Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18127–18141.
DOI: 10.3934/math.2022997
Received: 30 June 2022
Revised: 01 August 2022
Accepted: 01 August 2022
Published: 09 August 2022

Research article

Relationships between the discrete Riemann-Liouville and Liouville-Caputo fractional differences and their associated convexity results

Juan L. G. Guirao[1,2], **Pshtiwan Othman Mohammed**[3,*], **Hari Mohan Srivastava**[4,5,6,7], **Dumitru Baleanu**[8,9,10,*] **and Marwan S. Abualrub**[11]

[1] Departamento de Matemáca Aplicada y Estadística, Universidad Politécnica de Cartagena, ES-30202 Cartagena, Spain

[2] Nonlinear Analysis and Applied Mathematics (NAAM)−Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[3] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Iraq

[4] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

[5] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[6] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan

[7] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

[8] Department of Mathematics, Cankaya University, Balgat 06530, Ankara, Turkey

[9] Institute of Space Sciences, R76900 Magurele-Bucharest, Romania

[10] Department of Natural Sciences, School of Arts and Sciences, Lebanese American University, Beirut 11022801, Lebanon

[11] Mathematics Department (Prep. Program), Faculty of Science, Khalifa University, P.O. Box 127788, Abu Dhabi UAE

* **Correspondence:** Email: pshtiwansangawi@gmail.com, dumitru@cankaya.edu.tr.

Abstract: In this study, we have presented two new alternative definitions corresponding to the basic definitions of the discrete delta and nabla fractional difference operators. These definitions and concepts help us in establishing a relationship between Riemann-Liouville and Liouville-Caputo fractional differences of higher orders for both delta and nabla operators. We then propose and analyse some convexity results for the delta and nabla fractional differences of the Riemann-Liouville type. We also derive similar results for the delta and nabla fractional differences of Liouville-Caputo type by using the proposed relationships. Finally, we have presented two examples to confirm the main theorems.

Keywords: Riemann-Liouville fractional difference; Liouville-Caputo fractional difference; convexity analysis
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

Discrete fractional operators represent several fundamental models in order to understand the discrete fractional calculus phenomena. Furthermore, the study of these operators and their properties has motivated the innovation of new mathematical tools which have provided useful insights for a number of problems arising in various fields of application. On the other hand, several tools and techniques borrowed from mathematical analysis [1–3], stability analysis [4–6], probability theory [7–9], geometry [10, 11], ecology [12, 13] and topology [14–16] have contributed to a better understanding of the properties of these discrete fractional operators.

In addition to many experiments, the rules of discrete fractional operators have also been under active investigations emerging from theoretical and computational discrete fractional calculus. The discrete fractional calculus theory has been successfully applied to analyze the positivity and monotonicity of discrete delta and nabla fractional operators of the Riemann-Liouville, Liouville-Caputo, Atangana-Baleanu and Caputo-Fabrizio types (see [17–21]).

While, in recent years, several results have been obtained for monotonicity and positivity analysis involving discrete fractional operators, fewer results are available in the convexity analysis setting (see [22–24]). In this study, at first we establish a relationship between the Δ fractional difference of order β of the Riemann-Liouville type $\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})$ and the Liouville-Caputo type $\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})$, and a relationship between the ∇ fractional difference of order β of Riemann-Liouville type $\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z})$ and Liouville-Caputo type $\left({}^{\mathrm{LC}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z})$ for each $\aleph - 1 < \beta < \aleph$ with $\aleph \in \mathbb{S}_1$. We then establish some convexity results for the Δ and ∇ fractional difference of the Riemann-Liouville type as well as for the Liouville-Caputo type by using our derived relationships. For a systematic investigation of fractional calculus and its widespread applications, we refer the reader to the monograph by Kilbas et al. [25] (see also [26–29] for some recent developments based upon the Riemann-Liouville and Liouville-Caputo fractional integrals and fractional derivatives as well as their associated dufference operators).

The remainder of the study is organized as follows. Section 2 presents a brief overview of the delta and nabla fractional differences and a description of the essentials of the discrete fractional calculus methods. It also gives alternative discrete fractional definitions equivalent to the standard definitions and provides the relationships between the fractional Riemann-Liouville and Liouville-Caputo difference operators. Section 3 establishes several convexity results for the fractional Riemann-Liouville and Liouville-Caputo differences. Section 4 contains the application examples. These are achieved by using two basic formulas for Δ^2 and ∇^2. Several concluding remarks and thoughts on open questions are offered in Section 5.

2. Basic tools and fractional difference relationships

In this section, we consider the discrete fractional sums and differences in both of Riemann-Liouville and Liouville-Caputo senses. The reader can refer to [30–33] for the relevant details about these definitions and many other properties associated with them.

Let us define the sets $\mathbb{S}_{t_0} := \{t_0, t_0+1, t_0+2, \ldots\}$ and $\mathscr{D}_{t_0}(\mathrm{g}) := \{\mathrm{g}: \ \mathrm{g}: \mathbb{S}_{t_0} \to \mathcal{R} \quad \text{for} \quad a \in \mathcal{R}\}$. For $\mathrm{g} \in \mathscr{D}_{t_0+\beta}(\mathrm{g})$ with $\beta > 0$, the Δ fractional sum of order β can be expressed as follows:

$$\left({}_{t_0}\Delta^{-\beta}\mathrm{g}\right)(\mathrm{z}) = \frac{1}{\Gamma(\beta)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}-\beta}(\mathrm{z}-\mathrm{s}-1)^{\underline{\beta-1}}\mathrm{g}(\mathrm{s}), \quad \text{for z in } \mathbb{S}_{t_0+\beta}. \tag{2.1}$$

Moreover, for $\mathrm{g} \in \mathscr{D}_{t_0}(\mathrm{g})$, the ∇ fractional sum of order β can be expressed as follows:

$$\left({}_{t_0}\nabla^{-\beta}\mathrm{g}\right)(\mathrm{z}) = \frac{1}{\Gamma(\beta)}\sum_{\mathrm{s}=t_0+1}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{\beta-1}}\mathrm{g}(\mathrm{s}), \quad \text{for z in } \mathbb{S}_{t_0+1}. \tag{2.2}$$

Here, and in what follows, $\mathrm{z}^{\underline{\beta}}$ and $\mathrm{z}^{\overline{\beta}}$ are defined by

$$\mathrm{z}^{\underline{\beta}} = \frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+1-\beta)} \qquad \text{and} \qquad \mathrm{z}^{\overline{\beta}} = \frac{\Gamma(\mathrm{z}+\beta)}{\Gamma(\mathrm{z})}, \tag{2.3}$$

such that the right-hand sides of these identities are well defined. Besides, we use $\mathrm{z}^{\underline{\beta}} = 0$ and $\mathrm{z}^{\overline{\beta}} = 0$ when the numerators in each identities are well-defined, but the denominator is not defined. Further, we have

$$\Delta\,\mathrm{z}^{\underline{\beta}} = \beta\,\mathrm{z}^{\underline{\beta-1}} \qquad \text{and} \qquad \nabla\,\mathrm{z}^{\overline{\beta}} = \beta\,\mathrm{z}^{\overline{\beta-1}}. \tag{2.4}$$

Definition 2.1 (see [30–32]). Let $\mathrm{g} \in \mathscr{D}_{t_0}(\mathrm{g})$. Then $(\Delta\,\mathrm{g})(\mathrm{z}) := \mathrm{g}(\mathrm{z}+1) - \mathrm{g}(\mathrm{z})$, for $\mathrm{z} \in \mathbb{S}_{t_0}$, is the Δ difference operator and $(\nabla\,\mathrm{g})(\mathrm{z}) := \mathrm{g}(\mathrm{z}) - \mathrm{g}(\mathrm{z}-1)$, for $\mathrm{z} \in \mathbb{S}_{t_0+1}$, is the ∇ difference operator. In addition, the Δ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Riemann-Liouville type is defined by

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z}) = \frac{\Delta^{\aleph}}{\Gamma(\aleph-\beta)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-\aleph}(\mathrm{z}-\mathrm{s}-1)^{\underline{\aleph-\beta-1}}\mathrm{g}(\mathrm{s}), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}, \tag{2.5}$$

and the ∇ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Riemann-Liouville type is defined by

$$\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) = \frac{\nabla^{\aleph}}{\Gamma(\aleph-\beta)}\sum_{\mathrm{s}=t_0+1}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{\aleph-\beta-1}}\mathrm{g}(\mathrm{s}), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.6}$$

The following theorem is an alternative representation of the ∇ fractional difference (2.6).

Theorem 2.1 (see [21, Lemma 2.1]). *For* $\mathrm{g} \in \mathscr{D}_{t_0}(\mathrm{g})$ *and* $\aleph - 1 < \beta < \aleph$, *the* ∇ *fractional difference of order* β *of the Riemann-Liouville type can be expressed as follows*:

$$\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) = \frac{1}{\Gamma(-\beta)}\sum_{\mathrm{s}=t_0+1}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{-\beta-1}}\mathrm{g}(\mathrm{s}), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.7}$$

Furthermore, the following theorem is an alternative representation of the Δ fractional difference (2.5), which is also the generalization of the result established for $0 < \beta < 1$ in [18].

Theorem 2.2. *For* $g \in \mathscr{D}_{t_0+\beta}(g)$ *with* $\aleph - 1 < \beta < \aleph$, *the* Δ *fractional difference of order* β *of the Riemann-Liouville type can be expressed as follows*:

$$\left({}^{\mathrm{RL}}_{\ t_0}\Delta^{\beta} g\right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}. \tag{2.8}$$

Proof. The result was proved by Mohammed et al. in [18, Theorem 1] for $\aleph = 1$ (that is, for $0 < \beta < 1$), and their result is given below:

$$\left({}^{\mathrm{RL}}_{\ t_0}\Delta^{\beta} g\right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+1-\beta}. \tag{2.9}$$

For $\aleph = 2$ (that is, for $1 < \beta < 2$), by Definition (2.5) we find for each $z \in \mathbb{S}_{t_0+2-\beta}$ that

$$\begin{aligned}
\left({}^{\mathrm{RL}}_{\ t_0}\Delta^{\beta} g\right)(z) &= \Delta\left(\frac{\Delta}{\Gamma(-\beta+2)} \sum_{s=t_0}^{z+\beta-2} (z-s-1)^{\underline{1-\beta}} g(s)\right) \\
&\overset{by}{\underset{(2.9)}{=}} \Delta\left(\frac{1}{\Gamma(-\beta+1)} \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s)\right) \\
&= \frac{1}{\Gamma(-\beta+1)}\left\{\sum_{s=t_0}^{z+\beta} (z-s)^{\underline{-\beta}} g(s) - \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s)\right\} \\
&= \frac{1}{\Gamma(-\beta+1)}\left\{\sum_{s=t_0}^{z+\beta} (z-s)^{\underline{-\beta}} g(s) - \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta}} g(s)\right\} \\
&= \frac{1}{\Gamma(-\beta+1)} \sum_{s=t_0}^{z+\beta} \Delta\,(z-s-1)^{\underline{-\beta}} g(s) \\
&= \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta-1}} g(s),
\end{aligned}$$

where we have first used (see [18, Lemma 1])

$$(-\beta-1)^{\underline{-\beta}} = 0,$$

and then used

$$\Delta\left(z^{\underline{-\beta}}\right) = -\beta\, z^{\underline{-\beta-1}}.$$

The same procedure can be repeated $(\aleph-1)$ times to obtain the required result asserted by Theorem 2.2. □

Definition 2.2 (see [30–32]). For $g \in \mathscr{D}_{t_0+\beta}(g)$, the Δ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of Liouville-Caputo type is defined by

$$\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta} g\right)(z) = \frac{1}{\Gamma(\aleph-\beta)} \sum_{s=t_0}^{z+\beta-\aleph} (z-s-1)^{\underline{\aleph-\beta-1}} (\Delta^{\aleph} g)(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}, \tag{2.10}$$

and, for $g \in \mathscr{D}_{t_0}(g)$, the ∇ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Liouville-Caputo type is defined by

$$\left({}^{\mathrm{LC}}_{t_0}\nabla^{\beta} g\right)(z) = \frac{1}{\Gamma(\aleph-\beta)} \sum_{s=t_0+1}^{z} (z-s+1)^{\overline{\aleph-\beta-1}} (\nabla^{\aleph} g)(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.11}$$

The following proposition provides relationships between the Δ and ∇ fractional differences of the Riemann-Liouville and Liouville-Caputo types of the higher order β.

Proposition 2.1. *Let* $\beta \in (\aleph - 1, \aleph)$. *Then, for* $g \in \mathscr{D}_{t_0+\beta}(g)$,

$$\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta} g\right)(z) = \left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta} g\right)(z) - \sum_{J=0}^{\aleph-1} \frac{(z-t_0)^{\underline{-\beta+J}}}{\Gamma(-\beta+J+1)} (\Delta^{J} g)(t_0), \tag{2.12}$$

for z *in* $\mathbb{S}_{t_0+\aleph-\beta}$. *Moreover, for* $g \in \mathscr{D}_{t_0}(g)$, *it is asserted that*

$$\left({}^{\mathrm{LC}}_{t_0}\nabla^{\beta} g\right)(z) = \left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta} g\right)(z) - \sum_{J=0}^{\aleph-1} \frac{(z-t_0)^{\overline{-\beta+J}}}{\Gamma(-\beta+J+1)} (\nabla^{J} g)(t_0), \tag{2.13}$$

for z *in* $\mathbb{S}_{t_0+\aleph}$.

Proof. We only prove the first part and we omit the second part, because they have the same proof technique. Considering (2.10), for $\aleph = 1$ we have

$$\begin{aligned}
\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta} g\right)(z) &= \frac{1}{\Gamma(-\beta+1)} \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} (\Delta g)(s) \\
&= \frac{1}{\Gamma(-\beta+1)} \left\{ \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s+1) - \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s) \right\} \\
&= \frac{1}{\Gamma(-\beta+1)} \left\{ -(z-t_0)^{\underline{-\beta}} g(t_0) + \sum_{s=t_0}^{z+\beta} \Delta\, (z-s-1)^{\underline{-\beta}} g(s) \right\} \\
&= \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta-1}} g(s) - \frac{(z-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)} g(t_0),
\end{aligned} \tag{2.14}$$

where we have first used $(-\beta-1)^{\underline{-\beta}} = 0$ and then used $\Delta\left(z^{\underline{-\beta}}\right) = -\beta\, z^{\underline{-\beta-1}}$ as above.

Now, for $\aleph = 2$, we have

$$
\begin{aligned}
\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-2}(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(\mathrm{s})\\
&\underset{(2.14)}{\overset{by}{=}} \frac{1}{\Gamma(-\beta+1)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-1}(\mathrm{z}-\mathrm{s}-1)^{\underline{-\beta}}(\Delta\mathrm{g})(\mathrm{s}) - \frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\mathrm{g})(t_0)\\
&\underset{(2.14)}{\overset{\text{again by}}{=}} \frac{1}{\Gamma(-\beta)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}(\mathrm{z}-\mathrm{s}-1)^{\underline{-\beta-1}}\mathrm{g}(\mathrm{s}) - \frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0) - \frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\mathrm{g})(t_0).
\end{aligned}
$$

We can repeat the same procedure $(\aleph-1)$ times to get the required (2.12). Hence the proof is completed. □

3. Convexity results

We start with two lemmas concerning the Δ^2 and ∇^2 fractional differences which will be useful in the sequel.

Lemma 3.1. *Let* $\mathrm{g}\in\mathscr{D}_{t_0}(\mathrm{g}), \beta\in(2,3)$ *and* $\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})\geqq 0$ *for* $\mathrm{z}\in\mathbb{S}_{t_0+3-\beta}$. *Then, for* $\mathrm{z}:=t_0+-\beta+3+\eta$ *with* $\eta\in\mathbb{S}_0$,

$$
\begin{aligned}
(\Delta^2\mathrm{g})(t_0+\eta+1) \geqq& -\frac{(-\beta+3+\eta)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0) - \frac{(-\beta+3+\eta)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\mathrm{g})(t_0)\\
&-\frac{1}{\Gamma(-\beta+2)}\sum_{\imath=0}^{\eta}(-\beta+\eta+2-\imath)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(t_0+\imath),
\end{aligned}\tag{3.1}
$$

where

$$
\frac{(-\beta+\eta+2-\imath)^{\underline{1-\beta}}}{\Gamma(-\beta+2)} = \frac{(-\beta+2)(-\beta+3)\cdots(-\beta+\eta+2-\imath)}{(\eta-\imath+1)!} < 0,
$$

$$
\frac{(-\beta+3+\eta)^{\underline{-\beta}}}{\Gamma(-\beta+1)} > 0 \qquad \textit{and} \qquad \frac{(-\beta+3+\eta)^{\underline{1-\beta}}}{\Gamma(-\beta+2)} < 0.
$$

Proof. By considering Theorem 2.2 and (2.4), we have

$$
\begin{aligned}
\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{1}{\Gamma(-\beta)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}(\mathrm{z}-\mathrm{s}-1)^{\underline{-\beta-1}}\mathrm{g}(\mathrm{s})\\
&= \frac{1}{\Gamma(-\beta+1)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}\Delta\,(\mathrm{z}-\mathrm{s}-1)^{\underline{-\beta}}\mathrm{g}(\mathrm{s})\\
&= \frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0) + \frac{1}{\Gamma(-\beta+1)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}(\mathrm{z}-\mathrm{s}-1)^{\underline{-\beta}}(\Delta\,\mathrm{g})(\mathrm{s}),
\end{aligned}
$$

where we have used $(-\beta-1)^{\underline{-\beta}}=0$. By the same technique as in above, we can deduce

$$\begin{aligned}\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)+\frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}\Delta\,(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta\,\mathrm{g})(\mathrm{s})\\ &= \frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)+\frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\,\mathrm{g})(t_0)+\frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta}(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta^2\,\mathrm{g})(\mathrm{s}),\end{aligned}\tag{3.2}$$

where we have used $(-\beta-1)^{\underline{1-\beta}}=0$. Since $(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}=0$ at $\mathrm{s}=\mathrm{z}+\beta,\mathrm{z}+\beta-1$, (3.2) becomes

$$\begin{aligned}\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)+\frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\,\mathrm{g})(t_0)+\frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-2}(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(\mathrm{s})\\ &= (\Delta^2\mathrm{g})(\mathrm{z}+\beta-2)+\frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)+\frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\,\mathrm{g})(t_0)\\ &\quad+\frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-3}(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(\mathrm{s}).\end{aligned}$$

Considering the assumption that $\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})\geqq 0$, it follows that

$$(\Delta^2\mathrm{g})(\mathrm{z}+\beta-2)\geqq-\frac{(\mathrm{z}-t_0)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)-\frac{(\mathrm{z}-t_0)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\,\mathrm{g})(t_0)-\frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0}^{\mathrm{z}+\beta-3}(\mathrm{z}-\mathrm{s}-1)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(\mathrm{s}).$$

For $\mathrm{z}:=t_0+-\beta+3+\eta$ for $\eta\in\mathbb{S}_0$, it becomes

$$\begin{aligned}(\Delta^2\mathrm{g})(t_0+\eta+1)\geqq &-\frac{(-\beta+3+\eta)^{\underline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0)-\frac{(-\beta+3+\eta)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}(\Delta\,\mathrm{g})(t_0)\\ &-\frac{1}{\Gamma(-\beta+2)}\sum_{\iota=0}^{\eta}(2-\beta+\eta-\iota)^{\underline{1-\beta}}(\Delta^2\mathrm{g})(t_0+\iota),\end{aligned}$$

which is the required (3.1). Now, it is clear for $2<\beta<3$ that

$$\begin{aligned}\frac{(-\beta+\eta+2-\iota)^{\underline{1-\beta}}}{\Gamma(-\beta+2)} &= \frac{\Gamma(-\beta+3+\eta-\iota)}{\Gamma(-\beta+2)\Gamma(2+\eta-\iota)}\\ &= \frac{(-\beta+2)(-\beta+3)\cdots(-\beta+\eta+2-\iota)}{(\eta-\iota+1)!}<0,\end{aligned}$$

$$\begin{aligned}\frac{(-\beta+3+\eta)^{\underline{-\beta}}}{\Gamma(-\beta+1)} &= \frac{\Gamma(-\beta+4+\eta)}{\Gamma(4+\eta)\Gamma(-\beta+1)}\\ &= \frac{(-\beta+1)(-\beta+2)(-\beta+3)\cdots(\eta+1-\beta)(-\beta+\eta+2)(-\beta+\eta+3)}{(\eta+3)!}>0,\end{aligned}$$

and

$$\frac{(-\beta+3+\eta)^{\underline{1-\beta}}}{\Gamma(-\beta+2)}=\frac{\Gamma(-\beta+4+\eta)}{\Gamma(4+\eta)\Gamma(-\beta+2)}=\frac{(-\beta+2)(-\beta+3)\cdots(-\beta+\eta+2)(-\beta+\eta+3)}{(\eta+3)!}<0,$$

for $\iota=0,1,\ldots,\eta$ and $\eta\in\mathbb{S}_0$. Thus, our proof is complete. □

Lemma 3.2. *Let* $\mathrm{g} \in \mathscr{D}_{t_0+1}(\mathrm{g}), \beta \in (2,3)$ *and* $\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) \geqq 0$ *for* $\mathrm{z} \in \mathbb{S}_{t_0+1}$. *Then, for* $\mathrm{z} := t_0 + \eta$,

$$(\nabla^2\mathrm{g})(t_0+\eta) \geqq \frac{(\eta-2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1) - \frac{1}{\Gamma(-\beta+2)}\sum_{\iota=2}^{\eta-2}(\eta-\iota)^{\overline{1-\beta}}(\nabla^2\mathrm{g})(t_0+\iota+1), \tag{3.3}$$

where

$$\frac{(\eta-\iota)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} = \frac{(-\beta+2)(-\beta+3)\cdots(-\beta+\eta-\iota)(-\beta+\eta-\iota-1)}{(\eta-\iota-1)!} < 0$$

and

$$\frac{(\eta-2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1) \geqq 0,$$

for $\eta \in \mathbb{S}_4$.

Proof. By making use of Theorem 2.1 and (2.4), we find for $\mathrm{z} \in \mathbb{S}_{t_0+4}$ that

$$\begin{aligned}
\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{1}{\Gamma(-\beta)}\sum_{\mathrm{s}=t_0+1}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{-\beta-1}}\mathrm{g}(\mathrm{s}) \\
&= \frac{1}{\Gamma(-\beta+1)}\sum_{\mathrm{s}=t_0+1}^{\mathrm{z}+\beta}\nabla(\mathrm{z}-\mathrm{s}+1)^{\overline{-\beta}}\mathrm{g}(\mathrm{s}) \\
&= \frac{(\mathrm{z}-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1) + \frac{1}{\Gamma(-\beta+1)}\sum_{\mathrm{s}=t_0+2}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{-\beta}}(\nabla\,\mathrm{g})(\mathrm{s}),
\end{aligned}$$

where we used $(0)^{\overline{-\beta}} = 0$. We can continue by the same technique to get

$$\begin{aligned}
\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) &= \frac{(\mathrm{z}-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0) + \frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0+2}^{\mathrm{z}}\nabla(\mathrm{z}-\mathrm{s}+1)^{\overline{1-\beta}}(\nabla\,\mathrm{g})(\mathrm{s}) \\
&= \frac{(\mathrm{z}-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1) + \frac{(\mathrm{z}-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)}(\nabla\,\mathrm{g})(t_0+2) \\
&\quad + \frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0+3}^{\mathrm{z}}(\mathrm{z}-\mathrm{s}+1)^{\overline{1-\beta}}(\nabla^2\,\mathrm{g})(\mathrm{s}) \\
&= (\nabla^2\mathrm{g})(\mathrm{z}) + \frac{(\mathrm{z}-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1) + \frac{(\mathrm{z}-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)}(\nabla\,\mathrm{g})(t_0+2) \\
&\quad + \frac{1}{\Gamma(-\beta+2)}\sum_{\mathrm{s}=t_0+3}^{\mathrm{z}-1}(\mathrm{z}-\mathrm{s}+1)^{\overline{1-\beta}}(\nabla^2\,\mathrm{g})(\mathrm{s}),
\end{aligned}$$

where this time we used $(0)^{\overline{1-\beta}} = 0$. By using the assumption that $\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z}) \geqq 0$, it follows that

$$
\begin{aligned}
(\nabla^2 g)(z) &\geqq -\frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)} g(t_0+1) \\
&\quad - \frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} (\nabla\, g)(t_0+2) - \frac{1}{\Gamma(-\beta+2)} \sum_{s=t_0+3}^{z-1} (z-s+1)^{\overline{1-\beta}} (\nabla^2\, g)(s) \\
&= (z-t_0-2+\beta) \frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)} g(t_0+1) - \frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} g(t_0+2) \\
&\quad - \frac{1}{\Gamma(-\beta+2)} \sum_{s=t_0+3}^{z-1} (z-s+1)^{\overline{1-\beta}} (\nabla^2\, g)(s), \qquad (3.4)
\end{aligned}
$$

where we have used

$$
\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} - \frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)} = (z-t_0-2+\beta) \frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)}.
$$

Now, by using the assumption $\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta} g\right)(z) \geqq 0$ at $z = t_0+1$ and t_0+2 in Theorem 2.1, we have

$$
\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta} g\right)(t_0+1) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0+1}^{t_0+1} (t_0+2-s)^{\overline{-\beta-1}} g(s) = g(t_0+1) \geqq 0, \qquad (3.5)
$$

and

$$
\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta} g\right)(t_0+2) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0+1}^{t_0+2} (t_0+3-s)^{\overline{-\beta-1}} g(s) = g(t_0+2) - \beta\, g(t_0+1) \geqq 0. \qquad (3.6)
$$

By using (3.6) in (3.4), we get

$$
\begin{aligned}
(\nabla^2 g)(z) &\geqq (z-t_0-2+\beta) \frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)} g(t_0+1) - \beta \frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} g(t_0+1) \\
&\quad - \frac{1}{\Gamma(-\beta+2)} \sum_{s=t_0+3}^{z-1} (z-s+1)^{\overline{1-\beta}} (\nabla^2\, g)(s) \\
&= (z-t_0-2) \frac{\Gamma(z-t_0-\beta)}{\Gamma(-\beta+1)\Gamma(z-t_0)} g(t_0+1) - \frac{1}{\Gamma(-\beta+2)} \sum_{s=t_0+3}^{z-1} (z-s+1)^{\overline{1-\beta}} (\nabla^2\, g)(s), \qquad (3.7)
\end{aligned}
$$

where we used

$$
-\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} = -\frac{(-\beta+2)(-\beta+3)\cdots(-\beta+z-t_0-2)(-\beta+z-t_0-1)}{(z-t_0-1)!} > 0, \quad \text{for } 2 < \beta < 3.
$$

Set $z := t_0 + \eta$ for $\eta \in \mathbb{S}_4$ in (3.7), we can deduce

$$
(\nabla^2 g)(t_0+\eta) \geqq (\eta-2) \frac{\Gamma(\eta-\beta)}{\Gamma(-\beta+1)\Gamma(\eta)} g(t_0+1) - \frac{1}{\Gamma(-\beta+2)} \sum_{\iota=2}^{\eta-2} (\eta-\iota)^{\overline{1-\beta}} (\nabla^2 g)(t_0+\iota+1),
$$

which rearranges to the desired (3.3). Now, it is clear that

$$\begin{aligned}\frac{(\eta-\iota)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} &= \frac{\Gamma(\eta-\iota+1-\beta)}{\Gamma(-\beta+2)\Gamma(\eta-\iota)}\\ &= \frac{(-\beta+2)(-\beta+3)\cdots(-\beta+\eta-\iota)(-\beta+\eta-\iota-1)}{(\eta-\iota-1)!} < 0,\end{aligned}$$

and from (3.5), we see that

$$\begin{aligned}&\frac{(\eta-2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta+1)}\mathrm{g}(t_0+1)\\ &= (\eta-2)\frac{(-\beta+1)(-\beta+2)(-\beta+3)\cdots(-\beta+\eta-3)(-\beta+\eta-2)(-\beta+\eta-1)}{(\eta-1)!}\mathrm{g}(t_0+1)\\ &\geqq 0,\end{aligned}$$

for $2<\beta<3$, $\iota=2,3,\ldots,\eta-2$ and $\eta\in\mathbb{S}_4$. Thus, the proof is done. □

Based on the above lemmas, we can now present our Δ and ∇ convexity results.

Theorem 3.1. *If* $\beta\in(2,3)$ *and* $\mathrm{g}\in\mathscr{D}_{t_0}(\mathrm{g})$ *satisfies* $\left({}^{\mathrm{RL}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})\geqq 0$, *for all* $\mathrm{z}\in\mathbb{S}_{t_0+3-\beta}$, $\mathrm{g}(t_0)\leqq 0$, $(\Delta\mathrm{g})(t_0)\geqq 0$ *and* $(\Delta^2\mathrm{g})(t_0)\geqq 0$, *then* $(\Delta^2\mathrm{g})(\mathrm{z})\geqq 0$ *for* $\mathrm{z}\in\mathbb{S}_{t_0}$.

Proof. By using strong induction we will done this proof. From the assumption, we know that $(\Delta^2\mathrm{g})(t_0)\geqq 0$. We assume that $(\Delta^2\mathrm{g})(t_0+\iota)\geqq 0$ for $\iota=0,1,\ldots,\eta$. Then Lemma 3.1 gives that $(\Delta^2\mathrm{g})(t_0+\eta+1)\geqq 0$. Thus, the proof is done. □

Corollary 3.1. *If* $\beta\in(2,3)$ *and* $\mathrm{g}\in\mathscr{D}_{t_0}(\mathrm{g})$ *satisfies*

$$\left({}^{\mathrm{LC}}_{t_0}\Delta^{\beta}\mathrm{g}\right)(\mathrm{z})\geqq -\sum_{\jmath=0}^{2}\frac{(\mathrm{z}-t_0)^{\underline{-\beta+\jmath}}}{\Gamma(-\beta+\jmath+1)}(\Delta^{\jmath}\mathrm{g})(t_0),\quad \textit{for all } \mathrm{z}\in\mathbb{S}_{t_0+3-\beta},$$

$\mathrm{g}(t_0)\leqq 0$, $(\Delta\mathrm{g})(t_0)\geqq 0$ *and* $(\Delta^2\mathrm{g})(t_0)\geqq 0$, *then,* $(\Delta^2\mathrm{g})(\mathrm{z})\geqq 0$, *for* $\mathrm{z}\in\mathbb{S}_{t_0}$.

Proof. The result follows immediately from (2.12) with $\aleph=3$ and Theorem 3.1. □

Theorem 3.2. *If* $\beta\in(2,3)$ *and* $\mathrm{g}\in\mathscr{D}_{t_0+1}(\mathrm{g})$ *satisfies* $\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(\mathrm{z})\geqq 0$, *for all* $\mathrm{z}\in\mathbb{S}_{t_0+1}$, *then* $(\nabla^2\mathrm{g})(\mathrm{z})\geqq 0$ *for* $\mathrm{z}\in\mathbb{S}_{t_0+3}$.

Proof. The proof can be accomplished by using the principle of mathematical induction. Indeed, by using Theorem 2.1 with $\mathrm{z}=t_0+3$, we see that

$$\begin{aligned}\left({}^{\mathrm{RL}}_{t_0}\nabla^{\beta}\mathrm{g}\right)(t_0+3) &= \frac{1}{\Gamma(-\beta)}\sum_{\mathrm{s}=t_0+1}^{t_0+3}(t_0+4-\mathrm{s})^{\overline{-\beta-1}}\mathrm{g}(\mathrm{s})\\ &= \frac{\beta(\beta-1)}{2}\mathrm{g}(t_0+1)-\beta\,\mathrm{g}(t_0+2)+\mathrm{g}(t_0+3)\\ &= (\nabla^2\mathrm{g})(t_0+3)-(\beta-2)\mathrm{g}(t_0+2)-\mathrm{g}(t_0+1)+\frac{\beta(\beta-1)}{2}\mathrm{g}(t_0+1)\geqq 0.\end{aligned}$$

Thus, upon solving for $(\nabla^2 g)(t_0+3)$, we have

$$\begin{aligned}(\nabla^2 g)(t_0+3) &\geqq (\beta-2)g(t_0+2) - \frac{(\beta-2)(\beta+1)}{2}g(t_0+1)\\ &\underset{(3.6)}{\overset{by}{\geqq}} \beta(\beta-2)g(t_0+1) - \frac{(\beta-2)(\beta+1)}{2}g(t_0+1)\\ &= \frac{(\beta-2)(\beta-1)}{2}g(t_0+1) \geqq 0.\end{aligned}$$

Now, we assume that $(\nabla^2 g)(t_0+\imath) \geqq 0$ for $\imath = 3, 4, \ldots, \eta-1$. Then Lemma 3.2 leads to $(\Delta^2 g)(t_0+\eta) \geqq 0$. Hence, the proof is done. □

Corollary 3.2. *If* $\beta \in (2,3)$ *and* $g \in \mathscr{D}_{t_0}(g)$ *satisfies*

$$\left({}^{LC}_{t_0}\nabla^{\beta} g\right)(z) \geqq -\sum_{\jmath=0}^{2} \frac{(z-t_0)^{\overline{-\beta+\jmath}}}{\Gamma(-\beta+\jmath+1)}(\nabla^{\jmath} g)(t_0), \quad \textit{for all } z \in \mathbb{S}_{t_0+1},$$

then, $(\nabla^2 g)(z) \geqq 0$*, for* $z \in \mathbb{S}_{t_0+3}$.

Proof. The result follows immediately from (2.13) with $\aleph = 3$ and Theorem 3.2. □

4. Applications

We try to apply our main results on the function $g(z) = z^2$ with $\beta = \frac{5}{2}$ and $t_0 = 0$. Firstly, from the definition of delta fractional difference (2.8), we have

$$\begin{aligned}\left({}^{RL}_{0}\Delta^{\beta} g\right)(3-\beta) &= \frac{1}{\Gamma(-\beta)}\sum_{s=0}^{3}(2-\beta-s)^{\underline{-\beta-1}}g(s)\\ &= \frac{1}{\Gamma(-\beta)}\left[(2-\beta)^{\underline{-\beta-1}}g(0) + (1-\beta)^{\underline{-\beta-1}}g(1) + (-\beta)^{\underline{-\beta-1}}g(2) + (-\beta-1)^{\underline{-\beta-1}}g(3)\right]\\ &= \frac{7}{8} \geq 0,\end{aligned}$$

similarly, we can have

$$\begin{aligned}\left({}^{RL}_{0}\Delta^{\beta} g\right)(4-\beta) &= \frac{1}{\Gamma(-\beta)}\sum_{s=0}^{4}(3-\beta-s)^{\underline{-\beta-1}}g(s)\\ &= \frac{11}{16} \geq 0.\end{aligned}$$

We can continue by the same technique as above to get

$$\left({}^{RL}_{0}\Delta^{\beta} g\right)(\eta-\beta) \geq 0,$$

for each $\eta \in \mathbb{S}_3$. In addition, we have that

$$g(0) = 0 \leq 0, \quad (\Delta g)(0) = 1 \geq 0, \quad (\Delta^2 g)(0) = 2 \geq 0.$$

Hence, $g(z) = z^2$ is convex on $\mathbb{S}_0$ according to Theorem 3.1.

On the other hand, from the definition of nabla fractional difference (2.7), we have

$$\begin{aligned}\left({}^{RL}_{0}\nabla^{\beta}g\right)(1) &= \frac{1}{\Gamma(-\beta)}\sum_{s=1}^{1}(2-s)^{\overline{-\beta-1}}g(s)\\ &= \frac{1}{\Gamma(-\beta)}(1)^{\overline{-\beta-1}}g(1)\\ &= g(1) = 1 \geq 0,\end{aligned}$$

and

$$\begin{aligned}\left({}^{RL}_{0}\nabla^{\beta}g\right)(2) &= \frac{1}{\Gamma(-\beta)}\sum_{s=1}^{2}(3-s)^{\overline{-\beta-1}}g(s)\\ &= \frac{1}{\Gamma(-\beta)}\left[(2)^{\overline{-\beta-1}}g(1) + (1)^{\overline{-\beta-1}}g(2)\right]\\ &= \frac{3}{2} \geq 0.\end{aligned}$$

Proceeding by the same technique as above to get

$$\left({}^{RL}_{0}\nabla^{\beta}g\right)(\eta) \geq 0,$$

for all $\eta \in \mathbb{S}_1$. Therefore, $g(z) = z^2$ is convex on $\mathbb{S}_3$ by Theorem 3.2.

5. Conclusions and future directions

The outcomes of this article are briefly as follows:

(1) Alternative definitions have been presented to the basic definitions of the discrete delta and nabla fractional differences in Theorems 2.2 and 2.1, respectively.

(2) A relationship between the delta and nabla fractional differences of the Riemann-Liouville and Liouville-Caputo types of higher orders has been established in Proposition 2.1.

(3) The formulas for $(\Delta^2 g)$ and $(\nabla^2 g)$ have been represented in Lemmas 3.1 and 3.2, respectively.

(4) Based on Lemmas 3.1 and 3.2, some convexity results have been analyzed for the delta and nabla fractional differences of the Riemann-Liouville type in Theorems 3.1 and 3.2, respectively.

(5) Similar results have been obtained for the delta and nabla fractional differences of the Liouville-Caputo type by using the proposed relationships in Corollaries 3.1 and 3.2, respectively.

For the future direction of this work, we notice that the use of discrete fractional operators of the Caputo-Fabrizio and Atangana-Baleanu types to establish similar relationships and convexity results has the potential to lead to future investigations by the interested researchers (see [19, 32] for information about these discrete fractional operators).

Acknowledgements

This research work was funded by the Institutional Fund Projects under grant No. (IFPHI228-130-2020). The authors gratefully acknowledge technical and financial support from the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. F. M. Atıcı, M. Uyanik, Analysis of discrete fractional operators, *Appl. Anal. Discrete Math.*, **9** (2015), 139–149.
2. F. M. Atıcı, M. Atıcı, M. Belcher, D. Marshall, A new approach for modeling with discrete fractional equations, *Fund. Inform.*, **151** (2017), 313–324. https://doi.org/10.3233/FI-2017-1494
3. F. Atıcı, S. Sengul, Modeling with discrete fractional equations, *J. Math. Anal. Appl.*, **369** (2010), 1–9.
4. C. S. Goodrich, On discrete sequential fractional boundary value problems, *J. Math. Anal. Appl.*, **385** (2012), 111–124. https://doi.org/10.1016/j.jmaa.2011.06.022
5. C. R. Chen, M. Bohner, B. G. Jia, Ulam-Hyers stability of Caputo fractional difference equations, *Math. Meth. Appl. Sci.*, **42** (2019), 7461–7470. https://doi.org/10.1002/mma.5869
6. R. Dahal, C. S. Goodrich, Theoretical and numerical analysis of monotonicity results for fractional difference operators, *Appl. Math. Lett.*, **117** (2021), 107104. https://doi.org/10.1016/j.aml.2021.107104
7. C. Lizama, The Poisson distribution, abstract fractional difference equations, and stability, *Proc. Am. Math. Soc.*, **145** (2017), 3809–3827. https://doi.org/10.1090/proc/12895
8. H. M. Srivastava, P. O. Mohammed, C. S. Ryoo, Y. S. Hamed, Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations, *J. King Saud Univ.-Sci.*, **33** (2021), 101497. https://doi.org/10.1016/j.jksus.2021.101497
9. Q. Lu, Y. Zhu, Comparison theorems and distributions of solutions to uncertain fractional difference equations, *J. Comput. Appl. Math.*, **376** (2020), 112884. https://doi.org/10.1016/j.cam.2020.112884
10. F. M. Atıcı, P. W. Eloe, A transform method in discrete fractional calculus, *Int. J. Differ. Equ.*, **2** (2007), 165–176.
11. P. O. Mohammed, T. Abdeljawad, Discrete generalized fractional operators defined using *h*-discrete Mittag-Leffler kernels and applications to AB fractional difference systems, *Math. Meth. Appl. Sci.*, 2020. https://doi.org/10.1002/mma.7083
12. F. M. Atıcı, M. Atıcı, N. Nguyen, T. Zhoroev, G. Koch, A study on discrete and discrete fractional pharmaco kinetics pharmaco dynamics models for tumor growth and anti-cancer effects, *Comput. Math. Biophys.*, **7** (2019), 10–24. https://doi.org/10.1515/cmb-2019-0002
13. A. Silem, H. Wu, D. J. Zhang, Discrete rogue waves and blow-up from solitons of a nonisospectral semi-discrete nonlinear Schrödinger equation, *Appl. Math. Lett.*, **116** (2021), 107049. https://doi.org/10.1016/j.aml.2021.107049
14. R. A. C. Ferreira, D. F. M. Torres, Fractional *h*-difference equations arising from the calculus of variations, *Appl. Anal. Discrete Math.*, **5** (2011), 110–121. https://doi.org/10.2298/AADM110131002F

15. G. Wu, D. Baleanu, Discrete chaos in fractional delayed logistic maps, *Nonlinear Dyn.*, **80** (2015), 1697–1703. https://doi.org/10.1007/s11071-014-1250-3

16. J. W. He, L. Zhang, Y. Zhou, B. Ahmad, Existence of solutions for fractional difference equations via topological degree methods, *Adv. Differ. Equ.*, **2018** (2018), 153. https://doi.org/10.1186/s13662-018-1610-2

17. C. S. Goodrich, B. Lyons, Positivity and monotonicity results for triple sequential fractional differences via convolution, *Analysis*, **40** (2020), 89–103. https://doi.org/10.1515/anly-2019-0050

18. P. O. Mohammed, T. Abdeljawad, F. K. Hamasalh, On Riemann-Liouville and Caputo fractional forward difference monotonicity analysis, *Mathematics*, **9** (2021), 1303. https://doi.org/10.3390/math9111303

19. P. O. Mohammed, T. Abdeljawad, F. K. Hamasalh, On discrete delta Caputo-Fabrizio fractional operators and monotonicity analysis, *Fractal Fract.*, **5** (2021), 116. https://doi.org/10.3390/fractalfract5030116

20. T. Abdeljawad, D. Baleanu, Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel, *Chaos Solitons Fract.*, **102** (2017), 106–110. https://doi.org/10.1016/j.chaos.2017.04.006

21. X. Liu, F. Du, D. Anderson, B. Jia, Monotonicity results for nabla fractional h-difference operators, *Math. Meth. Appl. Sci.*, **44** (2020), 1207–1218. https://doi.org/10.1002/mma.6823

22. C. S. Goodrich, C. Lizama, Positivity, monotonicity, and convexity for convolution operators, *Discrete Cont. Dyn. Syst.*, **40** (2020), 4961–4983. http://dx.doi.org/10.3934/dcds.2020207

23. L. Erbe, C. S. Goodrich, B. Jia, A. C. Peterson, Survey of the qualitative properties of fractional difference operators: Monotonicity, convexity, and asymptotic behavior of solutions, *Adv. Differ. Equ.*, **2016** (2016), 43. https://doi.org/10.1186/s13662-016-0760-3

24. P. O. Mohammed, O. Almutairi, R. P. Agarwal, Y. S. Hamed, On convexity, monotonicity and positivity analysis for discrete fractional operators defined using exponential kernels, *Fractal Fract.*, **6** (2022), 55. https://doi.org/10.3390/fractalfract6020055

25. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, 2006.

26. H. M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, *Kyungpook Math. J.*, **60** (2020), 73–116.

27. P. O. Mohammed, H. M. Srivastava, D. Baleanu, E. E. Elattar, Y. S. Hamed, Positivity analysis for the discrete delta fractional differences of the Riemann-Liouville and Liouville-Caputo types, *Electron. Research Arch.*, **30** (2022), 3058–3070. https://doi.org/10.3934/era.2022155

28. H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, *J. Nonlinear Convex Anal.*, **22** (2021), 1501–1520.

29. H. M. Srivastava, An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions, *J. Adv. Eng. Comput.*, **5** (2021), 135–166. https://doi.org/10.55579/jaec.202153.340

30. C. S. Goodrich, A. C. Peterson, *Discrete fractional calculus*, Springer, 2015. https://doi.org/10.1007/978-3-319-25562-0

31. T. Abdeljawad, F. M. Atıcı, On the definitions of nabla fractional operators, *Abstr. Appl. Anal.*, **2012** (2012), 406757. https://doi.org/10.1155/2012/406757

32. T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, *Chaos Solit. Fract.*, **116** (2018), 146–156. https://doi.org/10.1016/j.chaos.2018.09.022

33. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, *Discrete Dyn. Nature Soc.*, **2013** (2013), 406910. https://doi.org/10.1155/2013/406910

AIMS Press