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*Research article*

# Existence of solutions for a semipositone fractional boundary value pantograph problem

**Hamid Boulares**[1], **Manar A. Alqudah**[2] **and Thabet Abdeljawad**[3,4,*]

[1] Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University 8 May 1945 Guelma, Algeria

[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia

[3] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[4] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

**Abstract:** The boundary value problem (BVP) for a nonlinear non positone or semi-positone multi-point Caputo-Hadamard fractional differential pantograph problem is addressed in this study.

$$\mathfrak{D}_1^{\upsilon} x(\mathfrak{t}) + \mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), x(1+\lambda\mathfrak{t})) = 0, \ \mathfrak{t} \in (1, \mathfrak{b})$$

$$x(1) = \delta_1, \ x(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i) + \delta_2, \ \delta_i \in \mathbb{R}, \ i = 1, 2,$$

where $\lambda \in \left(0, \frac{\mathfrak{b}-1}{\mathfrak{b}}\right)$. The novelty in our approach is to show that there is only one solution to this problem using the Schauder fixed point theorem. Our results expand some recent research in the field. Finally, we include an example to demonstrate our findings.

**Keywords:** Caputo-Hadamard fractional derivative; BVP; changing sign nonlinearity; pantograph problem; Schauder fixed point theorem
**Mathematics Subject Classification:** 34B10, 34B15, 26A33

## 1. Introduction

Fractional calculus has recently become popular as a method for computing the derivative of order real or complex. It was vital to the development of natural science by modelling a large variety of phenomena or real world problems. In fact, most of the problems arise in scientific fields. Fractional differential equations (FDEs) can be used to model chemistry and biology, physics, biomedical science, optics, biomedical research, and radiography [1–7]. Obtaining optimal solutions for FDEs expands the scope of the studies. This is why many scientists have focused on FDEs in recent years. Many papers, books, and other works about Caputo-Hadamard fractional derivatives have been produced to study the existence of solutions to certain fractional dynamic equations [8–12].

The pantograph equations are special cases of delay differential equations in the sense that the term $\tau = 0$ in the delay function $\theta(\mathfrak{t}) = \mathfrak{t} - \tau$. The delay or retarded types of equations have been extensively studied (for example see [13]). The generalised pantograph equation has a variety of uses that can be found in pure mathematics [14], electrodynamics [15] and current collection by an electric locomotive pantograph [16]. Over the last ten years or so problems (1.1) and (1.2), which are mentioned in abstract and below, have been studied widely by many authors. The technique used usually involves either the shooting method or phase plane methods. One of the difficulties encountered is that the norm of solutions of the "appropriate" family of problems considered is usually unbounded. Another difficulty that arises is that zero is not a lower solution (in fact it is sometimes an upper solution).

Many positive solutions or many solutions to nonlinear fractional BVP have been studied using fixed point theorems (pfts) such as Schauder's fpt, Guo-Krasnosel'skii fpt, and Leggett-Williams fpt. That was used by a group of experts. Recent research has focused on BVPs involving multipoint initial conditions and FDEs [17, 18].

Nonlinearity is usually nonnegative to assure the level of positive solutions for BVPs. The investigation of the problem would become significantly more complex if the nonlinearity changes sign. As a result, there are few research on the subject [19–23].

In [24], the authors considered the two- point Liouville-Caputo BVP of the form

$$^{C}\mathfrak{D}^{\upsilon}x(\mathfrak{t}) = -\mathfrak{f}(\mathfrak{t}, x(\mathfrak{t})),\ \mathfrak{t} \in (a, b),$$

$$x(a) = \delta_1,\ x(b) = \delta_2,\ \delta_i \in \mathbb{R},\ i = 1, 2,$$

where the Caputo fractional derivative of order $1 < \upsilon < 2$ is indicated by $^{C}\mathfrak{D}^{\upsilon}$ and $\mathfrak{f}$ is a continuous function.

The purpose of this study is to establish the existence of solution of the following $m$-point fractional BVP when the term of nonlinearity changes its sign

$$\mathfrak{D}_1^{\upsilon}x(\mathfrak{t}) + \mathfrak{f}(\mathfrak{t}, x(\mathfrak{t}), x(1 + \lambda\mathfrak{t})) = 0,\ \mathfrak{t} \in (1, \mathfrak{b}), \tag{1.1}$$

$$x(1) = \delta_1,\ x(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i) + \delta_2,\ \delta_i \in \mathbb{R},\ i = 1, 2, \tag{1.2}$$

where $\lambda \in \left(0, \frac{\mathfrak{b}-1}{\mathfrak{b}}\right)$, $\mathfrak{D}_1^{\upsilon}$ is the standard Caputo-Hadamard fractional derivative of order $1 < \upsilon \leq 2$, $\zeta_i$ $(1 \leq i \leq m-2)$ are positive real constants with $0 < \Sigma_{i=1}^{m-2}\zeta_i < 1$, $\eta_i \in (1, \mathfrak{b})$ and $\mathfrak{f} : [1, \mathfrak{b}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the sign of the continuous function can change. Problems of the above type are referred to in the literature as non positone or semi-positone boundary value problems. Our interest in semi positone problems and the existence of nonnegative solutions arises from the fact that these problems occur in many models. The following is the description to how this article is organised. The second section presents some fundamental ideas, definitions, lemmas, and arguments. In Section 3, we demonstrate the primary result, and to explain the main result, we give a specific example.

## 2. Preliminaries notation

This section contains certain basic definitions, and theorems in the field of fractional calculus, that will be used throughout this work. For more details about the theory of fractional calculus and its applications can be found in [5, 7].

**Definition 2.1** ( [5,7]). The Riemann-Liouville fractional integral of order $\upsilon > 0$ for a $x : [0, +\infty) \to \mathbb{R}$ is defined as

$$I^{\upsilon} x(\mathrm{t}) = \frac{1}{\Gamma(\upsilon)} \int_0^{\mathrm{t}} (\mathrm{t} - \mathfrak{s})^{\upsilon-1} x(\mathfrak{s}) d\mathfrak{s},$$

where $\Gamma$ is the Euler gamma function and it is defined by

$$\Gamma(\upsilon) = \int_0^{\infty} e^{-\mathrm{t}} \mathrm{t}^{\upsilon-1} d\mathrm{t}.$$

**Definition 2.2** ( [5, 7]). The Hadamard fractional integral of order $\upsilon > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{I}_1^{\upsilon} x(\mathrm{t}) = \frac{1}{\Gamma(\upsilon)} \int_1^{\mathrm{t}} \left(\log \frac{\mathrm{t}}{\mathfrak{s}}\right)^{\upsilon-1} x(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}}.$$

**Definition 2.3** ( [5,7]). The Caputo fractional derivative of order $\upsilon > 0$ for a function $x : [0, +\infty) \to \mathbb{R}$ supplied by

$$\mathfrak{D}^{\upsilon} x(\mathrm{t}) = \frac{1}{\Gamma(n-\upsilon)} \int_0^{\mathrm{t}} (\mathrm{t} - \mathfrak{s})^{n-\upsilon-1} x^{(n)}(\mathfrak{s}) d\mathfrak{s}, \ n-1 < \upsilon < n, \ n \in \mathbb{N}.$$

**Definition 2.4** ( [4]). The Hadamard fractional calculus of order $\upsilon > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is such as

$$\mathfrak{D}_1^{\upsilon} x(\mathrm{t}) = \frac{1}{\Gamma(n-\upsilon)} \int_1^{\mathrm{t}} \left(\log \frac{\mathrm{t}}{\mathfrak{s}}\right)^{n-\alpha-1} \delta^n x(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}}, \ n-1 < \alpha < n,$$

where $\delta^n = \left(\mathrm{t}\frac{d}{dt}\right)^n, n \in \mathbb{N}$.

**Lemma 2.5** ( [5,7]). *Let $n-1 < \upsilon \le n$, $n \in \mathbb{N}$. The equality $\left(\mathfrak{I}_1^{\upsilon} \mathfrak{D}_1^{\upsilon} x\right)(\mathrm{t}) = 0$ is valid iff*

$$x(\mathrm{t}) = \sum_{k=1}^{n} c_k (\log \mathrm{t})^{\upsilon-k} \ \textit{for each} \ \mathrm{t} \in [1, \infty),$$

*where $c_k \in \mathbb{R}$, $k = 1, ..., n$ are constants.*

**Lemma 2.6** ( [4]). *Let $m-1 < \upsilon \le m$, $m \in \mathbb{N}$ and $x \in C^{n-1}[1, \infty)$. Then*

$$\mathfrak{I}_1^{\upsilon}[\mathfrak{D}_1^{\upsilon} x(\mathrm{t})] = x(\mathrm{t}) - \sum_{k=0}^{m-1} \frac{\delta^k x(1)}{\Gamma(k+1)} (\log \mathrm{t})^k.$$

**Lemma 2.7** ( [5,7]). *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)} \int_1^{\mathrm{t}} \left(\log \frac{\mathrm{t}}{\mathfrak{s}}\right)^{\mu-1} (\log \mathfrak{s})^{\nu} \frac{d\mathfrak{s}}{\mathfrak{s}} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log \mathrm{t})^{\mu+\nu}.$$

**Lemma 2.8** ( [5,7]). *Let $x(\mathfrak{t}) = (\log(\mathfrak{t}))^{\mu}$, where $\mu \geq 0$ and let $m-1 < \upsilon \leq m$, $m \in \mathbb{N}$. Then*

$$\mathfrak{D}_1^{\upsilon} x(t) = \begin{cases} 0 & \textit{if } \mu \in \{0, 1, ..., m-1\}, \\ \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log \mathfrak{t})^{\mu-\nu} & \textit{if } \mu \in \mathbb{N}, \mu \geq m \textit{ or } \mu \notin \mathbb{N}, \mu > m-1. \end{cases}$$

To deal with the solution of the FDE (1.1) and (1.2) to consider the solution,

$$-\mathfrak{D}_1^{\upsilon} x(\mathfrak{t}) = h(\mathfrak{t}), \tag{2.1}$$

governed by the boundary condition (1.2).

Let's denote $\Delta := \log \mathfrak{b} - \Sigma_{i=1}^{m-2} \zeta_i \log \eta_i$.

**Lemma 2.9.** *Let $\upsilon \in (1, 2]$ and $\mathfrak{t} \in [1, \mathfrak{b}]$. Then, the The BVP (2.1) and (1.2) admits one x of the form*

$$x(\mathfrak{t}) = \left[1 + \frac{\Sigma_{i=1}^{m-2} \zeta_i - 1}{\Delta} \log \mathfrak{t}\right] \delta_1 + \frac{\log \mathfrak{t}}{\Delta} \delta_2 + \int_1^{\mathfrak{b}} \varpi(\mathfrak{t}, \mathfrak{s}) \mathrm{h}(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}},$$

*where*

$$\varpi(\mathfrak{t}, \mathfrak{s}) = \frac{1}{\Gamma(\upsilon)} \begin{cases} -(\log \frac{\mathfrak{t}}{\mathfrak{s}})^{\upsilon-1} + \frac{\log \mathfrak{t}}{\Delta} \left[(\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} - \Sigma_{j=i}^{m-2} \zeta_j (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1}\right], & \mathfrak{s} \leq \mathfrak{t}, \ \eta_{i-1} < \mathfrak{s} \leq \eta_i; \\ \frac{\log \mathfrak{t}}{\Delta} \left[(\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} - \Sigma_{j=i}^{m-2} \zeta_j (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1}\right], & \mathfrak{t} \leq \mathfrak{s}, \ \eta_{i-1} < \mathfrak{s} \leq \eta_i, \end{cases} \tag{2.2}$$

$i = 1, 2, ..., m-2$.

*Proof.* First the solution of $\mathfrak{D}_1^{\upsilon} x(\mathfrak{t}) = -h(\mathfrak{t})$ is given by

$$x(\mathfrak{t}) = -\frac{1}{\Gamma(\upsilon)} \int_1^{\mathfrak{t}} (\log \frac{\mathfrak{t}}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} + c_0 + c_1 \log \mathfrak{t}, \tag{2.3}$$

where $c_0, c_1 \in \mathbb{R}$.

By $x(1) = \delta_1$ and $x(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i) + \delta_2$, we have $c_0 = \delta_1$ and

$$\begin{aligned} c_1 = \frac{1}{\Delta} \Bigg( & -\frac{1}{\Gamma(\upsilon)} \sum_{i=1}^{m-2} \zeta_i \int_1^{\eta_j} (\log \frac{\eta_i}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} \\ & + \frac{1}{\Gamma(\upsilon)} \int_1^{\mathfrak{b}} (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} + \delta_1 \left[\sum_{i=1}^{m-2} \zeta_i - 1\right] + \delta_2 \Bigg). \end{aligned}$$

Substituting $c_0, c_1$ into Eq (2.3) we find,

$$\begin{aligned} x(\mathfrak{t}) &= \left[1 + \frac{\Sigma_{i=1}^{m-2} \zeta_i - 1}{\Delta} \log \mathfrak{t}\right] \delta_1 + \frac{\log \mathfrak{t}}{\Delta} \delta_2 - \frac{1}{\Gamma(\upsilon)} \Bigg( \int_1^{\mathfrak{t}} (\log \frac{\mathfrak{t}}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} \\ &\quad + \frac{\log \mathfrak{t}}{\Delta} \sum_{i=1}^{m-2} \zeta_i \int_1^{\eta_j} (\log \frac{\eta_i}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} - \frac{\log \mathfrak{t}}{\Delta} \int_1^{\mathfrak{b}} (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}} \Bigg) \\ &= \left[1 + \frac{\Sigma_{i=1}^{m-2} \zeta_i - 1}{\Delta} \log \mathfrak{t}\right] \delta_1 + \frac{\log \mathfrak{t}}{\Delta} \delta_2 + \int_1^{\mathfrak{b}} \varpi(\mathfrak{t}, \mathfrak{s}) h(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}}, \end{aligned}$$

where (2.2) is the expression of $\varpi(\mathfrak{t}, \mathfrak{s})$. The demonstration is finished. □

**Lemma 2.10.** *If* $0 < \sum_{i=1}^{m-2} \zeta_i < 1$, *so*

*i*) $\Delta > 0$,

*ii*) $(\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} - \sum_{j=i}^{m-2} \zeta_j (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1} > 0$.

*Proof.* *i*) That is clear to see

$$\eta_i < \mathfrak{b},$$

$$\zeta_i \log \eta_i < \zeta_i \log \mathfrak{b},$$

$$-\sum_{i=1}^{m-2} \zeta_i \log \eta_i > -\sum_{i=1}^{m-2} \zeta_i \log \mathfrak{b},$$

$$\log \mathfrak{b} - \sum_{i=1}^{m-2} \zeta_i \log \eta_i > \log \mathfrak{b} - \sum_{i=1}^{m-2} \zeta_i \log \mathfrak{b} = \log \mathfrak{b}[1 - \sum_{i=1}^{m-2} \zeta_i].$$

If $1 - \Sigma_{i=1}^{m-2} \zeta_i > 0$, then $\log \mathfrak{b} - \Sigma_{i=1}^{m-2} \zeta_i \log \eta_i > 0$. So we have $\Delta > 0$.

*ii*) Since $0 < \upsilon - 1 \leq 1$, we have $(\log \frac{\eta_i}{\mathfrak{s}})^{\upsilon-1} < (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1}$. Thus we have

$$\sum_{j=i}^{m-2} \zeta_j (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1} < \sum_{j=i}^{m-2} \zeta_j (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} \leq (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} \sum_{i=1}^{m-2} \zeta_j < (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1},$$

and so

$$(\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} - \sum_{j=i}^{m-2} \zeta_j (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1} > 0.$$

□

*Remark* 2.11. Regarding the Green's function $\varpi(\mathfrak{t}, \mathfrak{s})$ of the (1.1) and (1.2), it is simple to find

$$\begin{aligned}\int_1^{\mathfrak{b}} |\varpi(\mathfrak{t}, \mathfrak{s})| \frac{d\mathfrak{s}}{\mathfrak{s}} &\leq \frac{1}{\Gamma(\upsilon)} \int_1^{\mathfrak{t}} (\log \frac{\mathfrak{t}}{\mathfrak{s}})^{\upsilon-1} \frac{d\mathfrak{s}}{\mathfrak{s}} + \frac{\log \mathfrak{t}}{\Gamma(\upsilon)\Delta} \sum_{i=1}^{m-2} \zeta_i \int_1^{\eta_i} (\log \frac{\eta_j}{\mathfrak{s}})^{\upsilon-1} \frac{d\mathfrak{s}}{\mathfrak{s}} \\ &+ \frac{\log \mathfrak{t}}{\Delta\Gamma(\upsilon)} \int_1^{\mathfrak{b}} (\log \frac{\mathfrak{b}}{\mathfrak{s}})^{\upsilon-1} \frac{d\mathfrak{s}}{\mathfrak{s}} \\ &= \frac{(\log \mathfrak{t})^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{\log \mathfrak{t}}{\Delta\Gamma(\upsilon+1)} \sum_{i=1}^{m-2} \zeta_i (\log \eta_i)^{\upsilon} + \frac{\log \mathfrak{t}}{\Delta\Gamma(\upsilon+1)} (\log \mathfrak{b})^{\upsilon} \\ &\leq \frac{(\log \mathfrak{b})^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{\log \mathfrak{b}}{\Delta\Gamma(\upsilon+1)} \sum_{i=1}^{m-2} \zeta_i (\log \eta_i)^{\upsilon} + \frac{(\log \mathfrak{b})^{\upsilon+1}}{\Delta\Gamma(\upsilon+1)} = M. \end{aligned} \tag{2.4}$$

*Remark* 2.12. Suppose $p(\mathfrak{t}) \in L^1[1, \mathfrak{b}]$, and $w(\mathfrak{t})$ is a resolution of (2.5)

$$\begin{cases} \mathfrak{D}_1^{\upsilon} w(\mathfrak{t}) + p(\mathfrak{t}) = 0 \\ w(1) = 0, \ w(\mathfrak{b}) = \Sigma_{i=1}^{m-2} \zeta_i w(\eta_i) \end{cases}, \tag{2.5}$$

then $w(\mathfrak{t}) = \int_1^{\mathfrak{b}} \varpi(\mathfrak{t}, \mathfrak{s}) p(\mathfrak{s}) \frac{d\mathfrak{s}}{\mathfrak{s}}$.

The next fpt is essential to proceed in our main results.

**Theorem 2.13.** *[22] [Schauder fpt] Suppose that X is a Banach space. Suppose K is a convex, closed, bounded subset of X. T has a fixed point in K if T : K → K is compact.*

## 3. Existence results

Additionally, throughout this article, we present the following conditions:

($\Lambda$1) There exists a nonnegative function $p \in L^1[1, \mathfrak{b}]$ and $\int_1^{\mathfrak{b}} p(\mathfrak{t})d\mathfrak{t} > 0$ such that $\mathfrak{f}(\mathfrak{t}, x, v) \geq -p(\mathfrak{t})$ for all $(\mathfrak{t}, x, v) \in [1, \mathfrak{b}] \times \mathbb{R} \times \mathbb{R}$.

($\Lambda$2) $\mathfrak{f}(\mathfrak{t}, x, v) \neq 0$, for $(\mathfrak{t}, x, v) \in [1, \mathfrak{b}] \times \mathbb{R} \times \mathbb{R}$.

Suppose $B = \mathbb{C}([1, \mathfrak{b}], \mathbb{R})$ be the Banach space of all continuous function from $[1, \mathfrak{b}]$ to $\mathbb{R}$ endowed by the standard norm $\|x\| = \sup\{|x(\mathfrak{t})| : \mathfrak{t} \in [1, \mathfrak{b}]\}$.

We will start by showing that the fractional equation below is true

$$\mathfrak{D}_1^{\nu} x(\mathfrak{t}) + F\left(\mathfrak{t}, x^*\left(\mathfrak{t}\right), x^*\left(1 + \lambda\mathfrak{t}\right)\right) = 0,\ \mathfrak{t} \in [1, \mathfrak{b}]. \tag{3.1}$$

There is a solution with the boundary condition (1.2), where $F : [1, \mathfrak{b}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$F(\mathfrak{t}, z_1, z_2) = \begin{cases} \mathfrak{f}(\mathfrak{t}, z_1, z_2) + p(\mathfrak{t}),\ z_1, z_2 \geq 0, \\ \mathfrak{f}(\mathfrak{t}, 0, 0) + p(\mathfrak{t}),\ z_1 \leq 0 \text{ or } z_2 \leq 0, \end{cases} \tag{3.2}$$

and $x^*(\mathfrak{t}) = \max\{(x - w)(\mathfrak{t}), 0\}$ so that $w$ is the unique solution of the problem (2.5). The mapping $T : B \to B$ related with the (3.1) and (1.2) defined as

$$(Tx)(\mathfrak{t}) = \left[1 + \frac{\sum_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathfrak{t}\right]\delta_1 + \frac{\log\mathfrak{t}}{\Delta}\delta_2 + \int_1^{\mathfrak{b}} \varpi(\mathfrak{t}, \mathfrak{s})F(\mathfrak{t}, x^*(\mathfrak{s}), x^*(1 + \lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}}, \tag{3.3}$$

where the formula (2.2) is the definition of $\varpi(\mathfrak{t}, \mathfrak{s})$. The existence of a fixed point for the mapping $T$ means that the problems (3.1) and (1.2) has a solution.

**Theorem 3.1.** *Suppose that* ($\Lambda$1) *and* ($\Lambda$2) *are valid. If* $\rho > 0$ *valid*

$$\left[1 + \frac{\sum_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathfrak{b}\right]\delta_1 + \frac{\log\mathfrak{b}}{\Delta}\delta_2 + LM \leq \rho,$$

*where* $L \geq \max\{|F(\mathfrak{t}, x, v)| : \mathfrak{t} \in [1, \mathfrak{b}], |x|, |v| \leq \rho\}$ *and* $M$ *is given in (2.4) then the problems (3.1) and (3.2) has a solution* $x(\mathfrak{t})$.

*Proof.* Let's begin by defining $P := \{x \in B : \|x\| \leq \rho\}$. The Schauder fpt is applicable to $P$ because it is a closed, bounded, and convex subset of $B$ is described by (3.3). Define $T : P \to B$ by (3.3). $T : P \to B$ is easily observed to be continuous. Claims $T : P \to P$. Let $x \in P$. Suppose $x^*(\mathfrak{t}) \leq x(\mathfrak{t}) \leq \rho$ for all $\mathfrak{t} \in [1,\ \mathfrak{b}]$. So

$$\begin{aligned} |Tx(\mathfrak{t})| &= \left|\left[1 + \frac{\sum_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathfrak{t}\right]\delta_1 + \frac{\log\mathfrak{t}}{\Delta}\delta_2 + \int_1^{\mathfrak{b}} \varpi(\mathfrak{t}, \mathfrak{s})F(\mathfrak{s}, x^*(\mathfrak{s}), x^*(1 + \lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}}\right| \\ &\leq \left[1 + \frac{\sum_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathfrak{b}\right]\delta_1 + \frac{\log\mathfrak{b}}{\Delta}\delta_2 + LM \leq \rho, \end{aligned}$$

for all $\mathfrak{t} \in [1, \mathfrak{b}]$. This indicates that $\|Tx\| \leq \rho$. So $T : K \to K$ can be demonstrated to be a compact mapping using the Arzela-Ascoli theorem. As a consequence of the Schauder fpt, $T$ has a fixed point $x$ in $P$. This suggests that $x$ is a solution to the problem (3.1 and 1.2). $\square$

**Lemma 3.2.** *$x^*(\mathrm{t})$ is a solution of the fractional BVP (1.1) and (1.2) with $x(\mathrm{t}) > w(\mathrm{t})$ for all $\mathrm{t} \in [1, \mathfrak{b}]$ if and only if $x = x^* + w$ is the positive solution of fractional BVP (3.1) and (1.2).*

*Proof.* Let $x(\mathrm{t})$ be a solution of fractional BVP (3.1 and 1.2). Then

$$\begin{aligned}
x(\mathrm{t}) &= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})F(\mathfrak{s}, x^*(\mathfrak{s}), x^*(1+\lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}} \\
&= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})\left(\mathrm{f}(\mathfrak{s}, x^*(\mathfrak{s}), x^*(1+\lambda\mathfrak{s})) + p(\mathfrak{s})\right)\frac{d\mathfrak{s}}{\mathfrak{s}} \\
&= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})\mathrm{f}(\mathfrak{s}, (x-w)(\mathfrak{s}), (x-w)(1+\lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}} \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})p(\mathfrak{s})\frac{d\mathfrak{s}}{\mathfrak{s}} \\
&= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})\mathrm{f}(\mathfrak{s}, (x-w)(\mathfrak{s}), (x-w)(1+\lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}} + w(\mathrm{t})
\end{aligned}$$

or

$$\begin{aligned}
x(\mathrm{t}) - w(\mathrm{t}) &= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})\mathrm{f}(\mathfrak{s}, (x-w)(\mathfrak{s}), (x-w)(1+\lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}},
\end{aligned}$$

then we get

$$\begin{aligned}
x^*(\mathrm{t}) &= \left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\log\mathrm{t}\right]\delta_1 + \frac{\log\mathrm{t}}{\Delta}\delta_2 \\
&+ \frac{1}{\Gamma(\upsilon)}\int_1^{\mathfrak{b}} \varpi(\mathrm{t},\mathfrak{s})\mathrm{f}(\mathfrak{s}, x^*(\mathfrak{s}), x^*(1+\lambda\mathfrak{s}))\frac{d\mathfrak{s}}{\mathfrak{s}}.
\end{aligned}$$

Another way, if $x^*$ is a solution of the fractional BVP (1.1 and 1.2) so we obtain

$$\begin{aligned}
\mathfrak{D}_1^{\upsilon}(x^*(\mathrm{t}) + w(\mathrm{t})) &= \mathfrak{D}_1^{\upsilon}x^*(\mathrm{t}) + \mathfrak{D}_1^{\upsilon}w(\mathrm{t}) = -\mathrm{f}(\mathrm{t}, x^*(\mathrm{t}), x^*(1+\lambda\mathrm{t})) - p(\mathrm{t}) \\
&= -\left[\mathrm{f}(\mathrm{t}, x^*(\mathrm{t}), x^*(1+\lambda\mathrm{t})) + p(\mathrm{t})\right] = -F(\mathrm{t}, x^*(\mathrm{t}), x^*(1+\lambda\mathrm{t})),
\end{aligned}$$

it indicates that

$$\mathfrak{D}_1^{\upsilon}x(\mathrm{t}) = -F(\mathrm{t}, x^*(\mathrm{t}), x^*(1+\lambda\mathrm{t})).$$

We easily see that

$$x^*(1) = x(1) - w(1) = x(1) - 0 = \delta_1,$$

i.e., $x(1) = \delta_1$ and

$$x^*(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x^*(\eta_i) + \delta_2,$$

$$x(\mathfrak{b}) - w(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i) - \sum_{i=1}^{m-2} \zeta_j w(\eta_i) + \delta_2 = \sum_{i=1}^{m-2} \zeta_i (x(\eta_i) - w(\eta_i)) + \delta_2,$$

i. e.,

$$x(\mathfrak{b}) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i) + \delta_2.$$

Therefore $x(\mathrm{t})$ is a solution of the fractional BVP (3.1 and 3.2). □

## 4. An example

Consider the following specific fractional BVP

$$D^{\frac{5}{4}} x(\mathrm{t}) + \mathrm{f}(\mathrm{t}, x(\mathrm{t}), x(1 + 0.5\mathrm{t})) = 0, \ \mathrm{t} \in (1, e) \tag{4.1}$$

$$x(1) = 1, \ x(1) = \frac{1}{5} x(\frac{3}{2}) + \frac{1}{3} x(\frac{5}{4}) + \frac{1}{11} x(\frac{9}{4}) - 1 \tag{4.2}$$

with function $\mathrm{f}(\mathrm{t}, x(\mathrm{t}), x(1 + 0.5\mathrm{t})) = \frac{4\mathrm{t}}{1+\mathrm{t}} \arctan(x(\mathrm{t}) + x(1 + 0.5\mathrm{t}))$.

Taking $p(\mathrm{t}) = 2\mathrm{t}$ we get $\int_1^e 2\mathrm{t} d\mathrm{t} = e^2 - 1 > 0$, then the hypotheses $(\Lambda 1) - (\Lambda 2)$ hold. Calculating $\Delta \cong 0.771$, $M \cong 2.24$ along with observing $|F(\mathrm{t}, x, v)| < \pi + 2e = L$ such that $|x| \leq \rho$ where $\rho = 19$, we could simply confirm that

$$\left[1 + \frac{\Sigma_{i=1}^{m-2} \zeta_i - 1}{\Delta} \log \mathfrak{b}\right] \delta_1 + \frac{\log \mathfrak{b}}{\Delta} \delta_2 + LM \cong 18.43 \leq 19.$$

After that, by applying Theorem 3.1 there is a solution $x(\mathrm{t})$ to the problems (4.1) and (4.2).

## 5. Conclusions

In this research we have showed the existence of a soultion of the BVP for a nonlinear non positone or semi-positone multi-point Caputo-Hadamard fractional differential pantograph problems (1.1) and (1.2). The novelty in our approach is that we have shown there is only one solution to this problem. In our proofs, we used the Schauder fixed point theorem. The findings in this paper significantly generalize and improve the recent results about semi-positone multi-point Caputo-Hadamard fractional differential pantograph problems (1.1) and (1.2).

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## Conflict of interest

The authors declare that they have no competing interests

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