos r\sec^2 r.$$

Next, we have the following inequalities:

$$\Phi_1(0) = 1 > 0 \text{ and } \Phi_1(1) = -0.5934 < 0,$$

There is indeed a r_1 so that $\Phi_1(r) \geq 0$ for every $r \in [0, r_1]$, where r_1 is the smallest positive root of Eq (2.12). Thus, we have completed the proof of our result. □

Theorem 2.2. *Let $f(z) \in \mathfrak{A}$, $g \in \mathcal{F}_{\mathfrak{S}\mathcal{L}}$ and also assume that $f(z)$ is majorized by $g(z)$ in $\mathfrak{D}$. Then, for $|z| \leq r_2$,*

$$|f'(z)| \leq |g'(z)|,$$

where r_2 is the smallest positive root of the equation

$$\left(1 - 2r^2 - 5r\right)(1 - \tan r) - 2r\sec^2 r = 0. \tag{2.19}$$

Proof. If $g \in \mathcal{F}_{\Im\mathcal{L}}$. Then, a holomorphic function $u(z)$ in $\mathfrak{D}$ occurs with $u(0) = 0$ and $|u(z)| \leq |z|$ so that

$$\frac{g(z)}{z(q(z))} = \sqrt{1 + u(z)}.$$

After some simple computations, we now have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} + \frac{zu'(z)}{2(1 + u(z))}. \tag{2.20}$$

Utilizing (2.9), we get

$$\frac{|z|\,|u'(z)|}{2(1 - |u(z)|)} \leq \frac{|z|(1 + |u(z)|)}{2\left(1 - |z|^2\right)} \leq \frac{|z|(1 + |z|)}{2\left(1 - |z|^2\right)} = \frac{|z|}{2(1 - |z|)} \leq \frac{r}{2(1 - r)}.$$

By virtue of (2.9) and Lemma 2.1, we get

$$\begin{aligned} \left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{1 - \left|\frac{zq'(z)}{q(z)}\right| - \left|\frac{zu'(z)}{2(1+u(z))}\right|} \\ &\leq \frac{2r\left(1 - r^2\right)(1 - \tan r)}{2(1 - r^2)(1 - \tan r) - 2r\sec^2 r - r(1 - \tan r)(1 + r)}. \end{aligned} \tag{2.21}$$

Now, by using (2.21) and (2.17) in (2.16), we get

$$|f'(z)| \leq |g'(z)|\left[\phi(z) + \frac{2r\left(1 - |\phi'(z)|^2\right)(1 - \tan r)}{2(1 - r^2)(1 - \tan r) - 2r\sec^2 r - r(1 + r)(1 - \tan r)}\right].$$

The required results are obtained by the same computations as in Theorem 2.1, along with the use of (2.18). □

Theorem 2.3. *Let $f(z) \in \mathfrak{A}$, $g(z) \in \mathcal{F}_{\exp}$ and also assume that $f(z) \ll g(z)$ in $\mathfrak{D}$. Then, for $|z| \leq r_3$,*

$$|f'(z)| \leq |g'(z)|,$$

where r_3 is the smallest positive root of the equation

$$\left(\left(1 - r^2 - 3r\right)e^r\right)(1 - \tan r) - re^r\sec^2 r = 0. \tag{2.22}$$

Proof. If $g \in \mathcal{F}_{\exp}$. Then, by subordination relationship, we have

$$\frac{g(z)}{zq(z)} = e^{u(z)}.$$

Now, after some easy computations, we obtain

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} + \frac{zu'(z)\,e^{u(z)}}{e^{u(z)}}. \tag{2.23}$$

Now by using (2.7)–(2.9) in conjunction with Lemma 2.1, we have

$$\begin{aligned}\left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{\left|1+\frac{zq'(z)}{q(z)}+\frac{zu'(z)e^{u(z)}}{e^{u(z)}}\right|}\\ &= \frac{|z|}{1-\left|\frac{zq'(z)}{q(z)}\right|-\left|\frac{zu'(z)e^{u(z)}}{e^{u(z)}}\right|}\\ &\leq \frac{re^{r}\left(1-r^2\right)(1-\tan r)}{e^{r}(1-r^2)(1-\tan r)-re^{r}\sec^2 r-re^{r}(1-\tan r)}. \end{aligned} \tag{2.24}$$

Now, by using (2.24) and (2.17) in (2.16) , we get

$$|f'(z)| \leq |g'(z)|\left[\varphi(z)+\frac{re^{r}\left(1-|\varphi'(z)|^2\right)(1-\tan r)}{e^{r}(1-r^2)(1-\tan r)-re^{r}\sec^2 r-re^{r}(1-\tan r)}\right].$$

The required results is obtained by the same computations as in Theorem 2.1, along with the use of (2.18) . □

Theorem 2.4. *Let $f(z)\in\mathfrak{A}$, $g(z)\in\mathcal{F}_{car}$ and also assume that $f(z)\ll g(z)$ in $\mathfrak{D}$. Then, for $|z|\leq r_4$,*

$$|f'(z)|\leq|g'(z)|,$$

where r_4 is the smallest positive root of the equation

$$\left(\left(3\left(1-r^2\right)-6r\right)\Psi-4r(1+r)\right)(1-\tan r)-\Psi 3r\sec^2 r=0 \tag{2.25}$$

with

$$\Psi=\left(1+\frac{4}{3}r+\frac{2}{3}r^2\right).$$

Proof. If $g\in\mathcal{F}_{car}$. Then by subordination relationship, we have

$$\frac{g(z)}{zq(z)}=1+\frac{4}{3}u(z)+\frac{2}{3}(u(z))^2.$$

After some simple computations, we now have

$$\frac{zg'(z)}{g(z)}=1+\frac{zq'(z)}{q(z)}+\frac{\frac{4}{3}zu'(z)(1+u(z))}{1+\frac{4}{3}u(z)+\frac{2}{3}(u(z))^2}. \tag{2.26}$$

Now, by using (2.7)–(2.9) , we have

$$\begin{aligned}\left|\frac{g(z)}{g'(z)}\right| &= \frac{|z|}{\left|1+\frac{zq'(z)}{q(z)}+\frac{\frac{4}{3}zu'(z)(1+u(z))}{1+\frac{4}{3}u(z)+\frac{2}{3}(u(z))^2}\right|}\\ &\leq \frac{3r\left(1-r^2\right)(1-\tan r)\Psi}{3(1-r^2)(1-\tan r)\Psi-\Psi 3r\sec^2 r-4r(1+r)(1-\tan r)}, \end{aligned} \tag{2.27}$$

where

$$\Psi = \left(1 + \frac{4}{3}r + \frac{2}{3}r^2\right).$$

Now, by using (2.27) and (2.17) in (2.16) , we get

$$|f'(z)| \leq |g'(z)| \left[\varphi(z) + \frac{3r\left(1 - |\varphi'(z)|^2\right)(1 - \tan r)\Psi}{3(1 - r^2)(1 - \tan r)\Psi - \Psi 3r\sec^2 r - 4r(1 - \tan r)(1 + r)}\right].$$

We obtain the results directly utilizing the same calculation which are presented in the proof of our Theorem 2.1 in conjunctions with (2.18) . □

3. Conclusions

We investigated on majorization problems for a certain subfamilies of regular functions that are connected to distinct shapes domains. Other subfamilies of these problems can be investigated, such as the families of meromorphic functions and the families of harmonic functions. One may attempt the suggested results for different subclasses of q-starlike functions.

Acknowledgement

The first author (Huo Tang) was partly supported by the Natural Science Foundation of China under Grant 11561001, the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region under Grant NJYT18-A14, the Natural Science Foundation of Inner Mongolia of China under Grants 2022MS01004 and 2020MS01011, and the Higher School Foundation of Inner Mongolia of China under Grant NJZY20200, the Program for Key Laboratory Construction of Chifeng University (No. CFXYZD202004), the Research and Innovation Team of Complex Analysis and Nonlinear Dynamic Systems of Chifeng University (No. cfxykycxtd202005) and the Youth Science Foundation of Chifeng University (No. cfxyqn202133).

Conflict of interest

The authors declare that they have no conflicts of interest.

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