AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(11): 19649–19663.
DOI: 10.3934/math.20221078
Received: 21 June 2022
Revised: 27 August 2022
Accepted: 28 August 2022
Published: 06 September 2022

Research article

On the study of solutions of Bogoyavlenskii equation via improved G'/G^2 method and simplified $\tan(\phi(\xi)/2)$ method

Guowei Zhang[1], **Jianming Qi**[2,*] **and Qinghao Zhu**[2]

[1] School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000, China
[2] School of Business, Shanghai Dianji University, Shanghai, 201306, China

* **Correspondence:** Email: qijianmingsdju@163.com.

Abstract: The Bogoyavlenskii equation is used to describe some kinds of waves on the sea surface and discussed by many researchers. Recently, the G'/G^2 method and simplified $\tan(\frac{\phi(\xi)}{2})$ method are introduced to find novel solutions to differential equations. To the best of our knowledge, the Bogoyavlenskii equation has not been investigated by these two methods. In this article, we applied these two methods to the Bogoyavlenskii equation in order to obtain the novel exact traveling wave solutions. Consequently, we found that some new rational functions, trigonometric functions, and hyperbolic functions can be the traveling wave solutions of this equation. Some of these solutions we obtained have not been reported in the former literature. Through comparison, we see that the two methods are more effective than the previous methods for this equation. In order to make these solutions more obvious, we draw some 3D and 2D plots of them.

Keywords: soliton; traveling wave solution; Bogoyavlenskii equation
Mathematics Subject Classification: 34A05, 35C07, 35G20

1. Introduction

Many phenomena in physics, biology, and chemistry etc. can be studied with the aid of nonlinear differential equations. When the researchers investigate a system, they firstly seek the underlying feature, and then convert the problem into mathematical equations by mathematical modeling. Most of time these equations are nonlinear differential equation. Therefore, we need to use some suitable methods to get the exact solutions of these equations. In the past few years, many systematic methods have formed, for example [1–25].

We know that the mathematical equations describing waves in the ocean are nonlinear and their solutions are solitonic. In 1990, the following nonlinear partial differential equation is given [26], later called Bogoyavlenskii equation, to describe the fluctuation of sea waves.

$$\begin{cases} 4u_t + u_{xxy} - 4u^2u_y - 4u_xv = 0; \\ uu_y = v_x. \end{cases} \tag{1.1}$$

This classical equation has been studied for a long time. For the spectral parameter of (1.1), the authors of [26] studied nonisospectral condition. Kudryashov and Pickering [27] obtained the Schwarzian breaking soliton hierarchy of (1.1). It also has relation with non-isospectral scattering problems [28] and possesses Painlevé property [29]. As a modified breaking soliton equation, it describes the interaction of waves along the x-axis and y-axis. These results describe the phenomenon of waves on the sea surface.

In many years, seeking the exact solution of this nonlinear PDE by different methods is one of the main research aspects for researchers. Here we list some of the major approaches and results, but not all of them. By the singular manifold method, Peng and Shen [30] studied the analytical solutions of (1.1). In aid of modified extended tanh-function method the authors [31] got some exact traveling wave solutions. Using(G'/G)-expansion method Malik et al [32] investigated Eq (1.1), and Yu, Sun [33] also treated this equation by modified technique of simplest equation. In 2020, Yokus et al. [34] constructed some exact solutions of (1.1) by $(1/G')$-expansion and $(G'/G, 1/G)$-expansion method.

For the past few years, many researchers [35–37] have introduced the modified (G'/G^2) method, which is powerful and effective, to seek novel soliton solutions for different kinds of nonlinear PDE. Moreover, Manafian et al. [38, 39] proposed a powerful technique called the improved $\tan(\phi(\xi)/2)$ approach to get exact soliton solutions of various PDE. Later, in [40] this method has been simplified. This approach also can be used to treat the Kundu-Eckhaus equation [40], Konopelchenko-Dubrovsky equation and Boussinesq equation [41]. In this work we will apply these two methods to Bogoyavlenskii equation for finding new solutions. Some of these solutions we obtained have not been reported in former literature. Comparing these two methods to the previous results, we find that our approaches are more effective.

In the nonlinear waves theory, the investigation of traveling wave solutions with a fixed velocity is one of the most important aspect. We aim to look for the new traveling wave solution of (1.1) such as

$$u(x, y, t) = u(\xi), \ v(x, y, t) = v(\xi), \ \xi = x + y - kt. \tag{1.2}$$

We substitute (1.2) into (1.1) and integrate once the second equation of (1.1). For simplicity we choose the integral constant as zero, thus we obtain

$$\begin{cases} -4ku' + u''' - 4u^2u' - 4u'v = 0; \\ \dfrac{u^2}{2} = v. \end{cases} \tag{1.3}$$

Combining the two equations of (1.3) together, and then integrating it once, we obtain the following ordinary differential equation

$$u'' - 2u^3 - 4ku = 0. \tag{1.4}$$

Then we just need to find the solutions $u(\xi)$ of (1.4), then $v(\xi)$ is easy to obtain by the second equation of (1.3).

2. Via G'/G^2 method

At first, we introduce the steps of the modified (G'/G^2)-expansion approach, and then employ this method to find new traveling wave solutions of (1.1).

Step 1. We think about the following PDE

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yx}, u_{yt}, \cdots) = 0, \tag{2.1}$$

as to $u(x, y, t)$. In order to get the traveling wave solution of (1.1), we use the wave variables $\xi = x+y-kt$ to convert (2.1) into

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \cdots) = 0. \tag{2.2}$$

Step 2. We express the exact solution of (1.4) to be a polynomial in (G'/G^2), that is,

$$u(\xi) = A_0 + \sum_{i=1}^{n}\left[A_i\left(\frac{G'}{G^2}\right)^{i} + A_{-i}\left(\frac{G'}{G^2}\right)^{-i}\right], \tag{2.3}$$

here $G = G(\xi)$ is a solution of

$$\left(\frac{G'}{G^2}\right)' = a + b\left(\frac{G'}{G^2}\right) + c\left(\frac{G'}{G^2}\right)^2, \tag{2.4}$$

the coefficients a, b and c are any of the constants. Since the highest order derivatives terms and the non-linear terms should be homogenous balance in (2.2), we can obtain the value of integer number n.

Step 3. By taking (2.3) into (2.2) and utilizing (2.4), Eq (2.2) yields to a polynomial as to G'/G^2. Then we collect all the terms with the same power of G'/G^2 and let all the coefficients of this collected polynomial to be zero. Hence, we deduce a system algebraic equations for a, b, c, $A_0, A_i, (i = \pm1, \pm2, \cdots)$.

Step 4. By Solving this system of algebraic equations, we can get some families of the values of $A_0, A_i, (i = \pm1, \pm2, \cdots)$ and a, b, c. Since ordinary differential Eq (2.4) have five kinds solutions as follows:

Solution 1: If $ac > 0, b = 0$, then

$$\left(\frac{G'}{G^2}\right)(\xi) = \sqrt{\frac{a}{c}}\left(\frac{C_1 \cos\sqrt{ac}\xi + C_2 \sin\sqrt{ac}\xi}{C_2 \cos\sqrt{ac}\xi - C_1 \sin\sqrt{ac}\xi}\right);$$

Solution 2: If $ac < 0, b = 0$, then

$$\left(\frac{G'}{G^2}\right)(\xi) = -\sqrt{\left|\frac{a}{c}\right|}\left(\frac{C_1 \sinh 2\sqrt{|ac|}\xi + C_1 \cosh 2\sqrt{|ac|}\xi + C_2}{C_1 \cosh 2\sqrt{|ac|}\xi + C_1 \sinh 2\sqrt{|ac|}\xi - C_2}\right);$$

Solution 3: If $a = 0, c \neq 0, b = 0$, then

$$\left(\frac{G'}{G^2}\right)(\xi) = -\frac{C_1}{c(C_1\xi + C_2)};$$

Solution 4: If $b \neq 0, \Delta \geq 0$, then

$$\left(\frac{G'}{G^2}\right)(\xi) = -\frac{b}{2c} - \frac{\sqrt{\Delta}(C_1 \cosh(\frac{\sqrt{\Delta}}{2})\xi + C_2 \sinh(\frac{\sqrt{\Delta}}{2})\xi)}{2c(C_2 \cosh(\frac{\sqrt{\Delta}}{2})\xi + C_1 \sinh(\frac{\sqrt{\Delta}}{2})\xi)};$$

Solution 5: If $b \neq 0, \Delta < 0$, then

$$\left(\frac{G'}{G^2}\right)(\xi) = -\frac{b}{2c} - \frac{\sqrt{-\Delta}(C_1 \cos(\frac{\sqrt{-\Delta}}{2})\xi - C_2 \sin(\frac{\sqrt{-\Delta}}{2})\xi)}{2c(C_2 \cos(\frac{\sqrt{-\Delta}}{2})\xi + C_1 \sin(\frac{\sqrt{-\Delta}}{2})\xi)};$$

where C_1, C_2 are free constants and $\Delta = b^2 - 4ac$, then the exact solutions of (2.2) can be obtained.

Step 5. Applying the inverse transformation to $u(\xi)$ $(\xi = x + y - kt)$, then all exact solutions $u(x, y, t)$ of (2.1) can be gotten.

Since u'' and u^3 in (1.4) should be homogeneous balance, we get $n = 1$. Then $u(\xi)$ in (2.3) has the form

$$u(\xi) = A_1\left(\frac{G'}{G^2}\right) + A_0 + A_{-1}\left(\frac{G'}{G^2}\right)^{-1}, \tag{2.5}$$

where G'/G^2 satisfies Eq (2.4). For simplicity, set $g := G'/G^2$, then (2.4) can be rewritten as

$$g' = a + bg + cg^2. \tag{2.6}$$

Taking derivatives of (2.5), we obtain

$$u'(\xi) = A_1cg^2 + A_1bg + (A_1a - A_{-1}c) - A_{-1}bg^{-1} - A_{-1}ag^{-2} \tag{2.7}$$

and

$$\begin{aligned} u''(\xi) \;\; = \;\; & 2A_1c^2g^3 + 3A_1bcg^2 + (2A_1ac + A_1b^2)g + (A_1ab + A_{-1}bc) \\ & +(A_{-1}b^2 + 2A_{-1}ac)g^{-1} + (3A_{-1}ab)g^{-2} + 2A_{-1}a^2g^{-3}. \end{aligned} \tag{2.8}$$

By (2.5) and calculation we have

$$\begin{aligned} 2u^3 + 4ku \;\; = \;\; & 2A_1^3g^3 + 6A_0A_1^2g^2 + (6A_{-1}A_1^2 + 6A_0^2A_1 + 4kA_1)g + (12A_{-1}A_0A_1 + 2A_0^3 + 4kA_0) \\ & (6A_{-1}^2A_1 + 6A_{-1}A_0^2 + 4kA_{-1})g^{-1} + 6A_{-1}^2A_0g^{-2} + 2A_{-1}^3g^{-3}. \end{aligned} \tag{2.9}$$

Now, we put (2.8) and (2.9) into (1.4), and collect all the same power terms together. Then we extract its undetermined coefficients of the power of g, and set them to be zero. Therefore, we have the following equations

$$\begin{aligned} g^3 : & \quad 2A_1c^2 = 2A_1^3, \\ g^2 : & \quad 3A_1bc = 6A_0A_1^2, \\ g : & \quad 2A_1ac + A_1b^2 = 6A_{-1}A_1^2 + 6A_0^2A_1 + 4kA_1, \\ const : & \quad A_1ab + A_{-1}bc = 12A_{-1}A_0A_1 + 2A_0^3 + 4kA_0, \end{aligned}$$

$$
\begin{aligned}
g^{-1}: &\quad A_{-1}b^2 + 2A_{-1}ac = 6A_{-1}^2A_1 + 6A_{-1}A_0^2 + 4kA_{-1},\\
g^{-2}: &\quad 3A_{-1}ab = 6A_{-1}^2A_0,\\
g^{-3}: &\quad 2A_{-1}a^2 = 2A_{-1}^3.
\end{aligned}
$$

By solving the above equations, we have some cases as follows.

Case 1:

$$
\begin{cases}
A_1 = 0;\\
A_{-1} = 0;\\
A_0 = 0,\ \sqrt{-2k}, \text{or}, -\sqrt{-2k},
\end{cases}
\tag{2.10}
$$

then $u(\xi)$ is a constant.

Case 2:

$$
\begin{cases}
A_1 = 0;\\
A_{-1} = \pm a;\\
A_0 = \pm \dfrac{b}{2},
\end{cases}
\tag{2.11}
$$

where $k = -\frac{b^2}{8} + \frac{ac}{2}$, that is $\Delta = b^2 - 4ac = -8k$.

Subcase 2.1: If $ac > 0, b = 0$, then

$$
u(\xi) = \pm\sqrt{ac}\left(\frac{C_2\cos\sqrt{ac}\xi - C_1\sin\sqrt{ac}\xi}{C_1\cos\sqrt{ac}\xi + C_2\sin\sqrt{ac}\xi}\right).
\tag{2.12}
$$

Subcase 2.2: If $ac < 0, b = 0$, then

$$
u(\xi) = \mp\sqrt{|ac|}\left(\frac{C_1\cosh 2\sqrt{|ac|}\xi + C_1\sinh 2\sqrt{|ac|}\xi - C_2}{C_1\sinh 2\sqrt{|ac|}\xi + C_1\cosh 2\sqrt{|ac|}\xi + C_2}\right).
\tag{2.13}
$$

Subcase 2.3: If $a = 0, c \neq 0, b = 0$, then

$$
u(\xi) = 0;
\tag{2.14}
$$

Subcase 2.4: If $b \neq 0, \Delta = -8k \geq 0$, then

$$
u(\xi) = \pm\frac{b}{2} \mp \left(\frac{b}{2c} + \frac{\sqrt{\Delta}(C_1\cosh(\frac{\sqrt{\Delta}}{2})\xi + C_2\sinh(\frac{\sqrt{\Delta}}{2})\xi)}{2c(C_2\cosh(\frac{\sqrt{\Delta}}{2})\xi + C_1\sinh(\frac{\sqrt{\Delta}}{2})\xi)}\right)^{-1}.
\tag{2.15}
$$

Subcase 2.5: If $b \neq 0, \Delta = -8k < 0$, then

$$
u(\xi) = \pm\frac{b}{2} \mp \left(\frac{b}{2c} + \frac{\sqrt{-\Delta}(C_1\cos(\frac{\sqrt{-\Delta}}{2})\xi - C_2\sin(\frac{\sqrt{-\Delta}}{2})\xi)}{2c(C_2\cos(\frac{\sqrt{-\Delta}}{2})\xi + C_1\sin(\frac{\sqrt{-\Delta}}{2})\xi)}\right)^{-1}.
\tag{2.16}
$$

Case 3:

$$\begin{cases} A_1 = \pm c; \\ A_{-1} = 0; \\ A_0 = \pm \dfrac{b}{2}; \end{cases} \tag{2.17}$$

where $k = -\frac{b^2}{8} + \frac{ac}{2}$, that is $\Delta = b^2 - 4ac = -8k$.

Subcase 3.1: If $ac > 0, b = 0$, then

$$u(\xi) = \sqrt{ac}\left(\frac{C_1 \cos\sqrt{ac}\xi + C_2 \sin\sqrt{ac}\xi}{C_2 \cos\sqrt{ac}\xi - C_1 \sin\sqrt{ac}\xi}\right). \tag{2.18}$$

Subcase 3.2: If $ac < 0, b = 0$, then

$$u(\xi) = \mp\sqrt{|ac|}\left(\frac{C_1 \sinh 2\sqrt{|ac|}\xi + C_1 \cosh 2\sqrt{|ac|}\xi + C_2}{C_1 \cosh 2\sqrt{|ac|}\xi + C_1 \sinh 2\sqrt{|ac|}\xi - C_2}\right). \tag{2.19}$$

Subcase 3.3: If $a = 0, c \neq 0$, then

$$u(\xi) = \mp\frac{C_1}{C_1\xi + C_2}; \tag{2.20}$$

Subcase 3.4: If $b \neq 0, \Delta = -8k \geq 0$, then

$$u(\xi) = \mp\frac{\sqrt{\Delta}}{2} \cdot \frac{C_1 \cosh(\frac{\sqrt{\Delta}}{2})\xi + C_2 \sinh(\frac{\sqrt{\Delta}}{2})\xi}{C_2 \cosh(\frac{\sqrt{\Delta}}{2})\xi + C_1 \sinh(\frac{\sqrt{\Delta}}{2})\xi}. \tag{2.21}$$

Subcase 3.5: If $b \neq 0, \Delta = -8k < 0$, then

$$u(\xi) = \mp\frac{\sqrt{-\Delta}}{2} \cdot \frac{C_1 \cos(\frac{\sqrt{-\Delta}}{2})\xi - C_2 \sin(\frac{\sqrt{-\Delta}}{2})\xi}{C_2 \cos(\frac{\sqrt{-\Delta}}{2})\xi + C_1 \sin(\frac{\sqrt{-\Delta}}{2})\xi}. \tag{2.22}$$

3. Via $\tan(\frac{\phi(\xi)}{2})$ method

At first, we give the outline of $\tan(\frac{\phi(\xi)}{2})$ method.
Step 1. The same as G'/G^2 method.
Step 2. Suppose that the solutions can be written as follows

$$u(\xi) = \sum_{j=0}^{m} A_j\left(p + \tan(\frac{\phi(\xi)}{2})\right)^j + \sum_{j=1}^{m} A_{-j}\left(p + \tan(\frac{\phi(\xi)}{2})\right)^{-j}, \tag{3.1}$$

$\phi(\xi)$ satisfies

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c, \tag{3.2}$$

here a, b, c, A_j and A_{-j} are unknown constants. (3.2) has five kinds of solutions.

Solution 1: If $b = c, a = 0$, then

$$\tan\left(\frac{\phi}{2}\right) = b\xi + C_1 - p;$$

Solution 2: If $b = c$, $a \neq 0$, then

$$\tan\left(\frac{\phi}{2}\right) = C_1 \exp(a\xi) - \frac{b}{a};$$

Solution 3: If $b \neq c, \Delta = a^2 + b^2 - c^2 > 0$, then

$$\tan\left(\frac{\phi}{2}\right) = \frac{2}{b-c} \cdot \frac{C_1 r_1 \exp(r_1\xi) + C_2 r_2 \exp(r_2\xi)}{C_1 \exp(r_1\xi) + C_2 \exp(r_2\xi)} - p;$$

Solution 4: If $b \neq c, \Delta = a^2 + b^2 - c^2 = 0$, then

$$\tan\left(\frac{\phi}{2}\right) = \frac{a}{b-c} + \frac{2}{b-c} \cdot \frac{C_2}{C_1 + C_2\xi};$$

Solution 5: If $b \neq c, \Delta = a^2 + b^2 - c^2 < 0$, then

$$\tan\left(\frac{\phi}{2}\right) = \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c} \cdot \frac{-C_1 \sin(\frac{\sqrt{-\Delta}}{2}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{2}\xi)}{C_1 \cos(\frac{\sqrt{-\Delta}}{2}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{2}\xi)};$$

where C_1 and C_2 are arbitrary constants, $r_1 = \frac{a+p(b-c)+\sqrt{\Delta}}{2}, r_2 = \frac{a+p(b-c)-\sqrt{\Delta}}{2}$.

Step 3. We need to balance the highest order derivative and the nonlinear terms because of homogenous, then the value of positive integer m we will get. Substituting (3.1) into (2.2) yields a equation with the power of $\tan(\frac{\phi}{2})$. We firstly collect the terms with the same power of $\tan(\frac{\phi}{2})$, and then set the coefficients of it to be zero, then a system of equations for unknown A_j, B_j, a, b, c and p we will obtain.

Step 4. Solving the above equations we just obtain, and then substituting $A_0, A_1, B_1, \cdots, A_m, B_m, p$ into (3.1), we get the expression $u(\xi)$.

Step 5. This step is similar to Step 5 of G'/G^2.

In the following, we utilize $\tan(\frac{\phi(\xi)}{2})$ method to seek new solutions of (1.1). Substituting (3.1) into (1.4) and by Step 3 we get $m = 1$. Then (3.1) can be written as

$$u(\xi) = A_0 + A\left(p + \tan\left(\frac{\phi(\xi)}{2}\right)\right) + A_{-1}\left(p + \tan\left(\frac{\phi(\xi)}{2}\right)\right)^{-1}. \tag{3.3}$$

Substituting (3.3) into the left side and right side of Eq (1.4) and combining with (3.2), then we have

$$\begin{aligned} & (p+t)^3 \cdot u''(\xi) \\ = \; & (-b+c)^2 A\, t^6 2 \\ + \; & \left(\frac{3(-b+c)^2 Ap}{2} + \frac{3a(-b+c)A}{2}\right) t^5 \end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{(3A\,p^2 - A_{-1})(-b+c)^2}{2} + \frac{9Apa(-b+c)}{2}\right.\\
&+ \left.\frac{A(4a^2 - 2b^2 + 2c^2)}{4} + \frac{A_{-1}(-b+c)^2}{2}\right)t^4\\
&+ \left(\frac{(A\,p^3 - A_{-1}p)(-b+c)^2}{2} + \frac{3(3A\,p^2 - A_{-1})a(-b+c)}{2}\right.\\
&+ \left.\frac{3Ap(4a^2 - 2b^2 + 2c^2)}{4} + \frac{Aa(b+c)}{2} + 2A_{-1}a(-b+c)\right)t^3\\
&+ \left(\frac{3(A\,p^3 - A_{-1}p)a(-b+c)}{2} + \frac{(3A\,p^2 - A_{-1})(4a^2 - 2b^2 + 2c^2)}{4}\right.\\
&+ \left.\frac{3Apa(b+c)}{2} + \frac{A_{-1}(2(b+c)(-b+c) + 4a^2)}{2}\right)t^2\\
&+ \left(\frac{(A\,p^3 - A_{-1}p)(4a^2 - 2b^2 + 2c^2)}{4} + \frac{(3A\,p^2 - A_{-1})a(b+c)}{2} + 2A_{-1}a(b+c)\right)t\\
&+ \frac{(A\,p^3 - A_{-1}p)a(b+c)}{2} + \frac{A_{-1}(b+c)^2}{2};
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
& (p+t)^3 \cdot (2u^3(\xi) + 4ku(\xi))\\
=\ & 2A^3t^6\\
+\ & (12A^3p + 6A^2A_0)t^5\\
+\ & (30A^3p^2 + 30A^2A_0p + 6A^2A_{-1} + 6AA_0^2 + 4kA)t^4\\
+\ & (40A^3p^3 + 60A^2A_0p^2 + 24A^2A_{-1}p + 24AA_0^2p + 12AA_0A_{-1} + 16Apk + 2A_0^3 + 4A_0k)t^3\\
+\ & (30A^3p^4 + 60A^2A_0p^3 + 36A^2A_{-1}\,p^2 + 36AA_0^2p^2 + 36AA_0A_{-1}p + 24Ak\,p^2 + 6A_0^3p\\
+\ & 6A\,A_{-1}^2 + 6A_0^2A_{-1} + 12A_0kp + 4A_{-1}k)t^2\\
+\ & (12A^3p^5 + 30A^2A_0p^4 + 24A^2A_{-1}\,p^3 + 24AA_0^2p^3 + 36AA_0A_{-1}\,p^2 + 16Ak\,p^3\\
+\ & 6A_0^3p^2 + 12A\,A_{-1}^2p + 12A_0^2A_{-1}p + 12A_0k\,p^2 + 6A_0A_{-1}^2 + 8A_{-1}kp)t\\
+\ & 2A^3p^6 + 6A^2A_0p^5 + 6A^2A_{-1}\,p^4 + 6AA_0^2p^4 + 12AA_0A_{-1}\,p^3 + 4Ak\,p^4 + 2A_0^3p^3\\
+\ & 6A\,A_{-1}^2p^2 + 6A_0^2A_{-1}\,p^2 + 4A_0k\,p^3 + 6A_0A_{-1}^2p + 4A_{-1}k\,p^2 + 2A_{-1}^3.
\end{aligned}
\tag{3.5}
$$

Substituting (3.4) and (3.5) into (1.4), a system of algebraic equations for the unknowns A_0, A, A_{-1} will be obtained. Solving the algebraic equation by the aid of Maple software, we can have some cases as follows.

Case 1: If $b = c$, then we have two subcases.

Subcase 1.1

$$
\begin{cases}
A = 0;\\
A_0 = 0, \text{or} \pm \sqrt{-2k};\\
A_{-1} = 0,
\end{cases}
\tag{3.6}
$$

then $u(\xi)$ is a constant.

Subcase 1.2:

$$\begin{cases} A = 0; \\ A_0 = \pm\sqrt{-2k} = \dfrac{a}{2}; \\ A_{-1} = \pm\dfrac{3ac + 16kp - pa^2}{6\sqrt{-2k}} = c - ap, \end{cases} \tag{3.7}$$

with $a^2 = -8k$, c, p are arbitrary constant. If $a = 0$, then

$$u(\xi) = \frac{c}{c\xi + C_1}. \tag{3.8}$$

If $a \neq 0$,

$$u(\xi) = \frac{a}{2} + (c - ap)\left(\frac{pa - c}{a} + C_1 \exp(a\xi)\right)^{-1}. \tag{3.9}$$

Case 2: If $b \neq c$, then

$$\begin{cases} A = \pm\dfrac{c - b}{2}; \\ A_0 = \pm\dfrac{a - p(c - b)}{2} = 0; \\ A_{-1} = \pm\dfrac{a^2 + b^2 - c^2 + 8k}{6(b - c)}, \end{cases} \tag{3.10}$$

with $p = -a/(b - c)$ and $c^2 = a^2 + b^2 - 4k$, $c^2 = a^2 + b^2 + 2k$, or $c^2 = a^2 + b^2 + 8k$. then $\Delta = 4k, -2k$, or $-8k$. $A_{-1} = \pm\frac{2k}{b-c}, \pm\frac{k}{b-c}$ or 0.

Subcase 2.1: If $\Delta > 0$, then $r_1 = \frac{\sqrt{\Delta}}{2}, r_2 = -\frac{\sqrt{\Delta}}{2}$.

When $\Delta = 4k, k > 0$

$$u(\xi) = \mp\frac{C_1\frac{\sqrt{\Delta}}{2}\exp(\frac{\sqrt{\Delta}}{2}\xi) - C_2\frac{\sqrt{\Delta}}{2}\exp(-\frac{\sqrt{\Delta}}{2}\xi)}{C_1\exp(\frac{\sqrt{\Delta}}{2}\xi) + C_2\exp(-\frac{\sqrt{\Delta}}{2}\xi)} \pm k\frac{C_1\exp(\frac{\sqrt{\Delta}}{2}\xi) + C_2\exp(-\frac{\sqrt{\Delta}}{2}\xi)}{C_1\frac{\sqrt{\Delta}}{2}\exp(\frac{\sqrt{\Delta}}{2}\xi) - C_2\frac{\sqrt{\Delta}}{2}\exp(-\frac{\sqrt{\Delta}}{2}\xi)}. \tag{3.11}$$

When $\Delta = -2k, k < 0$

$$u(\xi) = \mp\frac{C_1\frac{\sqrt{\Delta}}{2}\exp(\frac{\sqrt{\Delta}}{2}\xi) - C_2\frac{\sqrt{\Delta}}{2}\exp(-\frac{\sqrt{\Delta}}{2}\xi)}{C_1\exp(\frac{\sqrt{\Delta}}{2}\xi) + C_2\exp(-\frac{\sqrt{\Delta}}{2}\xi)} \pm \frac{k}{2}\frac{C_1\exp(\frac{\sqrt{\Delta}}{2}\xi) + C_2\exp(-\frac{\sqrt{\Delta}}{2}\xi)}{C_1\frac{\sqrt{\Delta}}{2}\exp(\frac{\sqrt{\Delta}}{2}\xi) - C_2\frac{\sqrt{\Delta}}{2}\exp(-\frac{\sqrt{\Delta}}{2}\xi)}. \tag{3.12}$$

When $\Delta = -8k, k < 0$

$$u(\xi) = \mp\frac{C_1\frac{\sqrt{\Delta}}{2}\exp(\frac{\sqrt{\Delta}}{2}\xi) - C_2\frac{\sqrt{\Delta}}{2}\exp(-\frac{\sqrt{\Delta}}{2}\xi)}{C_1\exp(\frac{\sqrt{\Delta}}{2}\xi) + C_2\exp(-\frac{\sqrt{\Delta}}{2}\xi)}. \tag{3.13}$$

Subcase 2.2: If $\Delta = 0$, then $k = 0, A_{-1} = 0$.

$$u(\xi) = \mp\frac{C_2}{C_1 + C_2\xi}. \tag{3.14}$$

Subcase 2.3: When $\Delta = 4k, k < 0$, we have

$$u(\xi) = \mp\frac{\sqrt{-\Delta}}{2}\cdot\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\pm\frac{2k}{\sqrt{-\Delta}}\cdot\frac{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}. \quad (3.15)$$

When $\Delta = -2k, k > 0$, we have

$$u(\xi) = \mp\frac{\sqrt{-\Delta}}{2}\cdot\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\pm\frac{k}{\sqrt{-\Delta}}\cdot\frac{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}. \quad (3.16)$$

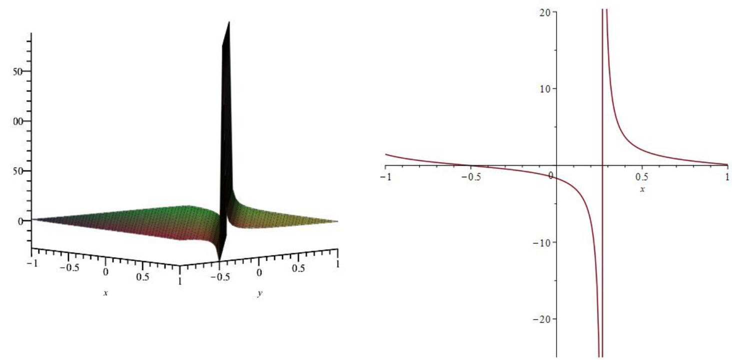
When $\Delta = -8k, k > 0$, we have

$$u(\xi) = \mp\frac{\sqrt{-\Delta}}{2}\cdot\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}. \quad (3.17)$$

C_1, C_2 are arbitrary constants when they appear.

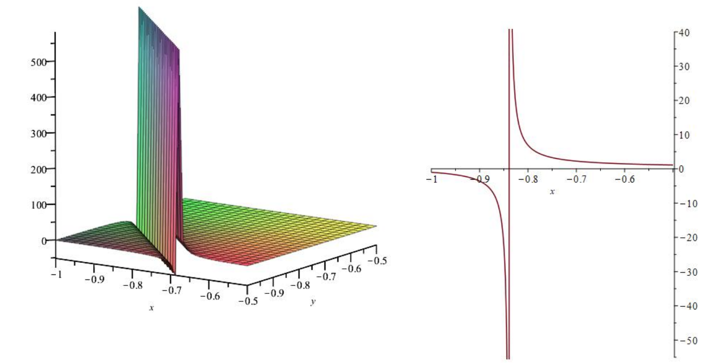
4. Conclusions and discussion

Traveling waves play an important role in solitary wave theory. To our knowledge, this work is the first time to use these two methods to study the Bogoyavlenskii equation. We have obtained some new traveling wave soliton solutions of the Bogoyavlenskii equation, these solutions are constructed by trigonometric, exponential and rational functions with arbitrary coefficients and parameters. We draw the figures of some solutions with special values of the coefficients and parameters. These coefficients and parameters have practical physical meanings, the coefficients and parameters depend on the initial value. It is easy for us to observe the soliton behaviors of these solutions, such as Figure 1 to Figure 6. Our results have verified that the new solutions of this equation do indeed have soliton phenomena, these will help us to study waves in the ocean. In the process of the use of the modified G'/G^2 and simplicity improved $\tan(\phi(\xi)/2)$ methods, we find that both methods are very useful, and can help us to get rational, trigonometric, exponential, hyperbolic solutions. By comparing these two methods, we found that the second method are more complicated, but we can get some additional solutions which we can not obtain by the modified G'/G^2 method, such as (3.11), (3.12), (3.15) and (3.16).



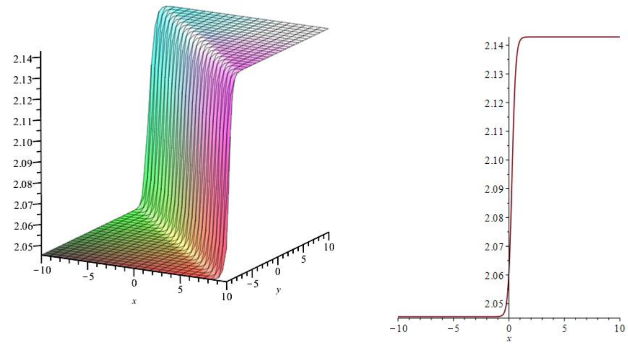
(a) (2.12) $a = 1, c = 1$ (b) (2.12) $a = 1, c = 1$,

Figure 1. 3D and 2D plot of (2.12) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.



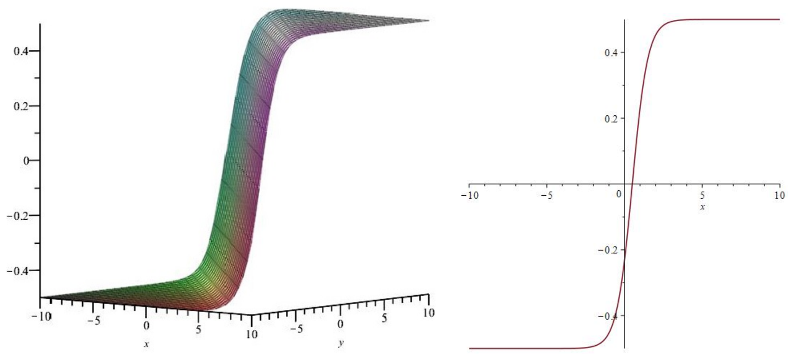
(a) $a = 1, b = 2, c = 2$ (b) $a = 1, b = 2, c = 2$

Figure 2. 3D and 2D plot of (2.16) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.



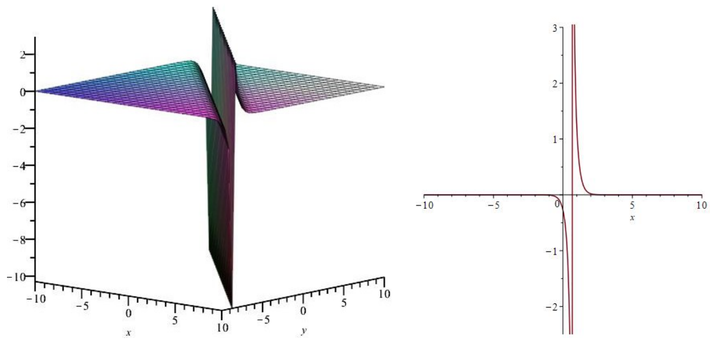
(a) $a = 1, b = 5, c = 1$ (b) $a = 1, b = 5, c = 1$

Figure 3. 3D and 2D plot of (2.15) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.



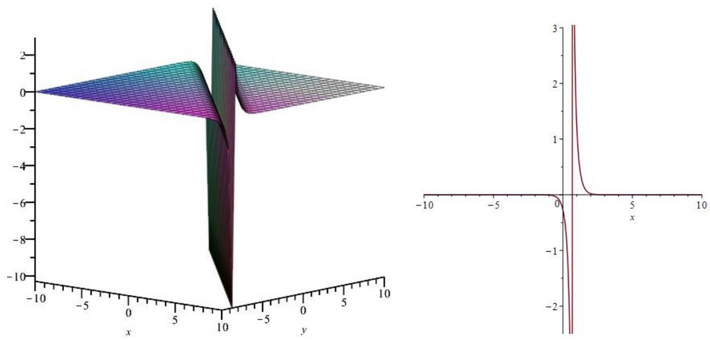
(a) $a = 1, c = 2, p = 3$ (b) $a = 1, c = 2, p = 3$

Figure 4. 3D and 2D plot of (3.9) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.



(a) $\Delta = 4$ (b) $\Delta = 4$

Figure 5. 3D and 2D plot of (3.11) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.



(a) $\Delta = 4$ (b) $\Delta = 4$

Figure 6. 3D and 2D plot of (3.11) with $C_1 = 1, C_2 = 2, k = 1, t = 1$.

Acknowledgments

This work was supported by the key scientific research project of colleges and universities in Henan province (No. 22A110004). Applied Mathematical subject of Shanghai Dianji University (16JCXK02) and science and technology innovation action plan biomedical science and technology support special project (19441914900): intelligent lower limb rehabilitation robot.

Conflict of interest

The authors declare that they have no competing interests.

References

1. J. Weiss, Bäcklund transformation and the Painlevé property, *J. Math. Phys.*, **27** (1986), 1293–1305. https://doi.org/10.1063/1.527134

2. M. L. Wang, Y. M. Wang, Y. B. Zhou, An auto-Backlund transformation and exact solutions to a generalized KdV equation with variable coefficients and their applications, *Phys. Lett. A*, **303** (2002), 45–51. https://doi.org/10.1016/S0375-9601(02)00975-1
3. D. G. Zhang, Integrability of fermionic extensions of the Burgers equation, *Phys. Lett. A*, **223** (1996), 436–438. https://doi.org/10.1016/S0375-9601(96)00773-6
4. Y. Y. Gu, Analytical solutions to the Caudrey-Dodd-Gibbon-Sawada-Kotera equation via symbol calculation approach, *J. Funct. Space.*, **2020** (2020), 5042724. https://doi.org/10.1155/2020/5042724
5. E. Zayed, Y. A. Amer, A. H. Arnous, Functional variable method and its applications for finding exact solutions of nonlinear PDEs in mathematical physics, *Sci. Res. Essays*, **8** (2013), 2068–2074. https://doi.org/10.5897/SRE2013.5725
6. M. N. Alam, M. A. Akbar, Exact traveling wave solutions of the KP-BBM equation by using the new approach of generalized (G/G)-expansion method, *SpringerPlus*, **2** (2013), 617. https://doi.org/10.1186/2193-1801-2-617
7. J. Satsuma, Hirota bilinear method for nonlinear evolution equations, In: *Direct and tnverse methods in nonlinear evolution equations*, Berlin: Springer, 2003. https://doi.org/10.1007/978-3-540-39808-0_4
8. J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.*, **24** (1983), 522. https://doi.org/10.1063/1.525721
9. W. Hereman, M. Takaoka, Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA, *J. Phys. A Math. Gen.*, **23** (1990), 4805. https://doi.org/10.1088/0305-4470/23/21/021
10. S. T. Mohyud-Din, A. Irshad, Solitary wave solutions of some nonlinear PDEs arising in electronics, *Opt. Quant. Electron.*, **49** (2017), 130. https://doi.org/10.1007/s11082-017-0974-y
11. A. Patel, V. Kumar, Dark and kink soliton solutions of the generalized ZKCBBM equation by iterative scheme, *Chinese J. Phys.*, **56** (2018), 819–829. https://doi.org/10.1016/j.cjph.2018.03.012
12. N. H. Aljahdaly, A. R. Seadawy, W. A. Albarakati, Applications of dispersive analytical wave solutions of nonlinear seventh order Lax and Kaup-Kupershmidt dynamical wave equations, *Results Phys.*, **14** (2019), 102372. https://doi.org/10.1016/j.rinp.2019.102372
13. N. H. Aljahdaly, A. R. Seadawy, W. A. Albarakati, Analytical wave solution for the generalized nonlinear seventh-order KdV dynamical equations arising in shallow water waves, *Mod. Phys. Lett. B*, **34** (2020), 2050279. https://doi.org/10.1142/S0217984920502796
14. M. I. Tarikul, M. A. Akbar, J. F. Gmez-Aguilar, E. Bonyahd, G. Fernandez-Anaya, Assorted soliton structures of solutions for fractional nonlinear Schrodinger types evolution equations, *J. Ocean Eng. Sci.*, 2021. https://doi.org/10.1016/j.joes.2021.10.006
15. M. M. A. Khater, A. Jhangeer, H. Rezazadeh, L. Akinyemi, M. A. Akbar, M. Inc, Propagation of new dynamics of longitudinal bud equation among a magneto-electro-elastic round rod, *Mod. Phys. Lett. B*, **35** (2021), 2150381. https://doi.org/10.1142/S0217984921503814

16. K. S. Nisar, K. K. Ali, M. Inc, M. S. Mehanna, H. Rezazadeh, L. Akinyemi, New solutions for the generalized resonant nonlinear Schrödinger equation, *Results Phys.*, **33** (2022), 105153. https://doi.org/10.1016/j.rinp.2021.105153

17. L. Akinyemi, M. Senol, E. Az-Zo'bi, P. Veeresha, U. Akpan, Novel soliton solutions of four sets of generalized (2+1)-dimensional Boussinesq-Kadomtsev-Petviashvili-like equations, *Mod. Phys. Lett. B*, **36** (2022), 2150530 https://doi.org/10.1142/S0217984921505308

18. M. S. Osman, H. I. Abdel-Gawad, M. A. El Mahdy, Two-layer-atmospheric blocking in a medium with high nonlinearity and lateral dispersion, *Results Phys.*, **8** (2018), 1054–1060. https://doi.org/10.1016/j.rinp.2018.01.040

19. A. A. J. Gaber, Solitary and periodic wave solutions of (2+1)-dimensions of dispersive long wave equations on shallow water, *J. Ocean Eng. Sci.*, **6** (2021), 292–298. https://doi.org/10.1016/j.joes.2021.02.002

20. S. F. Tian, J. M. Tu, T. T. Zhang, Y. R. Chen, Integrable discretizations and soliton solutions of an Eckhaus-Kundu equation, *Appl. Math. Lett.*, **122** (2021), 107507 https://doi.org/10.1016/j.aml.2021.107507

21. S. F. Tian, M. J. Xu, T. T. Zhang, A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation, *Proc. R. Soc. A*, **477** (2021), 20210455. https://doi.org/10.1098/rspa.2021.0455

22. S. F. Tian, Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation, *Appl. Math. Lett.*, **100** (2020), 106056. https://doi.org/10.1016/j.aml.2019.106056

23. M. Adel, D. Baleanu, U. Sadiya, M. A. Arefin, M. Hafiz Uddin, Mahjoub A. Elamin, et al., Inelastic soliton wave solutions with different geometrical structures to fractional order nonlinear evolution equations, *Results Phys.*, **38** (2022), 105661. https://doi.org/10.1016/j.rinp.2022.105661

24. A. Zafar, M. Raheel, M. Q. Zafar, K. S. Nisar, M. S. Osman, R. N. Mohamed, et al., Dynamics of different nonlinearities to the perturbed nonlinear schrödinger equation via solitary wave solutions with numerical simulation, *Fractal Fract.*, **5** (2021), 213. https://doi.org/10.3390/fractalfract5040213

25. M. Mirzazadeh, A. Akbulut, F. Tascan, L. Akinyemi, A novel integration approach to study the perturbed Biswas-Milovic equation with Kudryashov's law of refractive index, *Optik*, **252** (2022), 168529. https://doi.org/10.1016/j.ijleo.2021.168529

26. O. I. Bogoyavlenskii, Breaking solitons in 2+1-dimensional integrable equations, *Russ. Math. Surv.*, **45** (1990), 1. https://doi.org/10.1070/RM1990v045n04ABEH002377

27. N. Kudryashov, A. Pickering, Rational solutions for the Schwarzian integrable hierarchies, *J. Phys. A Math. Gen.*, **31** (1998), 9505. https://doi.org/10.1088/0305-4470/31/47/011

28. P. A. Clarkson, P. R. Gordoa, A. Pickering, Multicomponent equations associated to non-isospectral scattering problems, *Inverse Probl.*, **13** (1997), 1463. https://doi.org/10.1088/0266-5611/13/6/004

29. P. G. Estevez, J. Prada, A generalization of the sine-Gordon equation to (2+1)-dimensions, *J. Nonlinear Math. Phy.*, **11** (2004), 164–179. https://doi.org/10.2991/jnmp.2004.11.2.3

30. Y. Z. Peng, S. Ming, On exact solutions of the bogoyavlenskii equation, *Pramana*, **67** (2006), 449–456. https://doi.org/10.1007/s12043-006-0005-1

31. E. Zahran, M. A. Khater, Modified extended tanh-function method and its applications to the Bogoyavlenskii equation, *Appl. Math. Model.*, **40** (2016), 1769–1775. https://doi.org/10.1016/j.apm.2015.08.018

32. A. Malik, F. Chand, H. Kumar, S. C. Mishra, Exact solutions of the Bogoyavlenskii equation using the multiple (G'/G)-expansion method, *Comput. Math. Appl.*, **64** (2012), 2850–2859. https://doi.org/10.1016/j.camwa.2012.04.018

33. J. Yu, Y. Sun, Modified method of simplest equation and its applications to the Bogoyavlenskii equation, *Comput. Math. Appl.*, **72** (2016), 1943–1955. https://doi.org/10.1016/j.camwa.2016.08.002

34. A. Yokus, H. Durur, H. Ahmad, P. Thounthong, Y. F. Zhang, Construction of exact traveling wave solutions of the Bogoyavlenskii equation by $(G'/G, 1/G)$-expansion and $(1/G')$-expansion techniques, *Results Phys.*, **19** (2020), 103409. https://doi.org/10.1016/j.rinp.2020.103409

35. N. H. Aljahdaly, Aly R. Seadawy, Wafaa A. Albarakati, Applications of dispersive analytical wave solutions of nonlinear seventh order Lax and Kaup-Kupershmidt dynamical wave equations, *Results Phys.*, **14** (2019), 102372. https://doi.org/10.1016/j.rinp.2019.102372

36. S. Behera, N. H. Aljahdaly, J. P. S. Virdi, On the modified (G'/G^2)-expansion method for finding some analytical solutions of the traveling waves, *J. Ocean Eng. Sci.*, **7** (2022), 313–320. https://doi.org/10.1016/j.joes.2021.08.013

37. H. Chen, Q. Zhu, J. Qi, Further results about the exact solutions of conformable space-time fractional Boussinesq equation (FBE) and breaking soliton (Calogero) equation, *Results Phys.*, **37** (2022), 105428. https://doi.org/10.1016/j.rinp.2022.105428

38. J. Manafian, M. Lakestani, Application of $\tan(\phi/2)$-expansion method for solving the Biswas-Milovic equation for Kerr law nonlinearity, *Optik*, **127** (2016), 2040–2054. https://doi.org/10.1016/j.ijleo.2015.11.078

39. J. Manafian, Optical soliton solutions for Schrödinger type nonlinear evolution equations by the $\tan(\phi/2)$-expansion method, *Optik*, **127** (2016), 4222–4245. https://doi.org/10.1016/j.ijleo.2016.01.078

40. H. Liu, T. Zhang, A note on the improved $\tan(\phi/2)$-expansion method, *Optik*, **131** (2017), 273–278. https://doi.org/10.1016/j.ijleo.2016.11.029

41. H. C. Yaslan, A. Girgin, SITEM for the conformable space-time fractional Boussinesq and (2+1)-dimensional breaking solition equations, *J. Ocean Eng. Sci.*, **6** (2021), 228–236. https://doi.org/10.1016/j.joes.2020.11.010