Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 19539–19552.
DOI: 10.3934/math.20221072
Received: 11 August 2022
Revised: 29 August 2022
Accepted: 30 August 2022
Published: 05 September 2022

Research article

New analytical method of solution to a nonlinear singular fractional Lane–Emden type equation

McSylvester Ejighikeme Omaba*

Department of Mathematics, College of Science, University of Hafr Al Batin, P.O. Box 1803, Hafr Al Batin 31991, Saudi Arabia

* **Correspondence:** Email: mcomaba@uhb.edu.sa.

Abstract: We consider a nonlinear singular fractional Lane–Emden type differential equation

$$ {}^{LC}\mathcal{D}^{\alpha}_{a^+}\varphi(t) + \frac{\lambda}{t^{\alpha-\beta}}\ {}^{LC}\mathcal{D}^{\beta}_{a^+}\varpi(t,\varphi(t)) = 0,\ \ 0<\beta<\alpha<1,\ \ 0<a<t\leq T, $$

with an initial condition $\varphi(a) = \nu$ assumed to be bounded and non-negative, $\varpi : [a, T] \times \mathbb{R} \to \mathbb{R}$ a Lipschitz continuous function, and ${}^{LC}\mathcal{D}^{\alpha}_{a^+}, {}^{LC}\mathcal{D}^{\beta}_{a^+}$ are Liouville–Caputo derivatives of orders $0 < \alpha, \beta < 1$. A new analytical method of solution to the nonlinear singular fractional Lane–Emden type equation using fractional product rule and fractional integration by parts formula is proposed. Furthermore, we prove the existence and uniqueness and the growth estimate of the solution. Examples are given to illustrate our results.

Keywords: well-posedness; growth bound; nonlinear fractional Lane–Emden equation; regularized incomplete beta function; new analytical method
Mathematics Subject Classification: 26A33, 28A80, 34A08, 34A09

1. Introduction

In 1870, an American astrophysicist by name Jonathan Homer Lane first published the Lane–Emden type equations [1] and were further explored by a Swiss theoretical physicist (astrophysicist and meterologist) Robert Emden in 1907 [2]. They used the equations to describe the internal structure of gaseous spheres.

The standard form of Lane–Emden differential equation is given by [3]

$$ \begin{cases} y''(t) + \frac{\lambda}{t}y'(t) + f(t, y(t)) = g(t),\ \ 0 < t \leq 1, \lambda \geq 0, \\ y(0) = A,\ y'(0) = B. \end{cases} \tag{1.1} $$

This equation is a singular initial value problems relating to second order differential equations, used to describe the theory of singular boundary value problem. The Lane–Emden equation best depicts and describes a wide range of phenomena in mathematical physics, chemistry, and astrophysics, specifically in the areas of theory of stellar structure, thermal explosion, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and thermionic currents [4, 5]. For recent stochastic model applications, the authors in [6] adopted a design of Morlet wavelet neutral network to find a solution of second order Lane–Emden equation. Other stochastic models for singular Lane–Emden equations include [7–10].

Next, we review some other numerical and analytical methods of solutions to both the standard and fractional Lane–Emden type equations in literature. In 2013, the authors [11] constructed a second kind Chebyshev operational matrix algorithm to give numerical solutions of a class of linear and nonlinear Lane–Emden type singular initial value problems; in 2014, the same authors in [12] used shifted ultraspherical operational matrices of derivatives to give solutions of singular Lane–Emden equations arising in astrophysics; and in 2018, another set of authors in [13] developed an algorithm based on operational matrix of integration for Jacobi polynomials and collocation method to obtain an approximate solution of nonlinear Lane–Emden type equations arising in astrophysics. In a recent development, the authors in [14] were able to successfully propose a computationally effective approximation technique based on Bessel matrix representation and collocation points to find numerical solution of a nonlinear, singular second-order Lane–Emden pantograph delay differential equation. Other recent numerical methods of solving Lane–Emden type equation include the use of ultraspherical wavelets methods [15], the use of spectral Legendre's derivative algorithms [16], etc.

Now, one could ask, why the fractional Lane–Emden differential equation? It is known that fractional derivatives are needed in order to best describe the dynamics of materials in fractal medium, to capture the long-term memory effect and long-range interactions of systems, the apparent importance of fractional derivatives in modeling mechanical and electrical properties of real materials, and in the description of properties of gases, liquids and rocks, see [17–20] and their references. Consequently, the authors [3] in 2012, generalized Eq (1.1) to a nonlinear-singular fractional Lane–Emden system

$$\begin{cases} D^{\alpha}y(t) + \frac{\lambda}{t^{\alpha-\beta}}D^{\beta}y(t) + f(t, y(t)) = g(t), \ 0 < t \leq 1, \\ y(0) = c_0, \ y(1) = d_0, \end{cases}$$

where $\lambda \geq 0, 0 < \alpha \leq 2, \ 0 < \beta \leq 1, \ f : [0, 1] \times \mathbb{R} \to \mathbb{R}, \ g : [0, 1] \to \mathbb{R}$ are continuous, A, B constants and D^{α}, D^{β} are Riemann–Liouville fractional derivatives. The fractional Lane–Emden equation is of a significant importance in the accurate modeling of real-life phenomena. For example, the nonlinear singular fractional Lane–Emden systems have been applied in a novel design of fractional Meyer wavelet neutral networks [21].

Many researchers have employed different approaches and methods in formulating the analytical solution to the fractional Lane–Emden equations. Recently in 2022, the author in [22] studied analytical solution to a class of fractional Lane–Emden equation using a power series method. Another analytical solution involves the method of Laplace transform [5]. The author in [3] imployed a numerical method of collocation to give approximate solution of the fractional Lane–Emden equation. Some other numerical methods have also been developed to give approximate solutions of fractional order Lane–Emden-type differential equations. These methods include matrix method in terms of generalized Bessel functions and based on suitable collocation points [23], the homotopy perturbation using Adomian decomposition method [24] and Polynomial Least Square Method (PLSM) which gives an analytical approximate polynomial solution of fractional Lane–Emden differential equations [25].

In this paper, as a motivation, we seek to find a simplified and an alternative formulation of analytical solution to a fractional Lane–Emden type equation and consider the following

$$\begin{cases} {}^{LC}\mathcal{D}^{\alpha}_{a^+}\varphi(t) + \frac{\lambda}{t^{\alpha-\beta}}\,{}^{LC}\mathcal{D}^{\beta}_{a^+}\varpi(t,\varphi(t)) = 0, \qquad 0 < a < t \le T, \\ \varphi(a) = \nu, \end{cases} \tag{1.2}$$

where $0 < \beta < \alpha < 1$, ${}^{LC}\mathcal{D}^{\alpha}_{a^+}$ and ${}^{LC}\mathcal{D}^{\beta}_{a^+}$ are Liouville–Caputo fractional derivatives, and $\varpi : [a, T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. In contrast to the known fractional Lane–Emden equations, Eq (1.2) is a singular initial value nonlinear fractional Lane–Emden type equation relating to first order differential equations.

Remark 1.1. *(1) The novelty of the paper is that it is the first to apply this analytical method in solving nonlinear singular fractional Lane–Emden type equation. The advantage of using the fractional product rule and fractional integration by parts formula is that it is simple, straightforward and less complicated; the only downside is that the method was unable to capture all the usual order* $0 < \alpha \le 2$*, because of the positive requirement of* $1 - \alpha$ *in* $\Gamma(1 - \alpha)$ *in our formulation.*

(2) Our results are performed at approximation of the singular point since we have some terms in our solution kernel that are not defined at 0*.*

The paper is organized as follows. Section 2 contains the preliminaries and formulation of the solution; and in Section 3, we give the main results of the paper. Section 4 contains some examples to illusrate our main results and Section 5 provides a short summary of the paper.

2. Preliminaries

Here, we present definitions of some basic concepts. See [26] for more concepts on fractional calculus.

Definition 2.1 ([27]). *Let $a < b$ be positive real numbers and $f : [a, b] \to \mathbb{R}$ be an integrable function. The left sided Katugampola fractional integral of order α and parameter ρ is given by*

$$\mathcal{I}^{\alpha,\rho}_{a^+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-1} f(s)ds.$$

Remark 2.2. *For $\rho = 1$, one gets the Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : [a, b] \to \mathbb{R}$ given by*

$$\mathcal{I}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s)ds, \tag{2.1}$$

provided that the integral exists and finite.

Definition 2.3 ([28]). *The Riemann–Liouville fractional derivative of order* $0 < \alpha < 1$ *of a function* $f : [a, b] \to \mathbb{R}$ *is given by*

$$\mathcal{D}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds,$$

provided that the integral exists and finite.

Definition 2.4 ([27]). *Let* $a < b$ *be positive real numbers,* $\rho > 0$, $\alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$ *such that* $n - 1 < \alpha < n$, *and* $f : [a, b] \to \mathbb{R}$ *is a class of* C^n *function. The left-sided Caputo–Katugampola fractional derivative of order* α *and parameter* ρ *is defined by*

$$^{C}\mathcal{D}^{\alpha,\rho}_{a^+} f(t) = \mathcal{I}^{\alpha,\rho}_{a^+}\left(t^{1-\rho}\frac{d}{dt}\right)^n f(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_a^t s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-1}\left(t^{1-\rho}\frac{d}{ds}\right)^n f(s) ds.$$

Remark 2.5. *For* $n = 1$ *and* $\rho = 1$, *then the Liouville–Caputo fractional derivative of order* $0 < \alpha < 1$ *of a function* $f : [a, b] \to \mathbb{R}$ *is given by*

$$^{LC}\mathcal{D}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds,$$

provided that the integral exists and finite.

Lemma 2.6 ([28]). *The relationship between the Liouville–Caputo and Riemann–Liouville derivatives is*

$$^{LC}\mathcal{D}^{\alpha}_{a^+} f(t) = \mathcal{D}^{\alpha}_{a^+} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a).$$

When $n = 1$, *it implies that* $k = 0$ *and*

$$^{LC}\mathcal{D}^{\alpha}_{a^+} f(t) = \mathcal{D}^{\alpha}_{a^+} f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a).$$

For an example, if a function $f : [a, b] \to \mathbb{R}$ *is given by* $f(t) = t^{\nu}$. *Then*

$$\mathcal{D}^{\alpha}_{a^+} f(t) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}.$$

Theorem 2.7 ([27]). *Let* $y \in C^n[a, b]$, *then we have*

$$\mathcal{I}^{\alpha,\rho}_{a^+}\, {}^{C}\mathcal{D}^{\alpha,\rho}_{a^+} y(t) = y(t) - \sum_{k=0}^{n-1} \frac{\rho^{-k}}{k!} (t^{\rho} - a^{\rho})^k y_{(k)}(a).$$

For a function $y \in C[a, b]$ *and* $\rho = 1$, *we have*

$$\mathcal{I}^{\alpha}_{a^+}\, {}^{LC}\mathcal{D}^{\alpha}_{a^+} y(t) = y(t) - y(a). \tag{2.2}$$

The following is a generalized fractional integration by parts formula:

Theorem 2.8 ([27]). *Let $f \in C[a,b]$ and $g \in C^n[a,b]$ be two functions. Then*

$$\begin{aligned}\int_a^b f(t)^C\mathcal{D}_{a^+}^{\alpha,\rho} g(t)dt &= \int_a^b t^{\rho-1} g(t)\mathcal{D}_{b^-}^{\alpha,\rho}(t^{1-\rho})f(t)dt \\ &+ \Big[\sum_{k=0}^{n-1}\Big(-t^{1-\rho}\frac{d}{dt}\Big)^k I_{b^-}^{n-\alpha,\rho}(t^{1-\alpha}f(t))g_{(n-k-1)}(t)\Big]_{t=a}^{t=b}.\end{aligned}$$

In particular, for $n = 1$ and $\rho = 1$,

$$\int_a^b f(t)^{LC}\mathcal{D}_{a^+}^{\alpha} g(t)dt = \int_a^b g(t)\mathcal{D}_{b^-}^{\alpha} f(t)dt. \tag{2.3}$$

Definition 2.9 ([29,30]). *For $\mu, \upsilon > 0$, one defines the incomplete beta function by*

$$B(\tau,\mu,\upsilon) = \int_0^\tau t^{\mu-1}(1-t)^{\upsilon-1}dt,\ \ \tau \in [0,1].$$

It also has a representation in terms of a hypergeometric function given by

$$B(\tau,\mu,\upsilon) = \frac{\tau^\mu}{\mu}\ {}_2F_1(\mu, 1-\upsilon; \mu+1; \tau).$$

Definition 2.10 ([31]). *The regularized incomplete beta function is defined by*

$$I(\tau,\mu,\upsilon) = \frac{B(\tau,\mu,\upsilon)}{B(\mu,\upsilon)} = \frac{1}{B(\mu,\upsilon)}\int_0^\tau \tau^{\mu-1}(1-\tau)^{\upsilon-1}d\tau,$$

satisfying the following properties:

- $I(\tau,\mu,\upsilon) = I(\tau,\mu+1,\upsilon-1) + \frac{\tau^\mu(1-\tau)^{\upsilon-1}}{\mu B(\mu,\upsilon)}$,
- $I(\tau,\mu,\upsilon) = I(\tau,\mu+1,\upsilon+1) - \frac{\tau^\mu(1-\tau)^{\upsilon-1}}{\upsilon B(\mu,\upsilon)}$,
- $I(\tau,\mu,\upsilon) = I(\tau,\mu+1,\upsilon) + \frac{\tau^\mu(1-\tau)^{\upsilon}}{\mu B(\mu,\upsilon)}$,
- $I(\tau,\mu,\upsilon) = I(\tau,\mu,\upsilon+1) - \frac{\tau^\mu(1-\tau)^{\upsilon}}{\upsilon B(\mu,\upsilon)}$,
- $I(\tau,\mu,\upsilon) + I(1-\tau,\upsilon,\mu) = 1$,
- $I(1,\mu,\upsilon) = 1$ *and* $I(\tau,\mu,\upsilon) \in [0,1]$.

Here, we make sense of the solution to Eq (1.2).

Lemma 2.11. *The solution to fractional Lane–Emden type equation* (1.2) *is given by*

$$\varphi(t) = \nu + \frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}\varpi(s,\varphi(s))ds - \lambda\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\int_a^t (t-s)^{\alpha-1}s^{-\alpha}\varpi(s,\varphi(s))ds.$$

Proof. Apply the fractional integral operator $\mathcal{I}_{a^+}^{\alpha}$ on both sides of Eq (1.2) to obtain

$$\mathcal{I}_{a^+}^{\alpha}\Big[\ {}^{LC}\mathcal{D}_{a^+}^{\alpha}\varphi(t)\Big] + \mathcal{I}_{a^+}^{\alpha}\Big[\frac{\lambda}{t^{\alpha-\beta}}\ {}^{LC}\mathcal{D}_{a^+}^{\beta}\varpi(t,\varphi(t))\Big] = 0.$$

From Eq (2.2) in Theorem 2.7 and Eq (2.1), we have

$$\varphi(t)-\varphi(a)+\frac{\lambda}{\Gamma(\alpha)}\int_a^t \frac{(t-s)^{\alpha-1}}{s^{\alpha-\beta}}\ {}^{LC}\mathcal{D}_{a^+}^{\beta}\varpi(s,\varphi(s))ds=0.$$

By Eq (2.3) in Theorem 2.8, we obtain

$$\varphi(t)-\varphi(a)+\frac{\lambda}{\Gamma(\alpha)}\int_a^t \varpi(s,\varphi(s))\mathcal{D}_{a^+}^{\beta}((t-s)^{\alpha-1}s^{\beta-\alpha})ds=0. \tag{2.4}$$

Apply the product rule on $\mathcal{D}_{a^+}^{\beta}((t-s)^{\alpha-1}s^{\beta-\alpha})$ as follows

$$\begin{aligned}
\mathcal{D}_{a^+}^{\beta}((t-s)^{\alpha-1}s^{\beta-\alpha}) &= (t-s)^{\alpha-1}\mathcal{D}_{a^+}^{\beta}s^{\beta-\alpha}+s^{\beta-\alpha}\mathcal{D}_{a^+}^{\beta}(t-s)^{\alpha-1}\\
&= (t-s)^{\alpha-1}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\alpha-\beta+1)}s^{\beta-\alpha-\beta}-s^{\beta-\alpha}\frac{\Gamma(\alpha-1+1)}{\Gamma(\alpha-1-\beta+1)}(t-s)^{\alpha-\beta-1}\\
&= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}(t-s)^{\alpha-1}s^{-\alpha}-\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-s)^{\alpha-\beta-1}s^{\beta-\alpha}.
\end{aligned}$$

Therefore, from Eq (2.4), one gets

$$\begin{aligned}
\varphi(t)-\varphi(a) &+ \frac{\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}\frac{\lambda}{\Gamma(\alpha)}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-1}s^{-\alpha}ds\\
&- \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\frac{\lambda}{\Gamma(\alpha)}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-\beta-1}s^{\beta-\alpha}ds=0.
\end{aligned}$$

Using the cosecant identity $\Gamma(\alpha)\Gamma(1-\alpha)=\pi\csc(\pi\alpha)=\frac{\pi}{\sin(\pi\alpha)}$, we obatin

$$\begin{aligned}
\varphi(t)-\varphi(a) &+ \Gamma(\beta-\alpha+1)\frac{\lambda\sin(\pi\alpha)}{\pi}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-1}s^{-\alpha}ds\\
&- \frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-\beta-1}s^{\beta-\alpha}ds=0.
\end{aligned}$$

Alternatively, by the property of Beta functions $B(1-\alpha,\alpha)=\frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha+\alpha)}=\Gamma(1-\alpha)\Gamma(\alpha)$, one writes the solution as

$$\begin{aligned}
\varphi(t)-\varphi(a) &+ \Gamma(\beta-\alpha+1)\frac{\lambda}{B(1-\alpha,\alpha)}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-1}s^{-\alpha}ds\\
&- \frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t \varpi(s,\varphi(s))(t-s)^{\alpha-\beta-1}s^{\beta-\alpha}ds=0,
\end{aligned}$$

and the desired result follows. □

Now, we define the norm of φ by

$$||\varphi||:=\sup_{a\le t\le T}|\varphi(t)|.$$

3. Main results

We start with the following global Lipschitz condition on $\varpi(.,\varphi)$ as follows:

Condition 3.1. *Let* $0 < \mathrm{Lip}_{\varpi} < \infty$, *and for all* $x,\ y \in \mathbb{R}$, *and* $t \in [a, T]$, *we have*

$$|\varpi(t, x) - \varpi(t, y)| \leq \mathrm{Lip}_{\varpi}|x - y|. \tag{3.1}$$

We set $\varpi(t, 0) = 0$ *for convenience only.*

3.1. Existence and uniqueness result

The existence and uniqueness of solution to our Eq (1.2) will be proved by Banach's fixed point theorem. To begin, we define the operator

$$\mathcal{A}\varphi(t) = \nu + \frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}\varpi(s,\varphi(s))ds - \lambda\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\int_a^t (t-s)^{\alpha-1}s^{-\alpha}\varpi(s,\varphi(s))ds, \tag{3.2}$$

and show that the fixed point of the operator $\mathcal{A}$ gives the solution to Eq (1.2).

Lemma 3.2. *Let* φ *be a solution to Eq* (1.2) *and suppose that Condition 3.1 holds. Then for* $\beta + 1 > \alpha$ *and* $0 < \beta < \alpha < 1$,

$$||\mathcal{A}\varphi|| \leq \nu + c_1\lambda \mathrm{Lip}_{\varpi}||\varphi||, \tag{3.3}$$

with positive constant

$$c_1 := \left[\frac{1}{\Gamma(\alpha-\beta)}[\pi\csc(\pi(\alpha-\beta)) + B(1-\alpha+\beta,\alpha-\beta)] + \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}[\pi\csc(\pi\alpha) + B(1-\alpha,\alpha)]\right] < \infty.$$

Proof. We take absolute value of the Eq (3.2) to obtain

$$|\mathcal{A}\varphi(t)| \leq\ \nu + \frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}|\varpi(s,\varphi(s))|ds + \lambda\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\int_a^t (t-s)^{\alpha-1}s^{-\alpha}|\varpi(s,\varphi(s))|ds.$$

Applying Eq (3.1) of Condition 3.1, we have

$$\begin{aligned}
|\mathcal{A}\varphi(t)| &\leq \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}|\varphi(s)|ds + \lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\int_a^t (t-s)^{\alpha-1}s^{-\alpha}|\varphi(s)|ds\\
&\leq \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}||\varphi||\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}ds + \lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}||\varphi||\int_a^t (t-s)^{\alpha-1}s^{-\alpha}ds\\
&= \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}||\varphi||\Big[\pi\csc(\pi(\alpha-\beta)) - B\Big(\frac{a}{t}, 1-\alpha+\beta, \alpha-\beta\Big)\Big]\\
&+\lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}||\varphi||\Big[\pi\csc(\pi\alpha) - B\Big(\frac{a}{t}, 1-\alpha, \alpha\Big)\Big]\\
&\leq \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}||\varphi||\Big[\pi\csc(\pi(\alpha-\beta)) + B\Big(\frac{a}{t}, 1-\alpha+\beta, \alpha-\beta\Big)\Big]\\
&+\lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}||\varphi||\Big[\pi\csc(\pi\alpha) + B\Big(\frac{a}{t}, 1-\alpha, \alpha\Big)\Big],
\end{aligned}$$

where $B(.,.,.)$ is an incomplete Beta function. Now, applying the property of a regularized incomplete Beta function $I(z,a,b) \leq 1$ for all $0 < z < 1$, we have $B(z,a,b) = B(a,b)I(z,a,b) \leq B(a,b)$. Thus, for all $0 < a < t$, we have

$$|\mathcal{A}\varphi(t)| \leq \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}\|\varphi\|\Big[\pi\csc(\pi(\alpha-\beta)) + B(1-\alpha+\beta,\alpha-\beta)\Big]$$
$$+\lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\|\varphi\|\Big[\pi\csc(\pi\alpha) + B(1-\alpha,\alpha)\Big].$$

Taking supremum over $t \in [a,T]$ on both sides, we get

$$\|\mathcal{A}\varphi\| \leq \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)}\|\varphi\|\Big[\pi\csc(\pi(\alpha-\beta)) + B(1-\alpha+\beta,\alpha-\beta)\Big]$$
$$+\lambda \mathrm{Lip}_{\varpi}\frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\|\varphi\|\Big[\pi\csc(\pi\alpha) + B(1-\alpha,\alpha)\Big],$$

and the result follows. □

Lemma 3.3. *Let ψ and φ be solutions to Eq* (1.2) *and suppose that Condition 3.1 holds. Then for $\beta+1>\alpha$ and $0<\beta<\alpha<1$,*

$$\|\mathcal{A}\psi - \mathcal{A}\varphi\| \leq c_1\lambda \mathrm{Lip}_{\varpi}\|\psi-\varphi\|. \tag{3.4}$$

Proof. Since the proof follows same steps as the proof of Lemma 3.2, we omit the details to avoid repetition. □

Theorem 3.4. *Let $\beta+1>\alpha$ and $0<\beta<\alpha<1$; and suppose Condition 3.1 holds. Then there exists a positive constant c_1 such that for $c_1 < \frac{1}{\lambda \mathrm{Lip}_{\varpi}}$, Eq* (1.2) *has a unique solution.*

Proof. By fixed point theorem, one has $\varphi(t) = \mathcal{A}\varphi(t)$. So, using Eq (3.3) of Lemma 3.2,

$$\|\varphi\| = \|\mathcal{A}\varphi\| \leq \nu + c_1\lambda \mathrm{Lip}_{\varpi}\|\varphi\|.$$

This gives $\|\varphi\|[1-c_1\lambda \mathrm{Lip}_{\varpi}] \leq \nu$ and therefore, $\|\varphi\| < \infty$ if and only if $c_1 < \frac{1}{\lambda \mathrm{Lip}_{\varpi}}$.

On the other hand, suppose $\psi \neq \varphi$ are two solutions to Eq (1.2). Then, from Eq (3.4) of Lemma 3.3, one obtains

$$\|\psi-\varphi\| = \|\mathcal{A}\psi - \mathcal{A}\varphi\| \leq c_1\lambda \mathrm{Lip}_{\varpi}\|\psi-\varphi\|.$$

Thus, $\|\psi-\varphi\|[1-c_1\lambda \mathrm{Lip}_{\varpi}] \leq 0$. But $1-c_1\lambda \mathrm{Lip}_{\varpi} > 0$, it follows that $\|\psi-\varphi\| < 0$, which is a contradiction and therefore, $\|\psi-\varphi\| = 0$. Hence, the existence and uniqueness result follows from contraction principle. □

3.2. Upper growth bound

In 2005, Agarwal et al. in [32], presented the following retarded Gronwall-type inequality:

$$u(t) \leq a(t) + \sum_{i=1}^{n}\int_{b_i(t_0)}^{b_i(t)} g_i(t,s)w_i(u(s))ds, \quad t_0 \leq t < t_1. \tag{3.5}$$

Theorem 3.5 (Theorem 2.1 of [32]). *Suppose that the hypotheses of (Theorem 2.1 of [32]) hold and $u(t)$ is a continuous and nonnegative function on $[t_0, t_1)$ satisfying (3.5). Then*

$$u(t) \leq \mathcal{W}_n^{-1}\Big[\mathcal{W}_n(r_n(t)) + \int_{b_n(t_0)}^{b_n(t)} \max_{t_0\leq\tau\leq t} g_n(\tau, s)ds\Big],\ t_0 \leq t \leq T_1,$$

where $r_n(t)$ is determined recursively by

$$r_1(t) := a(t_0) + \int_{t_0}^{t} |a'(s)|ds,$$

$$r_{i+1} := \mathcal{W}_i^{-1}\Big[\mathcal{W}_i(r_i(t)) + \int_{b_i(t_0)}^{b_i(t)} \max_{t_0\leq\tau\leq t} g_i(\tau, s)ds\Big],\ i = 1, ..., n-1,$$

and $\mathcal{W}_i(x, x_i) := \int_{x_i}^{x} \frac{dz}{w_i(z)}$.

Remark 3.6. *Now, consider the case where $n = 2$. If*

$$u(t) \leq a(t) + \int_{b_1(t_0)}^{b_1(t)} g_1(t, s)w_1(u(s))ds + \int_{b_2(t_0)}^{b_2(t)} g_2(t, s)w_2(u(s))ds,$$

then

$$u(t) \leq \mathcal{W}_2^{-1}\Big[\mathcal{W}_2(r_2(t)) + \int_{b_2(t_0)}^{b_2(t)} \max_{t_0\leq\tau\leq t} g_2(\tau, s)ds\Big],$$

with $r_2(t) = \mathcal{W}_1^{-1}\Big[\mathcal{W}_1(r_1(t)) + \int_{b_1(t_0)}^{b_1(t)} \max_{t_0\leq\tau\leq t} g_1(\tau, s)ds\Big]$.

Here, take $w_1(u(s)) = w_2(u(s)) = u(s)$, $b_1(t_0) = b_2(t_0) = t_0 = a$ and $b_1(t) = b_2(t) = t$.

Thus, we estimate the upper growth bound on the solution.

Theorem 3.7. *Given that Condition 3.1 holds. Then for all $t \in [a, T]$, $a > 0$ and $c_2, c_3 > 0$, we have*

$$|\varphi(t)| \leq \frac{\nu}{\exp\left(c_2(a-t)^{\alpha-\beta} + c_3(a-t)^{\alpha}\right)},$$

with $c_2 = \frac{\lambda \mathrm{Lip}_\varpi}{\Gamma(1+\alpha-\beta)}\frac{1}{a^{\alpha-\beta}}$, $c_3 = \lambda \mathrm{Lip}_\varpi \frac{\Gamma(\beta-\alpha+1)}{\alpha B(1-\alpha,\alpha)}\frac{1}{a^{\alpha}}$, *for* $0 < \beta < \alpha < 1$ *and* $\beta + 1 > \alpha$.

Proof. From the proof of Lemma 3.2, it was obtained that

$$\begin{aligned}|\varphi(t)| &\leq \nu + \frac{\lambda \mathrm{Lip}_\varpi}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}s^{\beta-\alpha}|\varphi(s)|ds + \lambda \mathrm{Lip}_\varpi \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\int_a^t (t-s)^{\alpha-1}s^{-\alpha}|\varphi(s)|ds \\ &\leq \nu + \frac{\lambda \mathrm{Lip}_\varpi}{\Gamma(\alpha-\beta)}\sup_{a\leq s\leq t} s^{\beta-\alpha}\int_a^t (t-s)^{\alpha-\beta-1}|\varphi(s)|ds + \lambda \mathrm{Lip}_\varpi \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}\sup_{a\leq s\leq t} s^{-\alpha}\int_a^t (t-s)^{\alpha-1}|\varphi(s)|ds.\end{aligned}$$

Since $s^{\beta-\alpha}$ and $s^{-\alpha}$ are both decreasing for $0 < \beta < \alpha$, we have

$$|\varphi(t)| \leq \nu + \frac{\lambda \mathrm{Lip}_\varpi}{\Gamma(\alpha-\beta)}a^{\beta-\alpha}\int_a^t (t-s)^{\alpha-\beta-1}|\varphi(s)|ds + \lambda \mathrm{Lip}_\varpi \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)}a^{-\alpha}\int_a^t (t-s)^{\alpha-1}|\varphi(s)|ds.$$

Let $h(t) := |\varphi(t)|$, for $t \in [0, T]$ to get

$$h(t) \le \nu + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} \int_0^t (t-s)^{\alpha-\beta-1} h(s) ds + \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)} \frac{1}{a^{\alpha}} \int_0^t (t-s)^{\alpha-1} h(s) ds. \quad (3.6)$$

Now, we apply Theorem 3.5 to (3.6). For $\mathcal{W}_2$, we have that

$$\mathcal{W}_2(x, x_2) = \int_{x_2}^{x} \frac{dz}{z} = \ln x - \ln x_2.$$

For convenience, we take $x_2 = 1$ and $\mathcal{W}_2(x) = \ln x$ with the inverse $\mathcal{W}_2^{-1}(x) = e^x$. Similarly, $\mathcal{W}_1(x) = \ln x$ with its inverse $\mathcal{W}_1^{-1}(x) = e^x$.

Also, $a(t) = \nu$ and $a'(t) = 0$, so $r_1(t) = \nu$. Next, define non-negative functions g_1, $g_2 : [a, T] \times [a, T] \to \mathbb{R}_+$ as follows:

$$g_1(\tau, s) := \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} (\tau - s)^{\alpha-\beta-1},$$

and

$$g_2(\tau, s) := \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)} \frac{1}{a^{\alpha}} (\tau - s)^{\alpha-1}.$$

For $a \le s < \tau$ and given that $\alpha - \beta - 1 < 0$, then g_1 is continuous and decreasing, hence,

$$\max_{a \le \tau \le t} g_1(\tau, s) = \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} (a - s)^{\alpha-\beta-1},$$

and we have

$$r_2(t) = \exp\left[\ln(\nu) + \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} \int_a^t (a-s)^{\alpha-\beta-1} ds \right] = \exp\left[\ln(\nu) - \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} \frac{(a-t)^{\alpha-\beta}}{\alpha-\beta} \right].$$

Also, for $a \le s < \tau$, and for all $\alpha < 1$, g_2 is continuously decreasing, and

$$\max_{a \le \tau \le t} g_2(\tau, s) = \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)} \frac{1}{a^{\alpha}} (a - s)^{\alpha-1}.$$

Thus,

$$\begin{aligned} h(t) &\le \exp\left[\ln(r_2(t)) + \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)} \frac{1}{a^{\alpha}} \int_a^t (a-s)^{\alpha-1} ds \right] \\ &= \exp\left[\ln(\nu) - \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(1+\alpha-\beta)} \frac{1}{a^{\alpha-\beta}} (a-t)^{\alpha-\beta} - \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{B(1-\alpha,\alpha)} \frac{1}{a^{\alpha}} \frac{(a-t)^{\alpha}}{\alpha} \right] \\ &= \nu \exp\left[- \frac{\lambda \mathrm{Lip}_{\varpi}}{\Gamma(1+\alpha-\beta)} \frac{(a-t)^{\alpha-\beta}}{a^{\alpha-\beta}} - \lambda \mathrm{Lip}_{\varpi} \frac{\Gamma(\beta-\alpha+1)}{\alpha B(1-\alpha,\alpha)} \frac{(a-t)^{\alpha}}{a^{\alpha}} \right], \end{aligned}$$

and this completes the proof. □

4. Examples

Here, we give examples to illustrate Theorem 3.4.

(1) Let $\alpha = \frac{3}{5}$ and $\beta = \frac{2}{5}$ and define the nonlinear Lipschitz continuous function $\varpi : [a, T] \times \mathbb{R} \to \mathbb{R}$ by $\varpi(t, \varphi(t)) = \sin(\varphi(t))$ with Lipschitz constant $\mathrm{Lip}_{\varpi} = 1$. Then the fractional Lane–Emden type equation

$$\begin{cases} {}^{LC}\mathcal{D}^{\frac{3}{5}}_{a^+}\varphi(t) + \frac{\lambda}{\sqrt[5]{t}}\, {}^{LC}\mathcal{D}^{\frac{2}{5}}_{a^+}\sin(\varphi(t)) = 0, & 0 < a < t \leq T, \\ \varphi(a) = \nu, \end{cases}$$

has a unique solution whenever

$$c_1 := \frac{1}{\Gamma(\frac{1}{5})}\left[\pi \csc\left(\frac{\pi}{5}\right) + B\left(\frac{4}{5}, \frac{1}{5}\right)\right] + \frac{\Gamma(\frac{4}{5})}{B(\frac{2}{5}, \frac{3}{5})}\left[\pi \csc\left(\frac{3\pi}{5}\right) + B\left(\frac{2}{5}, \frac{3}{5}\right)\right] = 4.65692 < \frac{1}{\lambda},$$

or for all λ such that $0 < \lambda < 0.214734$.

(2) Suppose $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$ and define $\varpi : [a, T] \times \mathbb{R} \to [0, \infty)$ by $\varpi(t, \varphi(t)) = |\varphi(t)|$ with $\mathrm{Lip}_{\varpi} = 1$. Then the fractional Lane–Emden type equation

$$\begin{cases} {}^{LC}\mathcal{D}^{\frac{2}{3}}_{a^+}\varphi(t) + \frac{\lambda}{\sqrt[3]{t}}\, {}^{LC}\mathcal{D}^{\frac{1}{3}}_{a^+}|\varphi(t)| = 0, & 0 \leq a < t \leq T, \\ \varphi(a) = \nu, \end{cases}$$

has a unique solution for

$$c_1 := \frac{1}{\Gamma(\frac{1}{3})}\left[\pi \csc\left(\frac{\pi}{3}\right) + B\left(\frac{2}{3}, \frac{1}{3}\right)\right] + \frac{\Gamma(\frac{2}{3})}{B(\frac{1}{3}, \frac{2}{3})}\left[\pi \csc\left(\frac{2\pi}{3}\right) + B\left(\frac{1}{3}, \frac{2}{3}\right)\right] = 5.41648 < \frac{1}{\lambda},$$

or for all λ in $0 < \lambda < 0.184622$.

5. Conclusions

A new analytical technique of solution to a nonlinear singular fractional Lane–Emden type differential equation which involves the use of fractional product rule and fractional integration by parts formula applying the fractional integral operator was considered. Our proposed analytical method is easier and straightforward to apply when compared to other analytical methods of solutions. Furthermore, we study the estimation of the upper growth bound (exponential growth in time) of the solution using retarded Gronwall-type inequality, and the existence and uniqueness of solution to the nonlinear fractional Lane–Emden type differential equation using Banach's fixed point theorem. For further studies, one can investigate the asymptotic behaviour of the solution, estimate the lower growth bound of the solution, the continuous dependence on the initial condition and the stability of the solution. Moreso, one can seek to extend this method for the singular IVPs relating to second order differential equation, that is, for $0 < \alpha \leq 2$.

Acknowledgements

The author gratefully acknowledges technical and financial support from the Agency for Research and Innovation, Ministry of Education and University of Hafr Al Batin, Saudi Arabia. The author also acknowledges the reviewers for their comments and suggestions.

This research is funded by the University of Hafr Al Batin, Institutional Financial Program under project number IFP-A-2022-2-1-09.

Conflict of interest

The author declares no conflicts of interest.

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