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Research article

The Gauss sums involving 24-order character and their recursive properties

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Abstract: The main purpose of this paper is to use elementary and analytic methods to study the calculating problem of one kind of Gauss sums and obtain an exact computational formula for it.

Keywords: one kind of Gauss sums; elementary methods; analytic methods; computational formula
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1. Introduction

Let $q > 1$ be an integer. For any Dirichlet character χ modulo q, the classical Gauss sums $G(m,\chi;q)$ is defined as follows (see Section 5 of Chapter 8 in [1]).

$$G(m,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma}{q}\right),$$

where m is any integer, $e(y) = e^{2\pi iy}$ and $i^2 = -1$.

For convenience, we write $\tau(\chi) = G(1,\chi;q)$. If χ is a primitive character modulo q or $(m,q) = 1$, then we have (see [1,2]): $G(m,\chi;q) = \overline{\chi}(m)\tau(\chi)$ and the identity $|\tau(\chi)| = \sqrt{q}$. The study of the classical Gauss sums $G(m,\chi;q)$ has received considerable attention in past decades. For example, B. C. Berndt and R. J. Evans [3] studied the properties of some special Gauss sums, and obtained the following interesting results:

$$\tau^3(\chi_3) + \tau^3\left(\overline{\chi_3}\right) = dp, \tag{1.1}$$

where p is a prime with $p \equiv 1 \bmod 3$, χ_3 is any three-order character modulo p, and d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

L. Chen [4] obtained another identity for the six-order character modulo p. That is, she proved the following conclusion: Let p be a prime with $p \equiv 1 \bmod 6$, then for any six-order character χ_6 modulo p, we have

$$\tau^3(\chi_6) + \tau^3\left(\overline{\chi_6}\right) = \begin{cases} p^{\frac{1}{2}} \cdot \left(d^2 - 2p\right), & \text{if } p \equiv 1 \bmod 12; \\ -i \cdot p^{\frac{1}{2}} \cdot \left(d^2 - 2p\right), & \text{if } p \equiv 7 \bmod 12, \end{cases} \tag{1.2}$$

where d is the same as defined in (1.1).

As an application of (1.2), L. Chen [4] proved the following conclusion: Let p be a prime with $p \equiv 1 \bmod 12$. Then for any three-order character χ_3 modulo p and integer $n \geq 0$, one has the identity

$$U_n(p) = \frac{\tau^{3n}(\chi_3)}{\tau^{3n}\left(\overline{\chi_3}\right)} + \frac{\tau^{3n}\left(\overline{\chi_3}\right)}{\tau^{3n}(\chi_3)} = \left(\frac{d^2 - 2p + 3dbi\sqrt{3}}{2p}\right)^n + \left(\frac{d^2 - 2p - 3dbi\sqrt{3}}{2p}\right)^n;$$

If $p \equiv 7 \bmod 12$, then one has the identity

$$U_n(p) = i^n \left(\frac{2p - d^2 + \sqrt{8p^2 - 4d^2p + d^4}}{2p}\right)^n + i^n \left(\frac{2p - d^2 - \sqrt{8p^2 - 4d^2p + d^4}}{2p}\right)^n,$$

where d and b are the same as defined in (1.1).

Z. Y. Chen and W. P. Zhang [5] studied the case of the four-order character modulo p, and obtained the following conclusion: Let p be a prime with $p \equiv 1 \bmod 4$. Then for any four-order character χ_4 modulo p, one has the identity

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p} \cdot \alpha \quad \text{and} \quad \alpha = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a + \overline{a}}{p}\right), \tag{1.3}$$

where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre's symbol modulo p. It is clear that the constant $\alpha = \alpha(p)$ in (1.3) is closely related to prime p. In fact, we have the expression (For this see Theorem 4–11 in [8])

$$p = \alpha^2 + \beta^2 \equiv \left(\frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a + \overline{a}}{p}\right)\right)^2 + \left(\frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a + r\overline{a}}{p}\right)\right)^2, \tag{1.4}$$

where r is any quadratic non-residue modulo p. That is, $\chi_2(r) = -1$.

T. T. Wang and G. H. Chen [6] studied the Gauss sums for 12-order character χ_{12} modulo p, and proved that

$$\frac{\tau^{6n}(\chi_{12})}{\tau^{6n}\left(\chi_{12}^5\right)} + \frac{\tau^{6n}\left(\overline{\chi_{12}}\right)}{\tau^{6n}\left(\overline{\chi_{12}^5}\right)} = \left(\frac{\delta + \sqrt{\delta^2 - 4}}{2}\right)^n + \left(\frac{\delta - \sqrt{\delta^2 - 4}}{2}\right)^n,$$

where $p \equiv 1 \bmod 12$, $\delta = \frac{2p^2 - 4pd^2 + d^4}{p^2}$, and d is the same as in (1.1).

Some other results related to various Gauss sums and their recursion properties can also be found in [7, 9–15], see the Gauss sums for 8-order character modulo p in [7, 12] for instance. The main result considered in this paper is motivated by these references.

The main purpose of this paper is to consider the computational problem of the Gauss sums for 24-order character modulo p. To be exact, for any prime p with $p \equiv 1 \bmod 24$, let χ_3, χ_8 be a three-order and eight-order character modulo p, respectively. For any integer $n \geq 0$, we write

$$A_n(p) = \frac{\tau^{4n}\left(\overline{\chi}_8\chi_3\right)}{\tau^{4n}\left(\chi_8\chi_3\right)} + \frac{\tau^{4n}\left(\chi_8\chi_3\right)}{\tau^{4n}\left(\overline{\chi}_8\chi_3\right)} = \frac{\tau^{4n}\left(\overline{\chi}_8\overline{\chi_3}\right)}{\tau^{4n}\left(\chi_8\overline{\chi_3}\right)} + \frac{\tau^{4n}\left(\chi_8\overline{\chi_3}\right)}{\tau^{4n}\left(\overline{\chi}_8\overline{\chi_3}\right)}. \tag{1.5}$$

Our goal is to give an exact computational formula for (1.5).

As far as this problem is concerned, no one has studied it, at least we have not seen any related results in the related literature.

In this paper, we will use the analytic methods and the properties of the classical Gauss sums to give an exact computational formulas for (1.5). That is, we shall prove the following:

Theorem. Let p be an odd prime with $p \equiv 1 \bmod 24$. Then for any integer $n \geq 0$, we have the identity

$$\begin{aligned} A_n(p) &= \left(\frac{2\alpha^2 - p + 2\alpha\beta i}{p}\right)^n + \left(\frac{2\alpha^2 - p - 2\alpha\beta i}{p}\right)^n \\ &= \left(\frac{\alpha^2 - \beta^2 + 2\alpha\beta i}{p}\right)^n + \left(\frac{\alpha^2 - \beta^2 - 2\alpha\beta i}{p}\right)^n, \end{aligned}$$

where α and β are the same as defined as in (1.3) and (1.4), and $i^2 = -1$.

From this theorem we may immediate deduce the follow:

Corollary. Let p be an odd prime with $p \equiv 1 \bmod 24$, χ_8 be any eight-order character and χ_3 be any three-order character modulo p. Then we have the identity

$$\frac{\tau^4\left(\overline{\chi}_8\chi_3\right)}{\tau^4\left(\chi_8\chi_3\right)} = \frac{\alpha^2 - \beta^2}{p} \pm \frac{2\alpha\beta}{p} \cdot i.$$

Some notes. In fact, the sequence $A_n(p)$ satisfies the second-order linear recurrence formula:

$$A_{n+1}(p) = \frac{2(\alpha^2 - \beta^2)}{p} \cdot A_n(p) - A_{n-1}(p), \; n \geq 1$$

with the initial values $A_0(p) = 2$ and $A_1(p) = \frac{2(\alpha^2 - \beta^2)}{p}$.

For general positive integer k, let p be a prime with $p \equiv 1 \bmod 3 \cdot 2^k$, then for integer $n \geq 0$, whether there exists an exact computational formula for the sums

$$B_n(p) = \frac{\tau^{4n}\left(\overline{\chi}_{2^k}\chi_3\right)}{\tau^{4n}\left(\chi_{2^k}\chi_3\right)} + \frac{\tau^{4n}\left(\chi_{2^k}\chi_3\right)}{\tau^{4n}\left(\overline{\chi}_{2^k}\chi_3\right)} \; ?$$

where χ_{2^k} is a 2^k-order character modulo p, χ_3 is a three-order character modulo p.

This is an open problem. It remains to be further studied.

Of course, how to determine the plus or minus signs in the corollary is also a meaningful problem. Interested readers may consider it.

Notation. Before proceeding, we fixed some notation used throughout the paper. p is always reserved for a prime number. We use χ, χ_k to denote any non-principal character of modulo p and the k-order character of modulo p, respectively. $\tau(\chi) = G(1, \chi; q)$ means the classical Gauss sum, and $\tau^h(\chi)$ denotes $(\tau(\chi))^h$. As is usual, we abbreviate $e^{2\pi i y}$ to $e(y)$, where $i^2 = -1$.

2. Several lemmas

In this section, we give several simple but necessary lemmas. Many of the statement in this section are standard, and the readers can refer to many classical monographs, such as [1, 2, 8].

Lemma 1. Let p be an odd prime. Then for any non-principal character χ modulo p, we have the identity

$$\tau\left(\overline{\chi}^2\right) = \frac{\overline{\chi}^2(2)\chi_2(-1)\overline{\chi}(-1)\cdot\tau(\chi_2)\cdot\tau\left(\overline{\chi}\chi_2\right)}{\tau\left(\chi\right)}.$$

Proof. From the properties of the classical Gauss sums we infer

$$\begin{aligned}
&\sum_{a=0}^{p-1}\chi\left(a^2-1\right) = \sum_{a=0}^{p-1}\chi\left((a+1)^2-1\right) = \sum_{a=1}^{p-1}\chi(a)\chi\left(a+2\right)\\
=\;& \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}(b)\sum_{a=1}^{p-1}\chi(a)e\left(\frac{b(a+2)}{p}\right) = \frac{\tau\left(\chi\right)}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}(b)\overline{\chi}(b)e\left(\frac{2b}{p}\right)\\
=\;& \frac{\tau\left(\chi\right)}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}^2(b)e\left(\frac{2b}{p}\right) = \frac{\chi^2(2)\cdot\tau\left(\chi\right)\cdot\tau\left(\overline{\chi}^2\right)}{\tau\left(\overline{\chi}\right)}.
\end{aligned}\tag{2.1}$$

On the other hand, for any integer b with $(b, p) = 1$, note that the identity

$$\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1+\chi_2(a)\right)e\left(\frac{ba}{p}\right) = \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ba}{p}\right) = \chi_2(b)\cdot\tau(\chi_2),$$

we also have

$$\begin{aligned}
&\sum_{a=0}^{p-1}\chi\left(a^2-1\right) = \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{b(a^2-1)}{p}\right)\\
=\;& \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{-b}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right) = \frac{\tau(\chi_2)}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}(b)\chi_2(b)e\left(\frac{-b}{p}\right)\\
=\;& \frac{\chi_2(-1)\overline{\chi}(-1)\tau(\chi_2)\cdot\tau\left(\overline{\chi}\chi_2\right)}{\tau\left(\overline{\chi}\right)}.
\end{aligned}\tag{2.2}$$

Formulas (2.1) and (2.2) yield

$$\tau\left(\overline{\chi}^2\right) = \frac{\overline{\chi}^2(2)\chi_2(-1)\overline{\chi}(-1)\cdot\tau(\chi_2)\cdot\tau\left(\overline{\chi}\chi_2\right)}{\tau\left(\chi\right)}.$$

This proves Lemma 1.

Lemma 2. Let p be an odd prime with $p \equiv 1 \bmod 8$. Then for any eight-order character χ_8 modulo p, we have the identity

$$\frac{\tau^4\left(\chi_8^3\right)}{\tau^4\left(\chi_8\right)} + \frac{\tau^4\left(\chi_8\right)}{\tau^4\left(\chi_8^3\right)} = \frac{2\left(2\alpha^2-p\right)}{p} = \frac{2\left(\alpha^2-\beta^2\right)}{p},$$

where α is the same as defined as in (1.3).

Proof. Taking $\chi = \chi_8$ in Lemma 1, note that $\overline{\chi}_8\chi_2 = \chi_8^3$, $\tau(\chi_8)\tau(\overline{\chi}_8) = \overline{\chi}_8(-1)\cdot p$ and $\tau(\chi_2) = \sqrt{p}$, from Lemma 1 we derive that

$$\tau(\overline{\chi}_4) = \frac{\overline{\chi}_4(2)\overline{\chi}_8(-1)\cdot\sqrt{p}\cdot\tau\left(\chi_8^3\right)}{\tau(\chi_8)} \tag{2.3}$$

and

$$\tau(\chi_4) = \frac{\chi_4(2)\chi_8(-1)\cdot\sqrt{p}\cdot\tau\left(\overline{\chi}_8^3\right)}{\tau(\overline{\chi}_8)} = \frac{\chi_4(2)\chi_8(-1)\cdot\sqrt{p}\cdot\tau(\chi_8)}{\tau\left(\chi_8^3\right)}. \tag{2.4}$$

Combining formulae (1.3), (2.3) and (2.4), we obtain

$$p^2\cdot\left(\frac{\tau^4\left(\chi_8^3\right)}{\tau^4(\chi_8)} + \frac{\tau^4(\chi_8)}{\tau^4\left(\chi_8^3\right)}\right) = \tau^4(\chi_4) + \tau^4(\overline{\chi}_4)$$
$$= \left(\tau^2(\chi_4) + \tau^2(\overline{\chi}_4)\right)^2 - 2p^2 = 4p\alpha^2 - 2p^2,$$

which completes the proof readily in view of $\tau^2(\chi_4)\cdot\tau^2(\overline{\chi}_4) = p^2$.

Lemma 3. Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any character χ modulo p, we have the identity

$$\tau\left(\chi^3\right) = \frac{1}{p}\cdot\chi^3(3)\cdot\tau(\chi)\cdot\tau(\chi\chi_3)\cdot\tau(\chi\overline{\chi_3}),$$

where χ_3 is a three-order character modulo p.

Proof. For this see [16, 17]. The general result can also be found in [18].

Lemma 4. Let p be a prime with $p \equiv 1 \bmod 24$, χ_8 be any eight-order character and χ_3 be any three-order character modulo p. Then we have the identity

$$\frac{\tau^4(\overline{\chi}_8\chi_3)}{\tau^4(\chi_8\chi_3)} + \frac{\tau^4(\chi_8\chi_3)}{\tau^4(\overline{\chi}_8\chi_3)} = \frac{2\left(2\alpha^2 - p\right)}{p} = \frac{2\left(\alpha^2 - \beta^2\right)}{p}.$$

Proof. We consider $\chi = \chi_8$ in Lemma 3, so that

$$\tau\left(\chi_8^3\right) = \frac{\chi_8^3(3)}{p}\cdot\tau(\chi_8)\cdot\tau(\chi_8\overline{\chi_3})\cdot\tau(\chi_8\chi_3). \tag{2.5}$$

We can derive by adjusting both sides of the equation above that

$$\frac{\tau^4\left(\chi_8^3\right)}{\tau^4(\chi_8)} = \frac{\tau^4(\chi_8\chi_3)}{\tau^4(\overline{\chi}_8\chi_3)} \tag{2.6}$$

and

$$\frac{\tau^4(\chi_8)}{\tau^4\left(\chi_8^3\right)} = \frac{\tau^4(\overline{\chi}_8\chi_3)}{\tau^4(\chi_8\chi_3)}. \tag{2.7}$$

These two formulae follow by noting that $\tau\left(\chi_8\overline{\chi_3}\right)\tau\left(\overline{\chi}_8\chi_3\right)=\overline{\chi}_8(-1)\cdot p$ and $\chi_2(3)=1$.

Combining (2.6), (2.7) and Lemma 2 we can get

$$\frac{\tau^4\left(\overline{\chi}_8\chi_3\right)}{\tau^4\left(\chi_8\chi_3\right)}+\frac{\tau^4\left(\chi_8\chi_3\right)}{\tau^4\left(\overline{\chi}_8\chi_3\right)}=\frac{\tau^4\left(\chi_8^3\right)}{\tau^4\left(\chi_8\right)}+\frac{\tau^4\left(\chi_8\right)}{\tau^4\left(\chi_8^3\right)}=\frac{2\left(2\alpha^2-p\right)}{p}=\frac{2\left(\alpha^2-\beta^2\right)}{p}.$$

This completes the proof of Lemma 4.

3. Proof of the theorem

In this section we will use the lemmas from Section 2 to prove the theorem. From Lemma 4 we know that $A_0(p)=2$ and $A_1(p)=\frac{2\left(2\alpha^2-p\right)}{p}$. If $n\geq 1$, then from the definition of $A_n(p)$ we have the identity

$$\begin{aligned}A_1(p)\cdot A_n(p)&=\left(\frac{\tau^4\left(\overline{\chi}_8\chi_3\right)}{\tau^4\left(\chi_8\chi_3\right)}+\frac{\tau^4\left(\chi_8\chi_3\right)}{\tau^4\left(\overline{\chi}_8\chi_3\right)}\right)\cdot\left(\frac{\tau^{4n}\left(\overline{\chi}_8\chi_3\right)}{\tau^{4n}\left(\chi_8\chi_3\right)}+\frac{\tau^{4n}\left(\chi_8\chi_3\right)}{\tau^{4n}\left(\overline{\chi}_8\chi_3\right)}\right)\\ &=\frac{\tau^{4(n+1)}\left(\overline{\chi}_8\chi_3\right)}{\tau^{4(n+1)}\left(\chi_8\chi_3\right)}+\frac{\tau^{4(n+1)}\left(\chi_8\chi_3\right)}{\tau^{4(n+1)}\left(\overline{\chi}_8\chi_3\right)}+\frac{\tau^{4(n-1)}\left(\chi_8\chi_3\right)}{\tau^{4(n-1)}\left(\overline{\chi}_8\chi_3\right)}+\frac{\tau^{4(n-1)}\left(\overline{\chi}_8\chi_3\right)}{\tau^{4(n-1)}\left(\chi_8\chi_3\right)}\\ &=A_{n+1}(p)+A_{n-1}(p)\end{aligned}$$

or

$$A_{n+1}(p)=\frac{2\cdot\left(2\alpha^2-p\right)}{p}\cdot A_n(p)-A_{n-1}(p),\quad n\geq 1. \tag{3.1}$$

Let x_1 and x_2 be two roots of the equation $x^2-\frac{2\left(2\alpha^2-p\right)}{p}\cdot x+1=0$. Then from (1.4) we obtain

$$x_1=\frac{2\alpha^2-p+2\alpha\beta i}{p},\quad x_2=\frac{2\alpha^2-p-2\alpha\beta i}{p},$$

where α and β are the same as defined as in (1.3) and (1.4), and $i^2=-1$.

From (3.1) and the properties of the second order linear recursive sequence we derive that

$$A_n(p)=\left(\frac{2\alpha^2-p+2\alpha\beta i}{p}\right)^n+\left(\frac{2\alpha^2-p-2\alpha\beta i}{p}\right)^n,\quad n\geq 0.$$

Note that $p=\alpha^2+\beta^2$, this completes the proof of our theorem.

4. Conclusions

The main result of this paper is the theorem, an exact computational formula for one kind of Gauss sums is obtained. The result is not only closely related to the second order linear recursive sequence, but also makes a new contribution to the research in related fields.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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