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Research article

New results on a coupled system for second-order pantograph equations with $\mathcal{ABC}$ fractional derivatives

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Abstract: The aim of this paper is to demonstrate a coupled system of second-order fractional pantograph differential equations with coupled four-point boundary conditions. The proposed system involves Atangana-Baleanu-Caputo ($\mathcal{ABC}$) fractional order derivatives. We prove the solution formula for the corresponding linear version of the given system and then convert the system to a fixed point system. The existence and uniqueness results are obtained by making use of nonlinear alternatives of Leray-Schauder fixed point theorem, and Banach's contraction mapping. In addition, the guarantee of solutions for the system at hand is shown by the stability of Ulam-Hyers. Pertinent examples are provided to illustrate the theoretical results.

Keywords: pantograph system; $\mathcal{ABC}$ fractional derivative; existence; fixed point theorem
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1. Introduction

Fractional differential equations (FDEs) emerge in numerous scientific and engineering disciplines. The theory of FDEs with different kinds of boundary conditions (BCs) has been the topic of attention in the applied and pure sciences. Nevertheless, in the case of the classical two-point BCs, much consideration is paid to nonlocal multipoint BCs. Nonlocal BCs are utilized to describe specific advantages of chemical, physical, or other processes happening in the inward places of a given district. For more data and clarification, see, for instance [1–3]. One of the most popular and well-studied useful methods for generalizing equations and regular models uses fractional differential operators (FDOs). The FDOs include singular and nonsingular, local and nonlocal kernels type operators. These aspects were as of late featured in a few valuable articles. For subtleties, we refer the readers to [4–9].

Recently, FDOs have been studied in science and engineering for modeling system dynamics. In this regard, singular and nonsingular kernels are currently well deliberated in literature. It is challenging to say which one is the best at this moment, however, researchers and scientists generally look at sundry operators for new applications and advantages. For subtleties, we allude to the readers to [10–12].

Very recently, several researchers have become interested in the important progress of the theory of FDEs with various fractional derivatives (FDs). Specifically, examining the existence of unique solutions for FDEs and their stability are of big significance in more better comprehending the behavior of real phenomena. For instance, Ahmad and colleagues in [13, 14], proved the existence and uniqueness of solutions for the coupled system of nonlinear FDEs with BCs (and integral BCs) involving standard Riemann-Liouville (and Caputo) FD, respectively. The existence of the solution for a coupled system of FDEs with integral BCs involving standard Riemann-Liouville FDs has been studied by Mahmudov et al. [15]. Abdo et al. [16], established the existence and Ulam stability results of a coupled system for FDEs with a terminal BC involving a ψ-Hilfer FD. In [17], the author obtained the existence and uniqueness results of the coupled system of FDEs involving a ψ-Caputo FD. Zhou et al. [18, 19] discussed the existence and stability of some fractional problems involving ψ-Hilfer FD. Some recent results that dealt with the $\mathcal{ABC}$ problems can found in [20–26].

In this regard, the following nonlinear $\mathcal{ABC}$ type FD was studied (see [27])

$$^{\mathcal{ABC}}\mathbb{D}^{\alpha}_{0^+}\varphi(\varsigma) = f(\varsigma, \varphi(\varsigma)),\ \varphi(0) = \varphi_0,\ \ \varsigma \in [0, T].$$

Abdo et al. [28], studied the following nonlinear $\mathcal{ABC}$-implicit FDE

$$^{\mathcal{ABC}}\mathbb{D}^{\alpha}_{a^+}\varphi(\varsigma) = f(\varsigma, \varphi(\varsigma), {}^{\mathcal{ABC}}\mathbb{D}^{\alpha}_{a^+}\varphi(\varsigma)), \quad \varsigma \in [a, T],$$

with nonlinear integral conditions described by

$$\begin{cases} \varphi(a) - \varphi'(a) = \int_a^T g(s, \varphi(s))ds, & 0 < \alpha \leq 1, \\ \varphi(a) = 0,\ \ \varphi(T) = \int_a^T g(s, \varphi(s))ds, & 1 < \alpha \leq 2. \end{cases}$$

Asma and her coauthors [29], considered the following nonlinear $\mathcal{ABC}$-implicit FDE

$$\begin{cases} ^{\mathcal{ABC}}\mathbb{D}^{\alpha}_{a^+}\varphi(\varsigma) = f(\varsigma, \varphi(\varsigma), {}^{\mathcal{ABC}}\mathbb{D}^{\alpha}_{a^+}\varphi(\varsigma)), \quad \varsigma \in [0, b], \\ \varphi(0) = \varphi_0,\ \ \varphi(b) = \varphi_1. \end{cases}$$

Cui and Zou [30], discussed the Caputo-coupled four-point FDEs of the form:

$$\begin{cases} {}^{C}\mathbb{D}_{a^+}^{\alpha_1}\varphi_1(\varsigma) = f_1(\varsigma,\varphi_1(\varsigma),\varphi_2(\varsigma)),\ 0<\varsigma<1,\\ {}^{C}\mathbb{D}_{a^+}^{\alpha_2}\varphi_2(\varsigma) = f_2(\varsigma,\varphi_1(\varsigma),\varphi_2(\varsigma)),\ 0<\varsigma<1,\\ \varphi_1(0)=\varphi_2(0)=0,\\ \varphi_1(1)=a\varphi_2(\zeta),\ \ \varphi_2(1)=b\varphi_1(\eta), \end{cases} \tag{1.1}$$

where $1<\alpha_1,\alpha_2\le 2$, $0<\zeta,\eta<1$, $a,b>0$ with $ab<1$, and $f,g:(0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous.

One more significant category, known as pantograph differential equations (PDEs), has not been sufficiently investigated in the frame of FDs, especially for the derivatives of $\mathcal{ABC}$-type. A pantograph is essentially a gadget utilized for scaling and drawing. Nonetheless, nowadays, this gadget is utilized in electric trains [31–33]. PDEs comprise a subclass of deferral type differential equations which give changes in the dependent worth at a past time [34].

In this work, we consider the following coupled system involving $\mathcal{ABC}$-type nonlinear fractional pantograph equations:

$$\begin{cases} {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_1}\varphi_1(\varsigma) = f(\varsigma,\varphi_1(\mu\varsigma),\varphi_2(\varsigma)),\quad \varsigma\in\mathbb{J}:=[a,b],\\ {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2}\varphi_2(\varsigma) = g(\varsigma,\varphi_1(\varsigma),\varphi_2(\mu\varsigma)),\quad \varsigma\in\mathbb{J}:=[a,b],\\ \varphi_1(\varsigma)|_{\varsigma=a}=0,\quad \varphi_2(\varsigma)|_{\varsigma=a}=0,\\ \varphi_1(\varsigma)|_{\varsigma=b}=\lambda_1\,\varphi_2(\varsigma)|_{\varsigma=\zeta},\quad \varphi_2(\varsigma)|_{\varsigma=b}=\lambda_2\,\varphi_1(\varsigma)|_{\varsigma=\eta}, \end{cases} \tag{1.2}$$

where $1<\rho_1,\rho_2\le 2$, $0<\zeta,\eta<1$, $\lambda_1,\lambda_2>0$ with $\lambda_1\lambda_2<1$, $0<\mu<1$, ${}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho}$ represents the $\mathcal{ABC}$ type FD of order $\rho\in\{\rho_1,\rho_2\}$, and $f,g:\mathbb{J}\times\mathcal{C}\times\mathcal{C}\to\mathbb{R}$ are continuous on a Banach space $\mathcal{C}:=\mathcal{C}(\mathbb{J},\mathbb{R})$.

The literature shows that so few works are available on coupled systems of pantograph FDEs in the form of $\mathcal{ABC}$ derivatives; until now, no literature has been made available on the qualitative analysis of the $\mathcal{ABC}$ system (1.2) along with coupled four-point BCs. Therefore, this is the main motivation behind our work. We are also focusing on the topic of the new operator, it is one of the modern operators with helpful applications, in particular in biological systems [35–37]. So in order to enrich the literature, we will present new results on the existence, uniqueness, and Ulam-Hyers stability (UHS) of solutions to the $\mathcal{ABC}$ system (1.2), by using Banach's and Leray-Schauder's fixed point techniques, along with the Arzelá-Ascoli theorem.

The novelty of this work is that $\mathcal{ABC}$-fractional system involves two various FDs ${}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_1}$ and ${}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2}$ and that the nonlinear terms f,g in the system (1.2) involve unknown functions $\varphi_1(\varsigma)$ and $\varphi_2(\varsigma)$. Moreover, the results of the current paper unify several classes of FDEs. For instance, if we set $\mu=1$, and then we replace ${}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_1}, {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2}$ with ${}^{C}\mathbb{D}_{a^+}^{\rho_1}, {}^{C}\mathbb{D}_{a^+}^{\rho_2}$, then, our system reduces to the Caputo system (1.1), see [30]. The paper is organized as follows. Section 2 gives some fundamental results and basic definitions about $\mathcal{ABC}$ fractional calculus. Our main outcomes on problem (1.2) are addressed in Section 3. Some examples to justify the obtained results are constructed in Section 4.

2. Axiomatic results

In this section, we are rendering some results of $\mathcal{ABC}$ fractional calculus. Let us consider

$$\mathcal{C}=\left\{\varkappa:\mathbb{J}\to\mathbb{R};\ \ \|\varkappa\|_\infty=\max_{\varsigma\in\mathbb{J}}|\varkappa(\varsigma)|\right\}.$$

Obviously $(C, \|\varkappa\|_\infty)$ is Banach space. Hence the products $(C \times C, \|(\varkappa_1, \varkappa_2)\|_\infty)$ is also Banach spaces, where

$$\|(\varkappa_1, \varkappa_2)\|_\infty = \|\varkappa_1\|_\infty + \|\varkappa_2\|_\infty .$$

Definition 2.1. [2] Let $\rho > 0$. Then the Riemann–Liouville integral of order ρ of an integrable function ψ is defined by

$$^{\mathcal{RL}}\mathbb{I}^{\rho}_{\mathrm{a}^+}\, \psi(\varsigma) = \frac{1}{\Gamma(\rho)} \int_a^{\varsigma} (\varsigma - \xi))^{\rho-1} \psi(\xi) d\xi. \tag{2.1}$$

Definition 2.2. [10] Let $0 < \rho < 1$, and $\psi \in \mathrm{H}^1(a, b)$, $a < b$. The $\mathcal{ABC}$ type FD of order ρ for a function ψ is defined by

$$^{\mathcal{ABC}}\mathbb{D}^{\rho}_{\mathrm{a}^+}\psi(\varsigma) = \frac{\aleph(\rho)}{1-\rho} \int_a^{\varsigma} \psi(\xi) \mathrm{E}_{\rho}\left(\frac{-\rho(\varsigma - \xi)^{\rho}}{1-\rho}\right) d\xi.$$

Further, the Atangana-Baleanu FD in Riemann–Liouville sense is defined by

$$^{\mathcal{ABR}}\mathbb{D}^{\rho}_{\mathrm{a}^+}\psi(\varsigma) = \frac{\aleph(\rho)}{1-\rho} \frac{d}{d\varsigma} \int_a^{\varsigma} \psi(\xi) \mathrm{E}_{\rho}\left(\frac{-\rho(\varsigma - \xi)^{\rho}}{1-\rho}\right) d\xi. \tag{2.2}$$

Here, $\aleph(\rho) > 0$ satisfies $\aleph(0) = \aleph(1) = 1$ and E_{ρ} represents the Mittag-Leffler function.

Definition 2.3. [2] The Mittag-Leffler function E_{ρ} is defined by

$$\mathrm{E}_{\rho}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \quad (z \in C; Re(\rho) > 0). \tag{2.3}$$

Definition 2.4. [10] Let $0 < \rho < 1$ and ψ be function, then the Atangana-Baleanu type integral of order $\rho > 0$ is defined by

$$\begin{aligned} ^{\mathcal{AB}}\mathbb{I}^{\rho}_{\mathrm{a}^+}\psi(\varsigma) &= \frac{1-\rho}{\aleph(\rho)}\psi(\varsigma) + \frac{\rho}{\aleph(\rho)}\, ^{\mathcal{RL}}\mathbb{I}^{\rho}_{\mathrm{a}^+}\, \psi(\varsigma) \\ &= \frac{1-\rho}{\aleph(\rho)}\psi(\varsigma) + \frac{\rho}{\aleph(\rho)\Gamma(\rho)} \int_a^{\varsigma} \psi(\xi)(\varsigma - \xi)^{\rho-1} d\xi, \end{aligned} \tag{2.4}$$

where $^{\mathcal{RL}}\mathbb{I}^{\rho}_{\mathrm{a}^+}$ is defined by (2.1).

Lemma 2.5. *[38, 39] Let $0 < \rho \leq 1$ and $\psi \in \mathrm{H}^1(a, b)$. If $^{\mathcal{ABC}}\mathbb{D}^{\rho}_{\mathrm{a}^+}$ exists, then*

$$^{\mathcal{AB}}\mathbb{I}^{\rho}_{\mathrm{a}^+}\, ^{\mathcal{ABC}}\mathbb{D}^{\rho}_{\mathrm{a}^+}\, \psi(\varsigma) = \psi(\varsigma) - \psi(a).$$

Definition 2.6. [40] Let $n < \rho \leq n + 1$, and ψ be a function with $\psi^{(n)} \in \mathrm{H}^1(a, b)$. Then $\mathcal{ABC}$ type FD satisfies $^{\mathcal{ABC}}\mathbb{D}^{\rho}_{\mathrm{a}^+}\psi(\varsigma) = {}^{\mathcal{ABC}}\mathbb{D}^{\delta}_{\mathrm{a}^+}\, \psi^{(n)}(\varsigma)$, where $\delta = \rho - n$, and $n \in \mathbb{N}_0$.

Lemma 2.7. *[40] Let $\psi(\varsigma)$ be a function defined on (a, b) and $n < \rho \leq n + 1$, we have*

$$\left(^{\mathcal{AB}}\mathbb{I}^{\rho}_{\mathrm{a}^+}\, ^{\mathcal{ABC}}\mathbb{D}^{\rho}_{\mathrm{a}^+}\, \psi\right)(\varsigma) = \psi(\varsigma) - \sum_{j=0}^{n} \frac{\psi^{(j)}(a)}{j!} (\varsigma - a)^j ,$$

for some $n \in \mathbb{N}_0$, where $\psi^{(j)}(\varsigma) = \left(\frac{d}{d\varsigma}\right)^j \psi(\varsigma)$.

Lemma 2.8. *[40] Let* $1 < \rho \leq 2$ *and* $\psi(0) = 0$. *Then the solution of the next linear problem*

$$\begin{cases} {}^{\mathcal{ABC}}\mathbb{D}_{0^+}^{\rho}\, \varphi(\varsigma) = \psi(\varsigma), \\ \varphi(0) = c_1,\ \varphi'(0) = c_2, \end{cases}$$

is given by

$$\varphi(\varsigma) = c_1 + c_2\varsigma + \frac{2-\rho}{\aleph(\rho-1)}\int_0^{\varsigma}\psi(\xi)d\xi + \frac{\rho-1}{\aleph(\rho-1)}\int_0^{\varsigma}\frac{(\varsigma-\xi)^{\rho-1}}{\Gamma(\rho)}\psi(\xi)d\xi.$$

3. Main results

To obtain our desired results, we first outline the next suppositions and theorem:

(Hy$_1$) $f, g : \mathbb{J}\times C\times C \to \mathbb{R}$ are continuous and there exist $\kappa_f, \kappa_g, \overline{\kappa}_f, \overline{\kappa}_g > 0$ such that

$$\left|f(\varsigma,\varphi_1,\varphi_2) - f(\varsigma,\varphi_1^*,\varphi_2^*)\right| \leq \kappa_f\left|\varphi_1-\varphi_1^*\right| + \overline{\kappa}_f\left|\varphi_2-\varphi_2^*\right|,$$

$$\left|g(\varsigma,\varphi_1,\varphi_2) - g(\varsigma,\varphi_1^*,\varphi_2^*)\right| \leq \kappa_g\left|\varphi_1-\varphi_1^*\right| + \overline{\kappa}_g\left|\varphi_2-\varphi_2^*\right|,$$

for each $(\varsigma,\varphi_1,\varphi_2), \left(\varsigma,\varphi_1^*,\varphi_2^*\right) \in \mathbb{J}\times C\times C$.

(Hy$_2$) There exist two functions $\gamma_f, \gamma_g \in C$ and two nondecreasing functions $\psi_f, \psi_g : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $(\varsigma,\varphi_1,\varphi_2) \in \mathbb{J}\times\mathbb{R}\times\mathbb{R}$,

$$|f(\varsigma,\varphi_1,\varphi_2)| \leq \gamma_f(\varsigma)\psi_f(|\varphi_1| + |\varphi_2|),$$

$$|g(\varsigma,\varphi_1,\varphi_2)| \leq \gamma_g(\varsigma)\psi_g(|\varphi_1| + |\varphi_2|).$$

(Hy$_3$) There exists $M > 0$ such that

$$\frac{M}{\left(\Lambda_\eta b + \Lambda_b\right)\left\|\gamma_f\right\|\psi_f(M) + \Lambda_\zeta b\left\|\gamma_g\right\|\psi_g(M) + \left(\Delta_\zeta b + \Delta_b\right)\left\|\gamma_g\right\|\psi_g(M) + \Delta_\eta b\left\|\gamma_f\right\|\psi_f(M)} > 1,$$

where

$$\begin{aligned}
\Lambda_\eta &= \frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}(\eta-a) + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(\eta-a)^{\rho_1}\right)\\
\Lambda_\zeta &= \frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}(\zeta-a) + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(\zeta-a)^{\rho_2}\right)\\
\Lambda_b &= \frac{2-\rho_1}{\aleph(\rho_1-1)}(b-a) + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(b-a)^{\rho_1},
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\zeta &= \frac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}(\zeta-a) + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(\zeta-a)^{\rho_2}\right)\\
\Delta_\eta &= \frac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}(\eta-a) + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(\eta-a)^{\rho_1}\right)\\
\Delta_b &= \frac{2-\rho_2}{\aleph(\rho_2-1)}(b-a) + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(b-a)^{\rho_2}.
\end{aligned}$$

Theorem 3.1. *Let* $1 < \rho_1, \rho_2 \leq 2$ *and* $h_1, h_2 \in C$. *If* $(\varphi_1, \varphi_2) \in C \times C$ *satisfies*

$$\begin{cases} {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_1}\varphi_1(\varsigma) = h_1(\varsigma), \quad \varsigma \in \mathbb{J}, \\ {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2}\varphi_2(\varsigma) = h_2(\varsigma), \quad \varsigma \in \mathbb{J}, \\ \varphi_1(\varsigma)|_{\varsigma=a} = 0, \quad \varphi_2(\varsigma)|_{\varsigma=a} = 0, \\ \quad \varphi_1(\varsigma)|_{\varsigma=b} = \lambda_1\, \varphi_2(\varsigma)|_{\varsigma=\zeta}, \quad \varphi_2(\varsigma)|_{\varsigma=b} = \lambda_2\, \varphi_1(\varsigma)|_{\varsigma=\eta}, \end{cases} \tag{3.1}$$

then

$$\begin{aligned} \varphi_1(\varsigma) = {} & \tfrac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta h_1(\vartheta)d\vartheta + \tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta\right) \\ & + \tfrac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta h_2(\vartheta)d\vartheta + \tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta\right) \\ & + \tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\varsigma h_1(\vartheta)d\vartheta + \tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \varphi_2(\varsigma) = {} & \tfrac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta h_2(\vartheta)d\vartheta + \tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta\right) \\ & + \tfrac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta h_1(\vartheta)d\vartheta + \tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta\right) \\ & + \tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\varsigma h_2(\vartheta)d\vartheta + \tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta, \end{aligned} \tag{3.3}$$

where $\lambda_1\lambda_2\eta\zeta \neq 1$.

Proof. Applying the integrals ${}^{AB}\mathbb{I}_{a^+}^{\rho_1}$ and ${}^{AB}\mathbb{I}_{a^+}^{\rho_2}$ to the first two equations of (3.1) and then using Lemmas 2.7 and 2.8, we have

$$\varphi_1(\varsigma) = \varphi_1(b)\varsigma + \frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\varsigma h_1(\vartheta)d\vartheta + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta. \tag{3.4}$$

$$\varphi_2(\varsigma) = \varphi_2(b)\varsigma + \frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\varsigma h_2(\vartheta)d\vartheta + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta. \tag{3.5}$$

By the coupled BCs of problem (3.1), and using above equations, we have

$$\begin{aligned} \varphi_2(b) &= \lambda_2\varphi_1(\eta) = \lambda_2\varphi_1(b)\eta + \frac{2-\rho_1}{\aleph(\rho_1-1)}\lambda_2\int_a^\eta h_1(\vartheta)d\vartheta \\ &\quad + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\lambda_2\int_a^\eta(\eta-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \varphi_1(b) &= \lambda_1\varphi_2(\zeta) = \lambda_1\varphi_2(b)\zeta + \frac{2-\rho_2}{\aleph(\rho_2-1)}\lambda_1\int_a^\zeta h_2(\vartheta)d\vartheta \\ &\quad + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\lambda_1\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta, \end{aligned} \tag{3.7}$$

After simple calculations, we obtain

$$\begin{aligned} \varphi_2(b) &= \frac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta h_2(\vartheta)d\vartheta + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta\right) \\ &\quad + \frac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta h_1(\vartheta)d\vartheta + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta\right), \end{aligned} \tag{3.8}$$

$$\varphi_1(b) = \frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta h_1(\vartheta)d\vartheta + \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta (\eta-\vartheta)^{\rho_1-1}h_1(\vartheta)d\vartheta\right)$$
$$+\frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta}\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta h_2(\vartheta)d\vartheta + \frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta (\zeta-\vartheta)^{\rho_2-1}h_2(\vartheta)d\vartheta\right). \quad (3.9)$$

Substituting (3.9) into (3.4) and (3.8) into (3.5), respectively, we obtain (3.2) and (3.3). □

By Lemma 3.1, we give the fixed point problem equivalent to the considered problem as:

$$(\varphi_1,\varphi_2) = (\Pi_1,\Pi_2)(\varphi_1,\varphi_2), \quad (\varphi_1,\varphi_2)\in C\times C,$$

where $(\Pi_1,\Pi_2)(\varphi_1,\varphi_2) = \Pi(\varphi_1,\varphi_2) : C\times C \to C\times C$ is defined by

$$\begin{aligned}\Pi_1(\varphi_1,\varphi_2)(\varsigma) = {} & \tfrac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\Big(\tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta \\ & +\tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta\Big) \\ & + \tfrac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\Big(\tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta \\ & +\tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta\Big) \\ & + \tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\varsigma f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta \\ & + \tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_1-1}f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta.\end{aligned} \quad (3.10)$$

$$\begin{aligned}\Pi_2(\varphi_1,\varphi_2)(\varsigma) = {} & \tfrac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\Big(\tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta \\ & +\tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta\Big) \\ & + \tfrac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\Big(\tfrac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta \\ & +\tfrac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))d\vartheta\Big) \\ & + \tfrac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\varsigma g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta \\ & + \tfrac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_2-1}g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))d\vartheta.\end{aligned} \quad (3.11)$$

Distinctly, the existence of fixed points of Π is equivalent to the existence of solutions for the system (1.2).

For the first result, we apply the Leray-Schauder's theorem to prove the existence of solutions for the system (1.2).

3.1. Existence and uniqueness analysis

Theorem 3.2. *(Existence result) Let $f, g : \mathbb{J}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous. Assume that (Hy_2) and (Hy_3) are satisfied. Then, the system (1.2) has at least one solution.*

Proof. Consider the following closed ball

$$\mathcal{S}_\beta = \left\{(\varphi_1,\varphi_2)\in C\times C : \|(\varphi_1,\varphi_2)\| \le \beta,\ \|\varphi_1\| \le \frac{\beta}{2}, \|\varphi_2\| \le \frac{\beta}{2}\right\}.$$

Step1: Here, we show that the mapping $\Pi : C\times C \to C\times C$ is bounded and continuous.

Let $(\varphi_1, \varphi_2) \in \mathcal{S}_\beta$. Then we have

$$\|f(\cdot, \varphi_1(\cdot), \varphi_2(\cdot))\| \leq \left\|\gamma_f\right\| \psi_f\left(\|\varphi_1\| + \|\varphi_2\|\right) \leq \left\|\gamma_f\right\| \psi_f(\beta),$$

$$\|g(\cdot, \varphi_1(\cdot), \varphi_2(\cdot))\| \leq \left\|\gamma_g\right\| \psi_g\left(\|\varphi_1\| + \|\varphi_2\|\right) \leq \left\|\gamma_g\right\| \psi_g(\beta),$$

and

$$\begin{aligned}
|\Pi_1(\varphi_1, \varphi_2)(\varsigma)| \leq\ & \frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_1}{\aleph(\rho_1-1)} \int_a^\eta \left\|\gamma_f\right\| \psi_f(\beta)\, d\vartheta\right. \\
&+ \left.\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)} \int_a^\eta (\eta-\vartheta)^{\rho_1-1} \left\|\gamma_f\right\| \psi_f(\beta)\, d\vartheta\right) \\
&+ \frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_2}{\aleph(\rho_2-1)} \int_a^\zeta \left\|\gamma_g\right\| \psi_g(\beta)\, d\vartheta\right. \\
&+ \left.\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)} \int_a^\zeta (\zeta-\vartheta)^{\rho_2-1} \left\|\gamma_g\right\| \psi_g(\beta)\, d\vartheta\right) \\
&+ \frac{2-\rho_1}{\aleph(\rho_1-1)} \int_a^\varsigma \left\|\gamma_f\right\| \psi_f(\beta)\, d\vartheta \\
&+ \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)} \int_a^\varsigma (\varsigma-\vartheta)^{\rho_1-1} \left\|\gamma_f\right\| \psi_f(\beta)\, d\vartheta,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\Pi_1(\varphi_1, \varphi_2)\| \leq\ & \frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}(\eta-a) \left\|\gamma_f\right\| \psi_f(\beta)\right. \\
&+ \left.\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(\eta-a)^{\rho_1} \left\|\gamma_f\right\| \psi_f(\beta)\right) \\
&+ \frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}(\zeta-a) \left\|\gamma_g\right\| \psi_g(\beta)\right. \\
&+ \left.\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(\zeta-a)^{\rho_2} \left\|\gamma_g\right\| \psi_g(\beta)\right) \\
&+ \frac{2-\rho_1}{\aleph(\rho_1-1)}(b-a) \left\|\gamma_f\right\| \psi_f(\beta) \\
&+ \frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(b-a)^{\rho_1} \left\|\gamma_f\right\| \psi_f(\beta) \\
=\ & \left(\Lambda_\eta b + \Lambda_b\right) \left\|\gamma_f\right\| \psi_f(\beta) + \Lambda_\zeta b \left\|\gamma_g\right\| \psi_g(\beta).
\end{aligned}$$

Similarly

$$\|\Pi_2(\varphi_1, \varphi_2)(\varsigma)\| \leq \left(\Delta_\zeta b + \Delta_b\right) \left\|\gamma_g\right\| \psi_g(\beta) + \Delta_\eta b \left\|\gamma_f\right\| \psi_f(\beta).$$

It follows that $\Pi(\mathcal{S}_\beta)$ is bounded.

Step 2: Π maps bounded sets to equicontinuous sets of C.

Let $\varsigma_1, \varsigma_2 \in \mathbb{J}$ with $\varsigma_1 < \varsigma_2$ and for any $(\varphi_1, \varphi_2) \in \mathcal{S}_\beta$, we get

$$\begin{aligned}
&|\Pi_1(\varphi_1,\varphi_2)(\varsigma_2)-\Pi_1(\varphi_1,\varphi_2)(\varsigma_1)| \\
\leq\ & \Lambda_\eta(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta)+\Lambda_\zeta(\varsigma_2-\varsigma_1)\left\|\gamma_g\right\|\psi_g(\beta) \\
&+\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_{\varsigma_1}^{\varsigma_2}|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))|\,d\vartheta \\
&+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^{\varsigma_1}\left((\varsigma_2-\vartheta)^{\rho_1-1}-(\varsigma_1-\vartheta)^{\rho_1-1}\right)|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))|\,d\vartheta. \\
&+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_{\varsigma_1}^{\varsigma_2}(\varsigma_2-\vartheta)^{\rho_1-1}|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))|\,d\vartheta. \\
&\leq\Lambda_\eta(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta)+\Lambda_\zeta(\varsigma_2-\varsigma_1)\left\|\gamma_g\right\|\psi_g(\beta) \\
&+\frac{2-\rho_1}{\aleph(\rho_1-1)}(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta) \\
&+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}\left[\ (\varsigma_1-\varsigma_2)^{\rho_1}+(\varsigma_2-a)^{\rho_1}\right. \\
&\left.-(\varsigma_1-a)^{\rho_1}\right]\left\|\gamma_g\right\|\psi_g(\beta) \\
&\leq\Lambda_\eta(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta)+\Lambda_\zeta(\varsigma_2-\varsigma_1)\left\|\gamma_g\right\|\psi_g(\beta) \\
&+\frac{2-\rho_1}{\aleph(\rho_1-1)}(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta) \\
&+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}\left[(\varsigma_2-a)^{\rho_1}-(\varsigma_1-a)^{\rho_1}\right]\left\|\gamma_f\right\|\psi_f(\beta).
\end{aligned} \tag{3.12}$$

Similarly,

$$\begin{aligned}
&|\Pi_2(\varphi_1,\varphi_2)(\varsigma_2)-\Pi_2(\varphi_1,\varphi_2)(\varsigma_1)| \\
&\leq\Delta_\zeta(\varsigma_2-\varsigma_1)\left\|\gamma_g\right\|\psi_g(\beta)+\Delta_\eta(\varsigma_2-\varsigma_1)\left\|\gamma_f\right\|\psi_f(\beta) \\
&+\frac{2-\rho_2}{\aleph(\rho_2-1)}(\varsigma_2-\varsigma_1)\left\|\gamma_g\right\|\psi_g(\beta) \\
&+\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}\left[(\varsigma_2-a)^{\rho_2}-(\varsigma_1-a)^{\rho_2}\right]\left\|\gamma_g\right\|\psi_g(\beta).
\end{aligned} \tag{3.13}$$

Evidently, the right hand sides of (3.12) and (3.13) tend to zero independently of $(\varphi_1,\varphi_2)\in\mathcal{S}_\beta$ as $\varsigma_1\to\varsigma_2$. Thus, Π is equicontinuous. It follows from the Arzelá-Ascoli theorem that $\Pi: C\times C\to C\times C$ is completely continuous.

Step 3: We show that the set of all solutions to the equation $(\varphi_1,\varphi_2)=\lambda\Pi(\varphi_1,\varphi_2)$ is bounded, where $\lambda\in(0,1)$.

Let $(\varphi_1,\varphi_2)\in C\times C$ and $(\varphi_1,\varphi_2)=\lambda\Pi(\varphi_1,\varphi_2)$ for some $\lambda\in(0,1)$. Then, using calculations mentioned in Step 1, we obtain.

$$\begin{aligned}
|(\varphi_1,\varphi_2)(\varsigma)| &= \lambda|\Pi(\varphi_1,\varphi_2)(\varsigma)|\leq\lambda|\Pi_1(\varphi_1,\varphi_2)(\varsigma)|+\lambda|\Pi_2(\varphi_1,\varphi_2)(\varsigma)| \\
&\leq \left(\Lambda_\eta b+\Lambda_b\right)\left\|\gamma_f\right\|\psi_f(\|(\varphi_1,\varphi_2)\|)+\Lambda_\zeta b\left\|\gamma_g\right\|\psi_g(\|(\varphi_1,\varphi_2)\|) \\
&\quad+\left(\Delta_\zeta b+\Delta_b\right)\left\|\gamma_g\right\|\psi_g(\|(\varphi_1,\varphi_2)\|)+\Delta_\eta b\left\|\gamma_f\right\|\psi_f(\|(\varphi_1,\varphi_2)\|).
\end{aligned}$$

Consequently, we have

$$\frac{\|(\varphi_1,\varphi_2)\|}{\mathcal{A}_1\psi_f(\|(\varphi_1,\varphi_2)\|)+\mathcal{A}_2\psi_g(\|(\varphi_1,\varphi_2)\|)+\mathcal{A}_3\psi_g(\|(\varphi_1,\varphi_2)\|)+\mathcal{A}_4\psi_f(\|(\varphi_1,\varphi_2)\|)}\le 1,$$

where

$$\mathcal{A}_1:=\left(\Lambda_\eta b+\Lambda_b\right)\left\|\gamma_f\right\|,\ \ \mathcal{A}_2:=\Lambda_\zeta b\left\|\gamma_g\right\|,\ \ \mathcal{A}_3:=\left(\Delta_\zeta b+\Delta_b\right)\left\|\gamma_g\right\|$$

and

$$\mathcal{A}_4:=\Delta_\eta b\left\|\gamma_f\right\|.$$

By virtue of (Hy_2), there exists M such that $\|(\varphi_1,\varphi_2)\|\neq M$.

Set $U=\{(\varphi_1,\varphi_2)\in C\times C\colon \|(\varphi_1,\varphi_2)\|<M\}$. Observe that Π: $\overline{U}\to C$ is continuous and completely continuous. From the choice of U, there is no $(\varphi_1,\varphi_2)\in\partial U$ such that $(\varphi_1,\varphi_2)=\lambda\Pi(\varphi_1,\varphi_2)$ for some $\lambda\in(0,1)$. It follows from the Leray-Schauder fixed point theorem that Π has a fixed point $(\varphi_1,\varphi_2)\in U$ which is a solution of system (1.2). □

Theorem 3.3. *(Uniqueness result) Suppose that* (Hy_1) *holds. Then the system (1.2) has a unique solution on* $\mathbb{J}$*, provided that* $\max\{\sigma_1,\sigma_2\}=\sigma<1$, *where*

$$\begin{aligned}\sigma_1 &:= \left[\left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\kappa_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\kappa_g\right],\\ \sigma_2 &:= \left[\left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\overline{\kappa}_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\overline{\kappa}_g\right].\end{aligned}$$

Proof. For proof, it suffices to clear that Π is a contraction. For each $\varsigma\in\mathbb{J}$ and $(\varphi_1,\varphi_2),(\varphi_1^\star,\varphi_2^\star)\in C\times C$, we have

$$\left\|\Pi(\varphi_1,\varphi_2)-\Pi(\varphi_1^\star,\varphi_2^\star)\right\|\le\left\|\Pi_1(\varphi_1,\varphi_2)-\Pi_1(\varphi_1^\star,\varphi_2^\star)\right\|+\left\|\Pi_2(\varphi_2,\varphi_1)-\Pi_2(\varphi_2^\star,\varphi_1^\star)\right\|. \tag{3.14}$$

For Π_1, we have

$$\begin{aligned}&\left\|\Pi_1(\varphi_1,\varphi_2)-\Pi_1(\varphi_1^\star,\varphi_2^\star)\right\|\\ &=\max_{\varsigma\in\mathbb{J}}\left\{\left|\Pi_1(\varphi_1,\varphi_2)(\varsigma)-\Pi_1(\varphi_1^\star,\varphi_2^\star)(\varsigma)\right|\right\}\\ &\le\max_{\varsigma\in\mathbb{J}}\left\{\frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\eta\left|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))-f(\vartheta,\varphi_1^\star(\mu\vartheta),\varphi_2^\star(\vartheta))\right|d\vartheta\right.\right.\\ &+\left.\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\eta(\eta-\vartheta)^{\rho_1-1}\left|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))-f(\vartheta,\varphi_1^\star(\mu\vartheta),\varphi_2^\star(\vartheta))\right|d\vartheta\right)\\ &+\frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^\zeta\left|g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))-g(\vartheta,\varphi_1^\star(\vartheta),\varphi_2^\star(\mu\vartheta))\right|d\vartheta\right.\\ &+\left.\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^\zeta(\zeta-\vartheta)^{\rho_2-1}\left|g(\vartheta,\varphi_1(\vartheta),\varphi_2(\mu\vartheta))-g(\vartheta,\varphi_1^\star(\vartheta),\varphi_2^\star(\mu\vartheta))\right|d\vartheta\right)\\ &+\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^\varsigma\left|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))-f(\vartheta,\varphi_1^\star(\mu\vartheta),\varphi_2^\star(\vartheta))\right|d\vartheta\\ &+\left.\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^\varsigma(\varsigma-\vartheta)^{\rho_1-1}\left|f(\vartheta,\varphi_1(\mu\vartheta),\varphi_2(\vartheta))-f(\vartheta,\varphi_1^\star(\mu\vartheta),\varphi_2^\star(\vartheta))\right|d\vartheta\right\}.\end{aligned}$$

By (Hy_1), we obtain

$$
\begin{aligned}
& \left\|\Pi_1(\varphi_1,\varphi_2)-\Pi_1(\varphi_1^\star,\varphi_2^\star)\right\| \\
& \le \frac{\lambda_1\lambda_2\zeta}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}(\eta-a)\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right)\right. \\
& + \left.\frac{(\rho_1-1)(\eta-a)^{\rho_1}}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right)\right) \\
& + \frac{\lambda_1}{1-\lambda_1\lambda_2\eta\zeta} b\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}(\zeta-a)\left(\kappa_g\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_g\left\|\varphi_2-\varphi_2^\star\right\|\right)\right. \\
& + \left.\frac{(\rho_2-1)(\zeta-a)^{\rho_2}}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}\left(\kappa_g\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_g\left\|\varphi_2-\varphi_2^\star\right\|\right)\right) \\
& + \frac{2-\rho_1}{\aleph(\rho_1-1)}(b-a)\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right) \\
& + \frac{(\rho_1-1)(b-a)^{\rho_1}}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right) \\
= & \left(\Lambda_\eta b+\Lambda_b\right)\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right) \\
& + \Lambda_\zeta b\left(\kappa_g\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_g\left\|\varphi_2-\varphi_2^\star\right\|\right). \qquad (3.15)
\end{aligned}
$$

Similarly, for Π_2, we obtain

$$
\begin{aligned}
\left\|\Pi_2(\varphi_1,\varphi_2)-\Pi_2(\varphi_1^\star,\varphi_2^\star)\right\| \le & \left(\Delta_\zeta b+\Delta_b\right)\left(\kappa_g\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_g\left\|\varphi_2-\varphi_2^\star\right\|\right) \\
& +\Delta_\eta b\left(\kappa_f\left\|\varphi_1-\varphi_1^\star\right\|+\overline{\kappa}_f\left\|\varphi_2-\varphi_2^\star\right\|\right). \qquad (3.16)
\end{aligned}
$$

From (3.14)–(3.16), we get

$$
\begin{aligned}
\left\|\Pi(\varphi_1,\varphi_2)-\Pi(\varphi_1^\star,\varphi_2^\star)\right\| \le & \left[\left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\kappa_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\kappa_g\right]\left\|\varphi_1-\varphi_1^\star\right\| \\
& +\left[\left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\overline{\kappa}_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\overline{\kappa}_g\right]\left\|\varphi_2-\varphi_2^\star\right\| \\
= & \ \sigma_1\left\|\varphi_1-\varphi_1^\star\right\|+\sigma_2\left\|\varphi_2-\varphi_2^\star\right\|,
\end{aligned}
$$

which implies

$$
\left\|\Pi(\varphi_1,\varphi_2)-\Pi(\varphi_1^\star,\varphi_2^\star)\right\| \le \sigma\left\|(\varphi_1,\varphi_2)-(\varphi_1^\star,\varphi_2^\star)\right\|.
$$

Since $\sigma<1$, Π is a contraction mapping. Thus, by view of Banach contraction mapping, system (1.2) has a unique solution on $\mathbb{J}$. □

3.2. UHS analysis

In this portion, we discuss the UHS of the considered system via integral representation of its solution presented by

$$
\varphi_1(\varsigma)=\Pi_1(\varphi_1,\varphi_2)(\varsigma), \quad \varphi_2(\varsigma)=\Pi_2(\varphi_1,\varphi_2)(\varsigma), \qquad (3.17)
$$

where Π_1 and Π_2 are defined by (3.10) and (3.11).

Define the next nonlinear operators $\Phi_1,\Phi_2 : C\times C\to C$ such that

$$
\begin{cases}
\Phi_1(\varphi_1,\varphi_2)(\varsigma) = {}^{ABC}\mathbb{D}_{a^+}^{\rho_1}\varphi_1(\varsigma)-f(\varsigma,\varphi_1(\mu\varsigma),\varphi_2(\varsigma)), & \varsigma\in\mathbb{J}, \\
\Phi_2(\varphi_1,\varphi_2)(\varsigma) = {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2}\varphi_2(\varsigma)-g(\varsigma,\varphi_1(\varsigma),\varphi_2(\mu\varsigma)), & \varsigma\in\mathbb{J}.
\end{cases} \qquad (3.18)
$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we have the following inequalities:

$$|\Phi_1(\varphi_1, \varphi_2)(\varsigma)| \le \varepsilon_1, \quad |\Phi_2(\varphi_1, \varphi_2)(\varsigma)| \le \varepsilon_2. \tag{3.19}$$

Definition 3.4. System (1.2) is said to be Ulam-Hyers (UH)- stable if there exists constants $\Upsilon_1, \Upsilon_2 > 0$, such that for each $\varepsilon_1, \varepsilon_2 > 0$, and every solution $(\widetilde{\varphi_1}, \widetilde{\varphi_2}) \in C \times C$ of the inequalities (3.19), there exists a unique solution $(\varphi_1, \varphi_2) \in C \times C$ of the system (1.2) with

$$\|(\varphi_1, \varphi_2) - (\widetilde{\varphi_1}, \widetilde{\varphi_2})\| \le \Upsilon_1 \varepsilon_1 + \Upsilon_2 \varepsilon_2. \tag{3.20}$$

Theorem 3.5. *Apply the assumptions of the Theorem 3.3 , and let* $\Omega_1 \mho_1 - \Omega_2 \mho_2 \neq 0$. *Then the coupled system (1.2) is UH stable.*

Proof. Let $(\varphi_1, \varphi_2) \in C \times C$ be the unique solution of the coupled system (1.2) satisfying (3.10) and (3.11). Assume that $(\widetilde{\varphi_1}, \widetilde{\varphi_2}) \in C \times C$ be any solution satisfying (3.19) and

$$\begin{cases} {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_1} \widetilde{\varphi_1}(\varsigma) = f(\varsigma, \widetilde{\varphi_1}(\mu\varsigma), \widetilde{\varphi_2}(\varsigma)) + \Phi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\varsigma), & \varsigma \in \mathbb{J}, \\ {}^{\mathcal{ABC}}\mathbb{D}_{a^+}^{\rho_2} \widetilde{\varphi_2}(\varsigma) = g(\varsigma, \widetilde{\varphi_1}(\varsigma), \widetilde{\varphi_2}(\mu\varsigma)) + \Phi_2(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\varsigma), & \varsigma \in \mathbb{J}. \end{cases} \tag{3.21}$$

Thus

$$\begin{aligned} \widetilde{\varphi_1}(\varsigma) &= \Pi_1(\varphi_1, \varphi_2)(\varsigma) + \frac{\lambda_1 \lambda_2 \zeta}{1 - \lambda_1 \lambda_2 \eta \zeta} \varsigma \Bigg(\frac{2 - \rho_1}{\aleph(\rho_1 - 1)} \int_a^\eta \Phi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta \\ &+ \frac{\rho_1 - 1}{\aleph(\rho_1 - 1)\Gamma(\rho_1)} \int_a^\eta (\eta - \vartheta)^{\rho_1 - 1} \Phi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta \Bigg) \\ &+ \frac{\lambda_1}{1 - \lambda_1 \lambda_2 \eta \zeta} \varsigma \Bigg(\frac{2 - \rho_2}{\aleph(\rho_2 - 1)} \int_a^\zeta \Phi_2(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta \\ &+ \frac{\rho_2 - 1}{\aleph(\rho_2 - 1)\Gamma(\rho_2)} \int_a^\zeta (\zeta - \vartheta)^{\rho_2 - 1} \Phi_2(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta \Bigg) \\ &+ \frac{2 - \rho_1}{\aleph(\rho_1 - 1)} \int_a^\varsigma \Phi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta \\ &+ \frac{\rho_1 - 1}{\aleph(\rho_1 - 1)\Gamma(\rho_1)} \int_a^\varsigma (\varsigma - \vartheta)^{\rho_1 - 1} \Phi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\vartheta) d\vartheta. \end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned} &|\widetilde{\varphi_1}(\varsigma) - \Pi_1(\widetilde{\varphi_1}, \widetilde{\varphi_2})(\varsigma)| \\ &\le \frac{\lambda_1 \lambda_2 \zeta}{1 - \lambda_1 \lambda_2 \eta \zeta} \varsigma \left(\frac{2 - \rho_1}{\aleph(\rho_1 - 1)} (\eta - a) \varepsilon_1 + \frac{\rho_1 - 1}{\aleph(\rho_1 - 1)\Gamma(\rho_1 + 1)} (\eta - a)^{\rho_1} \varepsilon_1 \right) \\ &+ \frac{\lambda_1}{1 - \lambda_1 \lambda_2 \eta \zeta} \varsigma \left(\frac{2 - \rho_2}{\aleph(\rho_2 - 1)} (\zeta - a) \varepsilon_2 + \frac{\rho_2 - 1}{\aleph(\rho_2 - 1)\Gamma(\rho_2 + 1)} (\zeta - a)^{\rho_2} \varepsilon_2 \right) \\ &+ \frac{2 - \rho_1}{\aleph(\rho_1 - 1)} (\varsigma - a) \varepsilon_1 + \frac{\rho_1 - 1}{\aleph(\rho_1 - 1)\Gamma(\rho_1 + 1)} (\varsigma - a)^{\rho_1} \varepsilon_1 \\ &\le \left(\Lambda_\eta b + \Lambda_b \right) \varepsilon_1 + \Lambda_\zeta b \varepsilon_2. \end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned}
\widetilde{\varphi_2}(\varsigma) &= \Pi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)+\frac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^{\zeta}\Phi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta\right.\\
&\quad\left.+\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^{\zeta}(\zeta-\vartheta)^{\rho_2-1}\Phi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta\right)\\
&\quad+\frac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}\int_a^{\eta}\Phi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta\right.\\
&\quad\left.+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1)}\int_a^{\eta}(\eta-\vartheta)^{\rho_1-1}\Phi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta\right)\\
&\quad+\frac{2-\rho_2}{\aleph(\rho_2-1)}\int_a^{\varsigma}\Phi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta\\
&\quad+\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2)}\int_a^{\varsigma}(\varsigma-\vartheta)^{\rho_2-1}\Phi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\vartheta)d\vartheta
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
&\left|\widetilde{\varphi_2}(\varsigma)-\Pi_2\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)\right|\\
&\le\frac{\lambda_1\lambda_2\eta}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_2}{\aleph(\rho_2-1)}(\zeta-a)\varepsilon_2+\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(\zeta-a)^{\rho_2}\varepsilon_2\right)\\
&+\frac{\lambda_2}{1-\lambda_1\lambda_2\eta\zeta}\varsigma\left(\frac{2-\rho_1}{\aleph(\rho_1-1)}(\eta-a)\varepsilon_1+\frac{\rho_1-1}{\aleph(\rho_1-1)\Gamma(\rho_1+1)}(\eta-a)^{\rho_1}\varepsilon_1\right)\\
&+\frac{2-\rho_2}{\aleph(\rho_2-1)}(\varsigma-a)\varepsilon_2+\frac{\rho_2-1}{\aleph(\rho_2-1)\Gamma(\rho_2+1)}(\varsigma-a)^{\rho_2}\varepsilon_2\\
&\le\left(\Delta_\zeta b+\Delta_b\right)\varepsilon_2+\Delta_\eta b\varepsilon_1.
\end{aligned}\tag{3.25}$$

Therefore, by (3.10) and (3.23)

$$\begin{aligned}
|\varphi_1(\varsigma)-\widetilde{\varphi_1}(\varsigma)| &= \left|\varphi_1(\varsigma)-\Pi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)+\Pi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)-\widetilde{\varphi_1}(\varsigma)\right|\\
&\le \left|\Pi_1\left(\varphi_1,\varphi_2\right)(\varsigma)-\Pi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)\right|+\left|\Pi_1\left(\widetilde{\varphi_1},\widetilde{\varphi_2}\right)(\varsigma)-\widetilde{\varphi_1}(\varsigma)\right|\\
&\le \left(\Lambda_\eta b+\Lambda_b\right)\left(\kappa_f\left\|\varphi_1-\widetilde{\varphi_1}\right\|+\overline{\kappa}_f\left\|\varphi_2-\widetilde{\varphi_2}\right\|\right)\\
&\quad+\Lambda_\zeta b\left(\kappa_g\left\|\varphi_1-\widetilde{\varphi_1}\right\|+\overline{\kappa}_g\left\|\varphi_2-\widetilde{\varphi_2}\right\|\right)+\left(\Lambda_\eta b+\Lambda_b\right)\varepsilon_1+\Lambda_\zeta b\varepsilon_2\\
&= \left(\left(\Lambda_\eta b+\Lambda_b\right)\kappa_f+\Lambda_\zeta b\kappa_g\right)\left\|\varphi_1-\widetilde{\varphi_1}\right\|\\
&\quad+\left(\left(\Lambda_\eta b+\Lambda_b\right)\overline{\kappa}_f+\Lambda_\zeta b\overline{\kappa}_g\right)\left\|\varphi_2-\widetilde{\varphi_2}\right\|\\
&\quad+\left(\Lambda_\eta b+\Lambda_b\right)\varepsilon_1+\Lambda_\zeta b\varepsilon_2,
\end{aligned}$$

which implies

$$\mho_1\left\|\varphi_1-\widetilde{\varphi_1}\right\|\le\mho_2\left\|\varphi_2-\widetilde{\varphi_2}\right\|+\left(\Lambda_\eta b+\Lambda_b\right)\varepsilon_1+\Lambda_\zeta b\varepsilon_2 \tag{3.26}$$

where

$$\mho_1 \;:\; =1-\left(\left(\Lambda_\eta b+\Lambda_b\right)\kappa_f+\Lambda_\zeta b\kappa_g\right),$$

$$\mho_2 \;:\; = \left(\left(\Lambda_\eta b + \Lambda_b \right) \overline{\kappa}_f + \Lambda_\zeta b \overline{\kappa}_g \right).$$

Similarly

$$\begin{aligned}
|\varphi_2(\varsigma) - \widetilde{\varphi_2}(\varsigma)| &= |\varphi_2(\varsigma) - \Pi_2\left(\widetilde{\varphi_1}, \widetilde{\varphi_2}\right)(\varsigma) + \Pi_2\left(\widetilde{\varphi_1}, \widetilde{\varphi_2}\right)(\varsigma) - \widetilde{\varphi_2}(\varsigma)| \\
&\leq |\Pi_2\left(\varphi_1, \varphi_2\right)(\varsigma) - \Pi_2\left(\widetilde{\varphi_1}, \widetilde{\varphi_2}\right)(\varsigma)| + |\Pi_2\left(\widetilde{\varphi_1}, \widetilde{\varphi_2}\right)(\varsigma) - \widetilde{\varphi_2}(\varsigma)| \\
&\leq \left(\Delta_\zeta b + \Delta_b\right)\left(\kappa_g \left\|\varphi_1 - \varphi_1^\star\right\| + \overline{\kappa}_g \left\|\varphi_2 - \varphi_2^\star\right\|\right) \\
&\quad + \Delta_\eta b\left(\kappa_f \left\|\varphi_1 - \varphi_1^\star\right\| + \overline{\kappa}_f \left\|\varphi_2 - \varphi_2^\star\right\|\right) + \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1 \\
&= \left(\left(\Delta_\zeta b + \Delta_b\right)\kappa_g + \Delta_\eta b \kappa_f\right)\left\|\varphi_1 - \varphi_1^\star\right\| \\
&\quad + \left(\left(\Delta_\zeta b + \Delta_b\right)\overline{\kappa}_g + \Delta_\eta b \overline{\kappa}_f\right)\left\|\varphi_2 - \varphi_2^\star\right\| \\
&\quad + \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1,
\end{aligned}$$

which implies

$$\Omega_1 \left\|\varphi_2 - \varphi_2^\star\right\| \leq \Omega_2 \left\|\varphi_1 - \varphi_1^\star\right\| + \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1, \tag{3.27}$$

where

$$\begin{aligned}
\Omega_1 \;&:\; = 1 - \left(\left(\Delta_\zeta b + \Delta_b\right)\overline{\kappa}_g + \Delta_\eta b \overline{\kappa}_f\right), \\
\Omega_2 \;&:\; = \left(\left(\Delta_\zeta b + \Delta_b\right)\kappa_g + \Delta_\eta b \kappa_f\right).
\end{aligned}$$

From (3.26) and (3.27), we can write

$$\mho_1 \|\varphi_1 - \widetilde{\varphi_1}\| - \mho_2 \|\varphi_2 - \widetilde{\varphi_2}\| \leq \left(\Lambda_\eta b + \Lambda_b\right)\varepsilon_1 + \Lambda_\zeta b \varepsilon_2, \tag{3.28}$$

$$-\Omega_2 \|\varphi_1 - \widetilde{\varphi_1}\| + \Omega_1 \|\varphi_2 - \widetilde{\varphi_2}\| \leq \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1. \tag{3.29}$$

The matrix formula of (3.28) and (3.29) is

$$\begin{pmatrix} \mho_1 & -\mho_2 \\ -\Omega_2 & \Omega_1 \end{pmatrix} \begin{pmatrix} \|\varphi_1 - \widetilde{\varphi_1}\| \\ \|\varphi_2 - \widetilde{\varphi_2}\| \end{pmatrix} \leq \begin{pmatrix} \left(\Lambda_\eta b + \Lambda_b\right)\varepsilon_1 + \Lambda_\zeta b \varepsilon_2 \\ \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} \|\varphi_1 - \widetilde{\varphi_1}\| \\ \|\varphi_2 - \widetilde{\varphi_2}\| \end{pmatrix} \leq \frac{1}{\Delta} \begin{pmatrix} \Omega_1 & \mho_2 \\ \Omega_2 & \mho_1 \end{pmatrix} \begin{pmatrix} \left(\Lambda_\eta b + \Lambda_b\right)\varepsilon_1 + \Lambda_\zeta b \varepsilon_2 \\ \left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1 \end{pmatrix},$$

where $\Delta := \Omega_1 \mho_1 - \Omega_2 \mho_2 \neq 0$. Hence

$$\|\varphi_1 - \widetilde{\varphi_1}\| \leq \frac{\Omega_1 \left(\left(\Lambda_\eta b + \Lambda_b\right)\varepsilon_1 + \Lambda_\zeta b \varepsilon_2\right)}{\Delta} + \frac{\mho_2 \left(\left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1\right)}{\Delta}, \tag{3.30}$$

and

$$\|\varphi_2 - \widetilde{\varphi_2}\| \leq \frac{\Omega_2 \left(\left(\Lambda_\eta b + \Lambda_b\right)\varepsilon_1 + \Lambda_\zeta b \varepsilon_2\right)}{\Delta} + \frac{\mho_1 \left(\left(\Delta_\zeta b + \Delta_b\right)\varepsilon_2 + \Delta_\eta b \varepsilon_1\right)}{\Delta}. \tag{3.31}$$

□

Proof. By (3.30) and (3.31), we find that

$$\begin{aligned}
\|(\varphi_1,\varphi_2)-(\widetilde{\varphi_1},\widetilde{\varphi_2})\| &\le \|\varphi_1-\widetilde{\varphi_1}\|+\|\varphi_2-\widetilde{\varphi_2}\| \\
&\le \frac{\left(\Omega_1\left(\Lambda_\eta b+\Lambda_b\right)+\mho_2\Delta_\eta b\right)\varepsilon_1+\left(\Omega_1\Lambda_\zeta b+\mho_2\left(\Delta_\zeta b+\Delta_b\right)\right)\varepsilon_2}{\Delta} \\
&\quad +\frac{\left(\Omega_2\left(\Lambda_\eta b+\Lambda_b\right)+\mho_1\Delta_\eta b\right)\varepsilon_1+\left(\Omega_2\Lambda_\zeta b+\mho_1\left(\Delta_\zeta b+\Delta_b\right)\right)\varepsilon_2}{\Delta} \\
&\le \Upsilon_1\varepsilon_1+\Upsilon_2\varepsilon_2,
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon_1 &= \frac{(\Omega_1+\Omega_2)\left(\Lambda_\eta b+\Lambda_b\right)+(\mho_2+\mho_1)\,\Delta_\eta b}{\Delta}, \\
\Upsilon_2 &= \frac{(\mho_2+\mho_1)\left(\Delta_\zeta b+\Delta_b\right)+(\Omega_1+\Omega_2)\,\Lambda_\zeta b}{\Delta}.
\end{aligned}$$

Thus, the coupled system (1.2) is UH stable. □

4. Examples

Example 4.1. *Consider the following fractional system*

$$\begin{cases}
{}^{\mathcal{ABC}}\mathbb{D}_{0^+}^{\frac{4}{3}}\varphi_1(\varsigma)=\frac{1}{9}\frac{|\varphi_1(\mu\varsigma)|}{1+|\varphi_1(\mu\varsigma)|}+\frac{1}{6}\frac{|\varphi_2(\varsigma)|}{1+|\varphi_2(\varsigma)|}, \quad \varsigma\in[0,1], \\
{}^{\mathcal{ABC}}\mathbb{D}_{0^+}^{\frac{3}{2}}\varphi_2(\varsigma)=\frac{1}{9}(\sin\varphi_1(\varsigma)+(\cos\varsigma)\,\varphi_2(\mu\varsigma)), \quad \varsigma\in[0,1], \\
\varphi_1(0)=0, \ \ \varphi_2(0)=0, \\
\varphi_1(1)=\frac{1}{10}\varphi_2(\frac{1}{2}), \ \ \varphi_2(1)=\frac{1}{20}\varphi_1(\frac{1}{3}).
\end{cases} \tag{4.1}$$

By comparing the system (4.1) with the system (1.2), we notice that $\rho_1=\frac{4}{3}$, $\rho_2=\frac{3}{2}$, $a=0$, $b=1$, $\lambda_1=\frac{1}{10}$, $\lambda_2=\frac{1}{20}$, $\zeta=\frac{1}{2}$, $\eta=\frac{1}{3}$,

$$f(\varsigma,\varphi_1(\mu\varsigma),\varphi_2(\varsigma))=\frac{1}{9}\frac{|\varphi_1(\mu\varsigma)|}{1+|\varphi_1(\mu\varsigma)|}+\frac{1}{6}\frac{|\varphi_2|}{1+|\varphi_2|}, \ \text{for } \varphi_1,\varphi_2\in[0,\infty), \ \varsigma\in[0,1], \ 0<\mu<1,$$

and

$$g(\varsigma,\varphi_1(\varsigma),\varphi_2(\mu\varsigma))=\frac{1}{9}(\sin\varphi_1(\varsigma)+(\cos\varsigma)\,\varphi_2(\mu\varsigma)), \ \text{for } \varphi_1,\varphi_2\in[0,\infty), \ \varsigma\in[0,1], \ 0<\mu<1.$$

Set $\mu=\frac{1}{3}$, and for $\varphi_1,\varphi_2,\varphi_1^*,\varphi_2^*\in[0,\infty)$, $\varsigma\in[0,1]$, we have

$$\begin{aligned}
\left|f(\varsigma,\varphi_1(\mu\varsigma),\varphi_2(\varsigma))-f\left(\varsigma,\varphi_1^*(\mu\varsigma),\varphi_2^*(\varsigma)\right)\right| &\le \frac{1}{9}\left|\frac{\left|\varphi_1\left(\frac{\varsigma}{3}\right)\right|}{1+\left|\varphi_1\left(\frac{\varsigma}{3}\right)\right|}-\frac{\left|\varphi_1^*\left(\frac{\varsigma}{3}\right)\right|}{1+\left|\varphi_1^*\left(\frac{\varsigma}{3}\right)\right|}\right| \\
+\frac{1}{6}\left|\frac{|\varphi_2(\varsigma)|}{1+|\varphi_2(\varsigma)|}-\frac{\left|\varphi_2^*(\varsigma)\right|}{1+\left|\varphi_2^*(\varsigma)\right|}\right| &\le \frac{1}{9}\left|\varphi_1\left(\frac{\varsigma}{3}\right)-\varphi_1^*\left(\frac{\varsigma}{3}\right)\right|+\frac{1}{6}\left|\varphi_2(\varsigma)-\varphi_2^*(\varsigma)\right|,
\end{aligned}$$

$$\left|g(\varsigma,\varphi_1(\varsigma),\varphi_2(\mu\varsigma)-g\left(\varsigma,\varphi_1^*(\varsigma),\varphi_2^*(\mu\varsigma)\right)\right|\leq\frac{1}{9}\left|\varphi_1(\varsigma)-\varphi_1^*(\varsigma)\right|+\frac{1}{9}\left|\varphi_2(\frac{\varsigma}{3})-\varphi_2^*(\frac{\varsigma}{3})\right|.$$

Thus, (Hy_1) holds with $\kappa_f=\frac{1}{9},\overline{\kappa}_f=\frac{1}{6}$, and $\kappa_g=\overline{\kappa}_g=\frac{1}{9}$. From the above data, we get

$$\Lambda_\eta\approx 0.00072,\ \Lambda_\zeta\approx 0.038,\ \Lambda_b\approx 0.95,$$

$$\Delta_\zeta\approx 0.0006,\ \ \Delta_\eta\approx 0.014,\ \ \Delta_b\approx 0.88.$$

Additionally,

$$\begin{aligned}\sigma_1 &= \left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\kappa_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\kappa_g\approx 0.21,\\ \sigma_2 &= \left(\Lambda_\eta b+\Lambda_b+\Delta_\eta b\right)\overline{\kappa}_f+\left(\Lambda_\zeta b+\Delta_\zeta b+\Delta_b\right)\overline{\kappa}_g\approx 0.26.\end{aligned}$$

Hence, $\max_{\varsigma\in\mathbb{J}}\{\sigma_1,\sigma_2\}=\sigma\approx 0.26<1$. Thus, with the assistance of Theorem 3.3, the system (4.1) has a unique solution (φ_1,φ_2) on $[0,1]$. Furthermore, we have

$$\Omega_1=0.904,\ \Omega_2=0.0994,\ \mho_1=0.890,\ \mho_2=0.109.$$

Hence $\Delta:=\Omega_1\mho_1-\Omega_2\mho_2=0.79\neq 0$, which implies that the system (4.1) is UH stable.

Example 4.2. *Consider the following fractional system*

$$\begin{cases} {}^{\mathcal{ABC}}\mathbb{D}_{0.5^+}^{\frac{5}{4}}\varphi_1(\varsigma)=\frac{e^\varsigma}{7+e^\varsigma}\left(\frac{|\varphi_1(\mu\varsigma)|}{1+|\varphi_1(\mu\varsigma)|}+|\varphi_2(\varsigma)|+\frac{1}{10}\right), & \varsigma\in[\frac{1}{2},1],\ 0<\mu<1,\\ {}^{\mathcal{ABC}}\mathbb{D}_{0.5^+}^{\frac{7}{4}}\varphi_2(\varsigma)=\frac{e^\varsigma}{8+e^\varsigma}\left(|\varphi_1(\varsigma)|+\frac{|\varphi_2(\mu\varsigma)|}{1+|\varphi_2(\mu\varsigma)|}+\frac{1}{20}\right), & \varsigma\in[\frac{1}{2},1],\ 0<\mu<1,\\ \varphi_1(\frac{1}{2})=0,\ \ \varphi_2(\frac{1}{2})=0, & \\ \varphi_1(1)=\frac{1}{15}\varphi_2(\frac{1}{3}),\ \ \varphi_2(1)=\frac{1}{10}\varphi_1(\frac{1}{5}). & \end{cases}\tag{4.2}$$

By comparing thew system (4.2) with the system (1.2), we notice that $\rho_1=\frac{5}{4},\rho_2=\frac{7}{4},a=\frac{1}{2},b=1$, $\lambda_1=\frac{1}{15},\lambda_2=\frac{1}{10},\zeta=\frac{1}{3},\eta=\frac{1}{5}$. *Set* $\mu=\frac{1}{2}$, *then, for* $\varphi_1,\varphi_2\in[0,\infty)$, $\varsigma\in[\frac{1}{2},1]$,

$$|f(\varsigma,\varphi_1(\mu\varsigma)+\varphi_2(\varsigma))|\leq\frac{e^\varsigma}{7+e^\varsigma}\left(\left|\varphi_1\left(\frac{\varsigma}{2}\right)\right|+|\varphi_2(\varsigma)|+\frac{1}{10}\right),$$

and

$$|g(\varsigma,\varphi_1(\varsigma),\varphi_2(\mu\varsigma))|\leq\frac{e^\varsigma}{8+e^\varsigma}\left(|\varphi_1(\varsigma)|+\left|\varphi_2\left(\frac{\varsigma}{2}\right)\right|+\frac{1}{20}\right).$$

Obviously, f and g are continuous, and (Hy_2) is satisfied with

$$\gamma_f(\varsigma)=\frac{e^\varsigma}{7+e^\varsigma},\ \ \gamma_g(\varsigma)=\frac{e^\varsigma}{8+e^\varsigma},\ \ \psi_f(\cdot)=\cdot+\frac{1}{10}\ \text{ and }\ \psi_g(\cdot)=\cdot+\frac{1}{20}.$$

A simple calculation shows that condition (Hy_3) is satisfied for some constant $M>1$. Since all the assumptions of the Theorem 3.2 are fulfilled, then the problem (4.2) has a solution (φ_1,φ_2) on $[\frac{1}{2},1]$.

5. Conclusions

In last three decades, FDEs have gained the attention of many researchers because of their remarkable applications, especially, those involving $\mathcal{ABC}$ type FDs. As a supplemental contribution to this theme, we have investigated the existence, uniqueness, and UHS results of a coupled system for second-order fractional pantograph equations in the $\mathcal{ABC}$ sense. The analysis of acquired results has been based on using Leray-Schauder's and Banach's fixed point theorem's, along with Arzelà–Ascoli's theorem. It should be noticed that considering our acquired outcomes, our utilization of the $\mathcal{ABC}$ operator extends and advances many results in the literature. For future work, one can sum up various ideas for stability and existence analytic results to the present system along with the inclusion system.

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Conflict of interest

The authors declare that they have no competing interests.

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