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Research article

Existence of solutions for multi-point nonlinear differential system equations of fractional orders with integral boundary conditions

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Abstract: In this work, we study existence and uniqueness of solutions for multi-point boundary value problemS of nonlinear fractional differential equations with two fractional derivatives. By using a variety of fixed point theorems, such as Banach's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory, the existence of solutions is obtained. At the end, some illustrative examples are discussed.

Keywords: Riemann-Liouville integral; existence of solutions; fixed point theorem
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1. Introduction

Differential equations of arbitrary order have been shown to be useful in the study of models of many phenomena in various fields such as: Electrochemistry and material science, they are in fact described by differential equations of fractional order [9, 10, 15, 16, 25–29]. For more details, we refer the reader to the books of Hilfer [30], Podlubny [31], Kilbas et al. [34], Miller and Ross [22] and to the following research papers [1–8, 11, 12, 14, 16, 17, 19, 20, 24, 31, 35–42]. In this work, we discuss the existence and uniqueness of the solutions for multi-point boundary value problems of nonlinear fractional differential equations with two Riemann-Liouville fractionals:

$$\begin{cases} D^{\alpha}x(t) = \sum_{i=1}^{m} f_i(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t)), \alpha \in]1, 2], t \in [0, T] \\ D^{\beta}y(t) = \sum_{i=1}^{m} g_i(t, x(t), y(t), \varphi_2 x(t), \phi_2 y(t)), \beta \in]1, 2], t \in [0, T] \\ I^{2-\alpha}x(0) = 0,\ D^{\alpha-2}x(T) = \theta I^{\alpha-1}(x(\eta)),\ 0 < \eta < T, \\ I^{2-\beta}y(0) = 0,\ D^{\beta-2}x(T) = \omega I^{\beta-1}(x(\gamma)),\ 0 < \gamma < T, \end{cases} \tag{1.1}$$

where $D^{(.)}$, $I^{(.)}$ denote the Riemann-Liouville derivative and integral of fractional order (.), respectively, $f_i, g_i : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$, $i = 1, \cdots, m$ are continuous functions on $[0,T]$ and

$$(\varphi_1 x)(t) = \int_0^t A_1'(t,s)x(s)ds,\ (\phi_1 y)(t) = \int_0^t B_1'(t,s)y(s)ds,$$

$$(\varphi_2 x)(t) = \int_0^t A_2'(t,s)x(s)ds,\ (\phi_1 y)(t) = \int_0^t B_2'(t,s)y(s)ds,$$

with A_i and B_i being continuous functions on $[0,1] \times [0,1]$.

However, it is rare to find a work in nonlinear term f_i depends on fractional derivative of unknown functions $x(t), y(t), \varphi_1 x(t), \phi_1 y(t)$ and solutions for multi-order fractional differential equations on the infinite interval $[0,T)$. Motivated by [8, 11–14] and the references therein, we consider the existence and unicity of solution for multi-order fractional differential equations on infinite interval $[0,T)$.

The rest of this paper is organized as follow. In section 2, we present some preliminaries and lemmas. Section 3 is dedicated to showing the existence of a solution for problem (1.1). Finally, section 4 illustrated the proposed results with two examples.

Remark 1.1. *This work generalizes the work of Houas and Benbachir [14] on different boundary conditions and for another type of integral.*

2. Preliminaries

This section covers the basic concepts of Riemann-Liouville type fractional calculus that will be used throughout this paper.

Definition 2.1. *[31, 32] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$\begin{cases} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \\ J^0 f(t) = f(t), \end{cases}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2.2. *[31, 32] The Riemann-Liouville fractional derivative of order $\alpha > 0$, of a continuous function $h : (0,\infty) \to \mathbb{R}$ is defined as*

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} h(\tau)\, d\tau = \left(\frac{d}{dt}\right)^n I^{n-\alpha} h(\tau),$$

where $n = [\alpha] + 1$.

For $\alpha < 0$, we use the convention that $D^\alpha h = J^{-\alpha} h$. Also for $0 \leq \rho < \alpha$, it is valid that $D^\rho J^\alpha h = h^{\alpha-\rho}$.

We note that for $\varepsilon > -1$ and $\varepsilon \neq \alpha - 1, \alpha - 2, ..., \alpha - n$, we have

$$\begin{aligned} D^\alpha t^\varepsilon &= \frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon-\alpha+1)} t^{\varepsilon-\alpha}, \\ D^\alpha t^{\alpha-i} &= 0,\ i = 1, 2, ..., n. \end{aligned}$$

In particular, for the constant function $h(t) = 1$, we obtain

$$D^{\alpha}1 = \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}, \alpha \notin \mathbb{N}.$$

For $\alpha \in \mathbb{N}$, we obtain, of course, $D^{\alpha}1 = 0$ because of the poles of the gamma function at the points $0, -1, -2, ...$ For $\alpha > 0$, the general solution of the homgeneous equation $D^{\alpha}h(t) = 0$ in $C(0, T) \cap L(0, T)$ is

$$h(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + ... + c_{n-2}t^{\alpha-2} + c_{n-1}t^{\alpha-1},$$

where $c_i, i = 1, 2, ..., n-1$, are arbitrary real constants. Further, we always have $D^{\alpha}I^{\alpha}h = h$, and

$$D^{\alpha}I^{\alpha}h(t) = h(t) + c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + ... + c_{n-2}t^{\alpha-2} + c_{n-1}t^{\alpha-1}.$$

Lemma 2.1. *[33] Let E be Banach space. Assume that $T : E \longrightarrow E$ is a completely continuous operator. If the set $V = \{x \in E : x = \mu T x,\ 0 < \mu < 1\}$ is bounded, then T has a fixed point in E.*

To define the solution for problem (1.1). We consider the following lemma.

Lemma 2.2. *Suppose that $(H_i)_{i=1,...,m} \subset C([0, 1], \mathbb{R})$, and consider the problem*

$$D^{\alpha}h(t) - \sum_{i=1}^{m} H_i(t) = 0,\ t \in j,\ 1 < \alpha < 2,\ m \in \mathbb{N}^*, \tag{2.1}$$

with the conditions

$$I^{2-\alpha}h(0) = 0,\ D^{\alpha-2}h(T) = \theta I^{\alpha-1}(h(\eta)),\ 0 < \eta < T. \tag{2.2}$$

Then we have

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_0^t (t-\tau)^{\alpha-1}H_i(\tau)d\tau \\ &+\frac{t^{\alpha-1}}{\psi}\left(\sum_{i=1}^{m}\int_0^T (T-\tau)H_i(\tau)d\tau - \frac{\theta}{\Gamma(2\alpha)}\sum_{i=1}^{m}\int_0^{\eta}(\eta-\tau)^{2\alpha-2}H_i(\tau)d\tau\right) \end{aligned}$$

with $\psi = \theta\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\eta^{2\alpha-2} - \Gamma(\alpha)T.$

Proof. We have

$$h(t) = \sum_{i=1}^{m} I^{\alpha}H_i(t) + c_0 t^{\alpha-2} + c_1 t^{\alpha-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1.$

We obtain

$$\begin{aligned} I^{2-\alpha}h(\tau) &= \sum_{i=1}^{m} I^2 H_i(\tau) + c_0 I^{2-\alpha}\tau^{\alpha-2} + c_1 I^{2-\alpha}\tau^{\alpha-1} \\ &= \sum_{i=1}^{m} I^2 H_i(\tau) + c_0 + c_1\tau, \end{aligned}$$

$$
\begin{aligned}
I^{\alpha-1}h(\tau) &= \sum_{i=1}^{m} I^{2\alpha-1}H_i(\tau) + c_0 I^{\alpha-1}\tau^{\alpha-2} + c_1 I^{\alpha-1}\tau^{\alpha-1} \\
&= \sum_{i=1}^{m} I^{2\alpha-1}H_i(\tau) + c_0\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}\tau^{2\alpha-3} + c_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\tau^{2\alpha-2}, \\
D^{\alpha-2}h(\tau) &= \sum_{i=1}^{m} I^{2}H_i(\tau) + c_0\Gamma(\alpha-1) + c_1\Gamma(\alpha)\tau.
\end{aligned}
$$

Using the given conditions: $I^{2-\alpha}h(0) = 0$, we find that $c_0 = 0$, and since $D^{\alpha-2}h(T) - \theta I^{\alpha-1}(h(\eta)) = 0$, we have

$$
\sum_{i=1}^{m} I^2 h_i(T) + c_1\Gamma(\alpha)T - \theta\left[\sum_{i=1}^{m} I^{2\alpha-1}h_i(\eta) + c_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\eta^{2\alpha-2}\right] = 0,
$$

then

$$
c_1\left[\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\eta^{2\alpha-2} - \Gamma(\alpha)T\right] = \sum_{i=1}^{m} I^2 h_i(T) - \theta\sum_{i=1}^{m} I^{2\alpha-1}h_i(\eta)
$$

and

$$
\begin{aligned}
c_1 &= \frac{1}{\psi}\left(\sum_{i=1}^{m} I^2 H_i(T) - \theta\sum_{i=1}^{m} I^{2\alpha-1}H_i(\eta)\right) \\
&= \frac{1}{\psi}\left(\sum_{i=1}^{m}\int_0^T (T-\tau)H_i(\tau)d\tau - \frac{\theta}{\Gamma(2\alpha)}\sum_{i=1}^{m}\int_0^{\eta}(\eta-\tau)^{2\alpha-2}H_i(\tau)d\tau\right)
\end{aligned}
$$

with

$$
\psi = \theta\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\eta^{2\alpha-2} - \Gamma(\alpha)T.
$$

Finally, the solution of (2.1) and (2.2) is

$$
\begin{aligned}
h(t) = {} & \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_0^t (t-\tau)^{\alpha-1}H_i(\tau)d\tau + \frac{t^{\alpha-1}}{\psi}\left(\sum_{i=1}^{m}\int_0^T (T-\tau)H_i(\tau)d\tau\right. \\
& \left. -\frac{\theta}{\Gamma(2\alpha)}\sum_{i=1}^{m}\int_0^{\eta}(\eta-\tau)^{2\alpha-2}H_i(\tau)d\tau\right).
\end{aligned}
$$

□

3. Main results

We denote by

$$
E = \{x, y \in C([0,T], \mathbb{R}); \varphi_i x, \phi_i y \in C([0,T], \mathbb{R}) \quad i = 1, 2\},
$$

and the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by

$$
\|(x, y)\|_E = \max\left(\|x\|, \|y\|, \|\varphi_1 x\|, \|\phi_1 y\|, \|\varphi_2 x\|, \|\phi_2 y\|\right),
$$

where

$$\|x\| = \sup_{t\in j} |\varphi_i x(t)|, \quad \|y\| = \sup_{t\in j} |y(t)|,$$
$$\|\phi_i x\| = \sup_{t\in j} |\varphi_i x(t)|, \quad \|\phi_i y\| = \sup_{t\in j} |\phi_i y(t)|.$$

In this section, we prove some existence and uniqueness results to the nonlinear fractional coupled system (1.1).

For the sake of convenience, we impose the following hypotheses:

$(H1)$ For each $i = 1, 2, \cdots, m$, the functions f_i and g_i $: [0, T] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ are continuous.

$(H2)$ There exist nonnegative real numbers $\xi^i_k, \varphi^i_k, k = 1, 2, 3, 4, i = 1, 2, \cdots, m$, such that for all $t \in [0, T]$ and all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, we have

$$|f_i(t, x_1, x_2, x_3, x_4) - f_i(t, y_1, y_2, y_3, y_4)| \le \sum_{k=1}^{4} \xi^i_k |x_k - y_k|,$$

and

$$|g_i(t, x_1, x_2, x_3, x_4) - g_i(t, y_1, y_2, y_3, y_4)| \le \sum_{k=1}^{4} \chi^i_k |x_k - y_k|.$$

$(H3)$ There exist nonnegative constants (L_i) and (K_i) $i = 1, ..., m$, such that: For each $t \in [0, T]$ and all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

$$|f_i(t, x_1, x_2, x_3, x_4)| \le L_i, |g_i(t, x_1, x_2, x_3, x_4)| \le K_i, i = 1, ..., m.$$

We also consider the following quantities:

$$\begin{aligned}
A_1 &= \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} (\xi^i_1 + \xi^i_2 + \xi^i_3 + \xi^i_4),\\
A_2 &= \frac{T^\beta}{\Gamma(\beta+1)} \sum_{i=1}^{m} (\chi^i_1 + \chi^i_2 + \chi^i_3 + \chi^i_4),\\
A_3 &= \max_{t,s\in[0,1]} \|A'_1(t, s)\| \times A_1,\\
A_4 &= \max_{t,s\in[0,1]} \|A'_2(t, s)\| \times A_1,\\
A_5 &= \max_{t,s\in[0,1]} \|B'_1(t, s)\| \times A_2,\\
A_6 &= \max_{t,s\in[0,1]} \|B'_2(t, s)\| \times A_2,\\
\nu_1 &= \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\psi}\left(\frac{T^{\alpha+1}}{2} + \frac{\theta T^{3\alpha-2}}{(2\alpha-1)^2\Gamma(2\alpha-1)}\right)\right],\\
\nu_2 &= \left[\frac{T^\beta}{\Gamma(\beta+1)} + \frac{1}{\psi'}\left(\frac{T^{\beta+1}}{2} + \frac{\omega T^{3\beta-2}}{(2\beta-1)^2\Gamma(2\beta-1)}\right)\right],\\
\nu_3 &= \max_{t,s\in[0,1]} |A'_1(t, s)| \nu_1,\\
\nu_4 &= \max_{t,s\in[0,1]} |A'_2(t, s)| \nu_1,\\
\nu_5 &= \max_{t,s\in[0,1]} |B'_1(t, s)| \nu_2,\\
\nu_6 &= \max_{t,s\in[0,1]} |B'_2(t, s)| \nu_2.
\end{aligned}$$

3.1. Existence of solutions

The first result is based on Banach contraction principle. We have

Theorem 3.1. *Assume that* $(H2)$ *holds. If the inequality*

$$\max(A_1, A_2, A_3, A_4, A_5, A_6) < 1, \tag{3.1}$$

is valid, then the system (1.1) has a unique solution on $[0, T]$.

Proof. We define the operator $T : E \longrightarrow E$ by

$$T(x, y)(t) = (T_1(x, y)(t), T_2(x, y)(t)), t \in [0, T],$$

such that

$$\begin{aligned} T_1(x, y)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_0^t (t-\tau)^{\alpha-1} H_i(\tau) d\tau + \frac{t^{\alpha-1}}{\psi} \left(\sum_{i=1}^{m} \int_0^T (T-\tau) H_i(\tau) d\tau \right. \\ &\quad \left. - \frac{\theta}{\Gamma(2\alpha)} \sum_{i=1}^{m} \int_0^\eta (\eta-\tau)^{2\alpha-2} H_i(\tau) d\tau \right) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} T_2(x, y)(t) &= \frac{1}{\Gamma(\beta)} \sum_{i=1}^{m} \int_0^t (t-\tau)^{\beta-1} G_i(\tau) d\tau + \frac{t^{\beta-1}}{\psi'} \left(\sum_{i=1}^{m} \int_0^T (T-\tau) G_i(\tau) d\tau \right. \\ &\quad \left. - \frac{\omega}{\Gamma(2\beta)} \sum_{i=1}^{m} \int_0^\gamma (\gamma-\tau)^{2\beta-2} G_i(\tau) d\tau \right) \end{aligned} \tag{3.3}$$

where

$$H_i(\tau) = f_i(\tau, x(\tau), y(\tau), \varphi_1 x(\tau), \phi_1 y(\tau))$$

and

$$G_i(\tau) = g_i(\tau, x(\tau), y(\tau), \varphi_2 x(\tau), \phi_2 y(\tau)).$$

We obtain

$$\varphi_i T_1(x, y)(t) = \int_0^t A_i(t, s) T_1(x, y)(s) ds, \ \phi_i T_2(x, y)(t) = \int_0^t B_i(t, s) T_2(x, y)(s) ds$$

where $i = 1, 2$.

We shall now prove that T is contractive.

Let $T_1(x_1, y_1), T_2(x_2, y_2) \in E$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned} |T_1(x_1, y_1) - T_1(x_2, y_2)| &\leq \left[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_0^t (t-\tau)^{\alpha-1} d\tau + \frac{t^{\alpha-1}}{\psi} \left(\sum_{i=1}^{m} \int_0^T (T-\tau) d\tau \right. \right. \\ &\quad \left. \left. - \frac{\theta}{\Gamma(2\alpha)} \sum_{i=1}^{m} \int_0^\eta (\eta-\tau)^{2\alpha-2} d\tau \right) \right] \end{aligned}$$

$$
\begin{aligned}
&\times \max_{\tau\in[0,T]} \sum_{i=1}^{m} \left| \begin{pmatrix} f_i(\tau, x_1(\tau), y_1(\tau), \varphi_1 x_1(\tau), \phi_1 y_1(\tau)) \\ -f_i(\tau, x_2(\tau), y_2(\tau), \varphi_1 x_2(\tau), \phi_1 y_2(\tau)) \end{pmatrix} \right| \\
\leq\ & \frac{T^\alpha}{\Gamma(\alpha+1)} \max_{\tau\in[0,T]} \sum_{i=1}^{m} \left| \begin{pmatrix} f_i(\tau, x_1(\tau), y_1(\tau), \varphi_1 x_1(\tau), \phi_1 y_1(\tau)) \\ -f_i(\tau, x_2(\tau), y_2(\tau), \varphi_1 x_2(\tau), \phi_1 y_2(\tau)) \end{pmatrix} \right|.
\end{aligned}
$$

By $(H2)$, it follows that

$$
\begin{aligned}
\|T_1(x_1, y_1) - T_1(x_2, y_2)\| \leq\ & \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} (\xi_1^i + \xi_2^i + \xi_3^i + \xi_4^i) \times \max(\|x_1 - x_2\|, \|y_1 - y_2\|, \\
& \|\varphi_1(x_1 - x_2)\|, \|\varphi_2(x_1 - x_2)\|, \|\phi_1(y_1 - y_2)\|, \|\phi_2(y_1 - y_2)\|).
\end{aligned}
$$

Hence,

$$
\|T_1(x_1, y_1) - T_1(x_2, y_2)\| \leq A_1 \|x_1 - x_2, y_1 - y_2\|_E. \tag{3.4}
$$

With the same arguments as before, we can show that

$$
\|T_2(x_1, y_1) - T_2(x_2, y_2)\| \leq A_2 \|x_1 - x_2, y_1 - y_2\|_E. \tag{3.5}
$$

On the other hand, we have

$$
\begin{aligned}
\|\varphi_1(T_1(x_1, y_1) - T_1(x_2, y_2))\| \leq\ & \int_0^t \|A_1'(t,s)\| \|T_1(x_1, y_1) - T_1(x_2, y_2)\| ds \\
\leq\ & \max_{t,s\in[0,1]} \|A_1'(t,s)\| \times A_1 \|x_1 - x_2, y_1 - y_2\|_E.
\end{aligned}
$$

Hence,

$$
\|\varphi_1(T_1(x_1, y_1) - T_1(x_2, y_2))\| \leq A_3 \|x_1 - x_2, y_1 - y_2\|_E \tag{3.6}
$$

and

$$
\|\varphi_2(T_1(x_1, y_1) - T_1(x_2, y_2))\| \leq A_4 \|x_1 - x_2, y_1 - y_2\|_E. \tag{3.7}
$$

Also, we have

$$
\|\phi_1(T_2(x_1, y_1) - T_2(x_2, y_2))\| \leq A_5 \|x_1 - x_2, y_1 - y_2\|_E \tag{3.8}
$$

and

$$
\|\phi_2(T_2(x_1, y_1) - T_2(x_2, y_2))\| \leq A_6 \|x_1 - x_2, y_1 - y_2\|_E. \tag{3.9}
$$

Thanks to (3.4)–(3.9), we get

$$
\begin{aligned}
\|T(x_1, y_1) - T(x_2, y_2)\| \leq\ & \max(A_1, A_2, A_3, A_4, A_5, A_6) \\
& \times \|(x_1 - x_2, y_1 - y_2)\|_E.
\end{aligned} \tag{3.10}
$$

Thanks to (3.11), we conclude that T is a contractive operator. Therefore, by Banach fixed point theorem, T has a unique fixed point which is the solution of the system (1.1). □

3.2. Uniqueness of solutions

Our second main result is based on Lemma 2.1. We have

Theorem 3.2. *Assume that the hypotheses* $(H1)$ *and* $(H3)$ *are satisfied. Then, system (1.1) has at least a solution on* $[0, T]$.

Proof. The operator T is continuous on E in view of the continuity of f_i and g_i (hypothesis $(H1)$).

Now, we show that T is completely continuous:

(i) First, we prove that T maps bounded sets of E into bounded sets of E. Taking $\lambda > 0$, and $(x, y) \in \Omega_\lambda = \{(x, y) \in E; \|(x, y)\| \leq \lambda\}$, then for each $t \in [0, T]$, we have:

$$\begin{aligned} |T_1(x,y)| \leq & \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau + \frac{t^{\alpha-1}}{\psi}\left(\int_0^T (T-\tau)d\tau \right.\right. \\ & \left.\left. -\frac{\theta}{\Gamma(2\alpha)}\int_0^\eta (\eta-\tau)^{2\alpha-2}d\tau\right)\right] \\ & \times \sup_{t\in[0,T]} \sum_{i=1}^m |f_i(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t))| \\ \leq & \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\psi}\left(\frac{T^{\alpha+1}}{2} + \frac{\theta T^{3\alpha-2}}{(2\alpha-1)^2\Gamma(2\alpha-1)}\right)\right] \\ & \times \sup_{t\in[0,T]} \sum_{i=1}^m |f_i(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t))|, \end{aligned}$$

Thanks to $(H3)$, we can write

$$\|T_1(x,y)\| \leq \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\psi}\left(\frac{T^{\alpha+1}}{2} + \frac{\theta T^{3\alpha-2}}{(2\alpha-1)^2\Gamma(2\alpha-1)}\right)\right]\sum_{i=1}^m L_i.$$

Thus,

$$\|T_1(x,y)\| \leq \nu_1 \sum_{i=1}^m L_i. \tag{3.11}$$

As before, we have

$$\|T_2(x,y)\| \leq \nu_2 \sum_{i=1}^m K_i. \tag{3.12}$$

On the other hand, for all $j = 1, 2$, we get

$$|\phi_j T_1(x,y)(t)| = \left|\int_0^t A'_j(t,s)T_1(x,y)(s)ds\right| \leq \max_{t,s\in[0,1]} |A'_j(t,s)|\nu_1 \sum_{i=1}^m L_i.$$

This implies that

$$\|\phi_1 T_1(x,y)(t)\| \leq \nu_3 \sum_{i=1}^m L_i, \tag{3.13}$$

$$\|\phi_2 T_1(x,y)(t)\| \leq \nu_4 \sum_{i=1}^{m} L_i. \tag{3.14}$$

Similarly, we have

$$\|\varphi_1 T_2(x,y)(t)\| \leq \nu_5 \sum_{i=1}^{m} K_i, \tag{3.15}$$

$$\|\varphi_2 T_2(x,y)(t)\| \leq \nu_6 \sum_{i=1}^{m} K_i. \tag{3.16}$$

It follows from (3.11)–(3.16) that:

$$\|T(x,y)\|_E \leq \max\begin{pmatrix} \nu_1 \sum_{i=1}^{m} L_i, \nu_2 \sum_{i=1}^{m} K_i, \nu_3 \sum_{i=1}^{m} L_i, \\ \nu_4 \sum_{i=1}^{m} L_i, , \nu_5 \sum_{i=1}^{m}, \nu_6 \sum_{i=1}^{m} \end{pmatrix}.$$

Thus,

$$\|T(x,y)\|_E < \infty.$$

(ii) Second, we prove that T is equi-continuous:
For any $0 \leq t_1 < t_2 \leq T$ and $(x,y) \in \Omega_\lambda$, we have

$$\begin{aligned}
&|T_1(x,y)(t_2) - T_1(x,y)(t_1)| \\
&\leq \left[\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}d\tau + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau \right.\\
&\quad \left. + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\psi}\left(\frac{T^2}{2} - \frac{\theta\eta^{2\alpha-1}}{\Gamma(2\alpha-1)^2\Gamma(2\alpha-1)}\right)\right] \\
&\times \sup_{t\in[0,T]} \sum_{i=1}^{m} |f_i(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t))| \\
&\leq \left[\frac{2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha-1} + (t_2^{\alpha-1} - t_1^{\alpha-1})\left[\frac{T^2}{2\psi}\right.\right. \\
&\quad \left.\left. - \frac{\theta\eta^{2\alpha-1}}{\psi\Gamma(2\alpha-1)^2\Gamma(2\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right]\right] \times \sum_{i=1}^{m} L_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|T_1(x,y)(t_2) - T_1(x,y)(t_1)\|_E \\
&\left[\frac{2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha-1} + (t_2^{\alpha-1} - t_1^{\alpha-1})\left[\frac{T^2}{2\psi} + \frac{1}{\Gamma(\alpha+1)}\right]\right] \times \sum_{i=1}^{m} L_i.
\end{aligned} \tag{3.17}$$

We also have

$$\begin{aligned}
&\|T_2(x,y)(t_2) - T_2(x,y)(t_1)\|_E \\
&\left[\frac{2}{\Gamma(\beta+1)}(t_2-t_1)^{\beta-1} + (t_2^{\beta-1} - t_1^{\beta-1})\left[\frac{T^2}{2\psi'} + \frac{1}{\Gamma(\beta+1)}\right]\right] \times \sum_{i=1}^{m} K_i.
\end{aligned} \tag{3.18}$$

On the other hand,

$$
\begin{aligned}
&|\phi_i T_1(x,y)(t_2) - \phi_i T_1(x,y)(t_1)| \\
&\leq \left[\max_{s\in[0,1]} |A_i'(t_2,s) - A_i'(t_1,s)| + (t_2 - t_1)\max_{s\in[0,1]} |A_i'(t_1,s)|\right] \times \sup_{s\in[0,1]} |T_1(x,y)(s)|.
\end{aligned}
$$

Consequently, for all $i = 1,2$, we obtain

$$
\begin{aligned}
&\|\phi_i T_1(x,y)(t_2) - \phi_i T_1(x,y)(t_1)\| \\
&\leq \left[\max_{s\in[0,1]} |A_i'(t_2,s) - A_i'(t_1,s)| + (t_2 - t_1)\max_{s\in[0,1]} |A_i'(t_1,s)|\right] \nu_1 \sum_{i=1}^{m} L_i.
\end{aligned}
\tag{3.19}
$$

Similarly,

$$
\begin{aligned}
&\|\varphi_i T_1(x,y)(t_2) - \varphi_i T_1(x,y)(t_1)\| \\
&\leq \left[\max_{s\in[0,1]} |B_i'(t_2,s) - B_i'(t_1,s)| + (t_2 - t_1)\max_{s\in[0,1]} |B_i'(t_1,s)|\right] \nu_2 \sum_{i=1}^{m} K_i.
\end{aligned}
\tag{3.20}
$$

where $i = 1,2$. Using (3.17)–(3.20), we deduce that

$$
\|T(x,y)(t_2) - T(x,y)(t_1)\|_E \longrightarrow 0
$$

as $t_2 \to t_1$.

Combining (i) and (ii), we conclude that T is completely continuous.

(iii) Finally, we shall prove that the set F defined by

$$
F = \{(x,y) \in E, (x,y) = \rho T(x,y),\ 0 < \rho < 1\}
$$

is bounded.

Let $(x,y) \in F$, then $(x,y) = \rho T(x,y)$, for some $0 < \rho < 1$. Thus, for each $t \in [0,T]$, we have:

$$
x(t) = \rho T_1(x,y)(t),\ y(t) = \rho T_2(x,y)(t). \tag{3.21}
$$

Thanks to $(H3)$ and using (3.11) and (3.12) , we deduce that

$$
\|x\| \leq \rho \nu_1 \sum_{i=1}^{m} L_i,\ \|y\| \leq \rho \nu_2 \sum_{i=1}^{m} K_i. \tag{3.22}
$$

Using (3.13)–(3.16), it yields that

$$
\begin{cases}
\|\phi_1 x\| \leq \rho \nu_3 \sum_{i=1}^{m} L_i \\
\|\phi_2 x\| \leq \rho \nu_4 \sum_{i=1}^{m} L_i \\
\|\varphi_1 y\| \leq \rho \nu_5 \sum_{i=1}^{m} K_i \\
\|\varphi_2 y\| \leq \rho \nu_6 \sum_{i=1}^{m} K_i
\end{cases}. \tag{3.23}
$$

It follows from (3.22) and (3.23) that

$$
\|T(x,y)\|_E \leq \rho \max \left(\begin{array}{l} \nu_1 \sum_{i=1}^{m} L_i, \nu_2 \sum_{i=1}^{m} K_i, \nu_3 \sum_{i=1}^{m} L_i, \\ \nu_4 \sum_{i=1}^{m} L_i, , \nu_5 \sum_{i=1}^{m}, \nu_6 \sum_{i=1}^{m} \end{array} \right).
$$

Consequently,

$$\|(x,y)\|_E < \infty.$$

This shows that F is bounded. By Lemma (2.1), we deduce that T has a fixed point, which is a solution of (1.1).

□

4. Related examples

To illustrate our main results, we treat the following examples.

Example 4.1. *Consider the following system:*

$$\begin{cases} D^{\frac{3}{2}}x(t) = \dfrac{\cos(\pi t)(x+y+\varphi_1 x(t)+\phi_1 y(t))}{10\pi(x+y+\varphi_1 x(t)+\phi_1 y(t))} + \dfrac{1}{32\pi^2 e} \\ \qquad \left(\cos x(t) + \cos y(t) + \dfrac{\varphi_1 x(t)+\phi_1 y(t)}{4\pi}\right), \\ D^{\frac{3}{2}}y(t) = \dfrac{1}{8\pi^3(t+1)}\left(\dfrac{x+y+\varphi_2 x(t)+\phi_2 y(t)}{3+x+y+\varphi_2 x(t)+\phi_2 y(t)}\right) + \dfrac{1}{(10\pi+e^t)e^{(t+1)}} \\ \qquad \left(\dfrac{\sin x(t)+\sin y(t)+\cos\varphi_2 x(t)+\cos\phi_2 y(t)}{2+\sin x(t)+\sin y(t)+\cos\varphi_2 x(t)+\cos\phi_2 y(t)}\right), \\ I^{\frac{1}{2}}x(0)=0,\ D^{-\frac{1}{2}}x(T) = I^{\frac{1}{2}}(x(1)), \\ I^{\frac{1}{2}}y(0)=0,\ D^{-\frac{1}{2}}y(T) = I^{\frac{1}{2}}(y(1)). \end{cases} \tag{4.1}$$

We have

$$\alpha = \frac{3}{2},\ \beta = \frac{3}{2},\ T=1,\ \theta=1,\ \omega=1,\ \gamma=1,\ m=2,\ \eta=1.$$

Also,

$$f_1(t,x(t),y(t),\varphi_1 x(t),\phi_1 y(t)) = \frac{\cos(\pi t)(x+y+\varphi_1 x(t)+\phi_1 y(t))}{10\pi(1+x+y+\varphi_1 x(t)+\phi_1 y(t))}, \tag{4.2}$$

$$f_2(t,x(t),y(t),\varphi_1 x(t),\phi_1 y(t)) = \frac{1}{32\pi^2 e}\left(\cos x(t)+\cos y(t)+\frac{\varphi_1 x(t)+\phi_1 y(t)}{4\pi}\right). \tag{4.3}$$

For $t \in [0,1]$ and $(x_1,y_1,\varphi_1 x_1,\phi_1 y_1),(x_2,y_2,\varphi_1 x_2,\phi_1 y_2) \in \mathbb{R}^4$, we have

$$\begin{aligned} &|f_1(t,x_1,y_1,\varphi_1 x_1,\phi_1 y_1) - f_1(t,x_2,y_2,\varphi_1 x_2,\phi_1 y_2)| \\ &\le \frac{|\cos(\pi t)|}{10\pi}\left|\frac{x_1+y_1+\varphi_1 x_1+\phi_1 y_1}{1+x_1+y_1+\varphi_1 x_1+\phi_1 y_1} - \frac{x_2+y_2+\varphi_1 x_2+\phi_1 y_2)}{1+x_2+y_2+\varphi_1 x_2+\phi_1 y_2)}\right| \\ &\le \frac{1}{10\pi}(|x_1-x_2|+|y_1-y_2|+|\varphi_1 x_1-\varphi_1 x_2|+|\phi_1 y_1-\phi_1 y_2|) \end{aligned} \tag{4.4}$$

and

$$|f_2(t,x_1,y_1,\varphi_1x_1,\phi_1y_1)-f_2(t,x_2,y_2,\varphi_1x_2,\phi_1y_2)|$$
$$\leq\frac{1}{32\pi e}(|x_1-x_2|+|y_1-y_2|+|\varphi_1x_1-\varphi_1x_2|+|\phi_1y_1-\phi_1y_2|). \tag{4.5}$$

So, we can take

$$\xi_1^1=\xi_2^1=\xi_3^1=\xi_4^1=\frac{1}{10\pi},$$
$$\xi_1^2=\xi_2^2=\xi_3^2=\xi_4^2=\frac{1}{32\pi e}.$$

We also have

$$g_1(t,x(t),y(t),\varphi_2x(t),\phi_2y(t)) = \frac{1}{8\pi^3(t+1)}\left(\frac{x+y+\varphi_2x(t)+\phi_2y(t)}{3+x+y+\varphi_2x(t)+\phi_2y(t)}\right)$$

and

$$g_2(t,x(t),y(t),\varphi_2x(t),\phi_2y(t))$$
$$=\frac{1}{(10\pi+e^t)e^{t+1}}\left(\frac{\sin x(t)+\sin y(t)+\cos\varphi_2x(t)+\cos\phi_2y(t)}{2+\sin x(t)+\sin y(t)+\cos\varphi_2x(t)+\cos\phi_2y(t)}\right) \tag{4.6}$$

For $t\in[0,1]$ *and* $(x_1,y_1,\varphi_2x_1,\phi_2y_1),(x_2,y_2,\varphi_2x_2,\phi_2y_2)\in\mathbb{R}^4$, *we can write*

$$|g_1(t,x_1,y_1,\varphi_2x_1,\phi_2y_1)-g_1(t,x_2,y_2,\varphi_2x_2,\phi_2y_2)|$$
$$\leq\frac{1}{8\pi^3}(|x_1-x_2|+|y_1-y_2|+|\varphi_2x_1-\varphi_2x_2|+|\phi_2y_1-\phi_2y_2|), \tag{4.7}$$

and

$$|g_2(t,x_1,y_1,\varphi_2x_1,\phi_2y_1)-g_2(t,x_2,y_2,\varphi_2x_2,\phi_2y_2)|$$
$$\leq\frac{1}{10\pi e^2}(|x_1-x_2|+|y_1-y_2|+|\varphi_2x_1-\varphi_2x_2|+|\phi_2y_1-\phi_2y_2|). \tag{4.8}$$

Hence,

$$\chi_1^1=\chi_2^1=\chi_3^1=\chi_4^1=\frac{1}{8\pi^3},$$
$$\chi_1^2=\chi_2^2=\chi_3^2=\chi_4^2=\frac{1}{10\pi e^2}.$$

Therefore,

$$A_1=0.0589009676,\quad A_2=0.0250930393.$$

Suppose

$$A_i'=B_i'=1,\ i=1,2,$$

so,

$$A_1=A_3=A_4,\quad A_2=A_5=A_6.$$

Thus,

$$\max(A_1,A_2,A_3,A_4,A_5,A_6)<1, \tag{4.9}$$

and by Theorem 3.1, we conclude that the system (4.1) has a unique solution on $[0,1]$.

Example 4.2.

$$
\begin{cases}
D^{\frac{3}{2}}x(t) = \dfrac{\pi(t+1)\sin(\varphi_1 x(t) + \phi_1 y(t))}{2-\cos(x(t)+y(t))} \\
\qquad\qquad + \dfrac{e^t}{2\pi + \cos(x(t)+\varphi_1 x(t)) + \sin(\sin(y(t)+\phi_1 y(t))}, \ t \in [0,1], \\
D^{\frac{4}{3}}y(t) = \dfrac{e^2 \sin(x(t)+y(t))}{2\pi + \cos(\varphi_2 x(t) + \phi_2 y(t))} \\
\qquad\qquad + \dfrac{3t^2 \cos y(t)}{e^{t^3+1} - \cos(x(t)+y(t)-\varphi_2 x(t) - \phi_2 y(t))}, \ t \in [0,1], \\
I^{\frac{1}{2}}x(0) = 0, \ D^{-\frac{1}{2}}x(T) = I^{\frac{1}{2}}(x(1)), \\
I^{\frac{2}{3}}y(0) = 0, \ D^{-\frac{2}{3}}y(T) = I^{\frac{1}{3}}(y(1)).
\end{cases}
\tag{4.10}
$$

We have

$$
\alpha = \frac{3}{2}, \ \beta = \frac{4}{3}, \ T = 1, \ \theta = 1, \ \omega = 1, \ \gamma = 1, \ m = 2, \ \eta = 1.
$$

Since

$$
\begin{aligned}
|f_1(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t))| &= \left| \frac{\pi(t+1)\sin(\varphi_1 x(t)+\phi_1 y(t))}{2-\cos(x(t)+y(t))} \right| \leq 2\pi, \\
|f_2(t, x(t), y(t), \varphi_1 x(t), \phi_1 y(t))| &= \left| \frac{e^t}{2\pi + \cos(x(t)+\varphi_1 x(t)) + \sin(\sin(y(t)+\phi_1 y(t))} \right| \leq \frac{e}{2\pi+2}, \\
|g_1(t, x(t), y(t), \varphi_2 x(t), \phi_2 y(t))| &= \left| \frac{e^2 \sin(x(t)+y(t))}{2\pi + \cos(\varphi_2 x(t)+\phi_2 y(t))} \right| \leq \frac{e^2}{2\pi+1}, \\
|g_2(t, x(t), y(t), \varphi_2 x(t), \phi_2 y(t))| &= \left| \frac{3t^2 \cos y(t)}{e^{t^3+1} - \cos(x(t)+y(t)-\varphi_2 x(t)-\phi_2 y(t))} \right| \leq \frac{3}{e-1}.
\end{aligned}
$$

The functions $f_1, \ f_2, \ g_1$ *and* g_2 *are continuous and bounded on* $[0,1] \times \mathbb{R}^4$. *So, by Theorem 3.2, the system (4.10) has at least one solution on* $[0,1]$.

5. Conclusions

We have proved the existence of solutions for fractional differential equations with integral and multi-point boundary conditions. The problem is solved by applying some fixed point theorems. We also provide examples to make our results clear.

Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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