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Research article

Intertwining relations for composition operators and integral-type operators between the Bloch-type spaces

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Abstract: In this paper, the compact intertwining relations of integral-type operators and composition operators between the Bloch-type spaces are investigated.

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1. Introduction

For two Banach spaces X and Y, $\mathcal{B}(X,Y)$ denotes the collection of all bounded linear operators from X to Y and $\mathcal{K}(X,Y)$ denotes the collection of all compact operators in $\mathcal{B}(X,Y)$. The Calkin algebra $\mathcal{Z}(X,Y)$ is the quotient Banach algebra $\mathcal{B}(X,Y)/\mathcal{K}(X,Y)$.

For bounded linear operators $A \in \mathcal{B}(X,X)$, $B \in \mathcal{B}(Y,Y)$ and $T \in \mathcal{B}(X,Y)$, we say "T intertwines A and B" if

$$TA = BT \quad with \quad T \neq 0.$$

When it is convenient to deemphasize the intertwining operator $T \in \mathcal{B}(X,Y)$, we write $A \propto B$ (sometimes we also use $A \propto B(T)$) as the intertwining relation above for simplicity. In [2] Bourdon and Shapiro showed that the intertwining relation is neither symmetric nor transitive. Furthermore, we say "T intertwines A and B in $\mathcal{Z}(X,Y)$" (or "T intertwines A and B compactly") if

$$TA = BT \quad \mathrm{mod}\ \mathcal{K}(X,Y) \quad \text{with} \quad T \neq 0.$$

For simplicity, the notation $A \propto_K B(T)$ represents the compact intertwining relations above. The relation $\propto_K$ turns to be symmetric when $T \in \mathcal{B}(X,Y)$ is invertible.

As usual, $S(\mathbb{D})$ denotes the collection of all analytic self-maps of the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. The composition operator C_φ induced by $\varphi \in S(\mathbb{D})$ is defined as $C_\varphi f = f \circ \varphi$ for each $f \in H(\mathbb{D})$, where $H(\mathbb{D})$ is the collection of all holomorphic functions on the unit disk.

We next recall the spaces to work on, one of which is a classical Banach space of analytic functions, the Bloch space, which is defined as

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \|f\|_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\}.$$

The Bloch space $\mathcal{B}$ is maximal among all *Möbius-invariant* Banach spaces of analytic functions on $\mathbb{D}$, which implies that $\|f\circ\varphi\|_1 = \|f\|_1$ holds for all $f \in \mathcal{B}$ and $\varphi \in Aut(\mathbb{D})$ with the seminorm $\|\cdot\|_1$. It is well-known that $\mathcal{B}$ is a Banach space endowed with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_1$.

For $0 < \alpha < \infty$, the α-Bloch space (or Bloch-type space) is defined as:

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}) : \|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty\}.$$

The little α-Bloch space defined as:

$$\mathcal{B}^\alpha_0 = \{f \in \mathcal{B}^\alpha : \lim_{|z|\to1}(1-|z|^2)^\alpha|f'(z)| = 0\}.$$

$\mathcal{B}^\alpha$ is a Banach space endowed with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$.

For $0 < \alpha,\beta < \infty$, the weighted logarithmic Bloch space and the little weighted logarithmic Bloch space were introduced in [13, 14]. It is defined as:

$$\mathcal{B}^\alpha_{\log^\beta} = \{f \in H(\mathbb{D}) : \|f\|^\alpha_{\log^\beta} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha(\log\frac{2}{1-|z|^2})^\beta|f'(z)| < \infty\}.$$

$$\mathcal{B}^\alpha_{\log^\beta,0} = \{f \in \mathcal{B}^\alpha_{\log^\beta} : \lim_{|z|\to1}(1-|z|^2)^\alpha(\log\frac{2}{1-|z|^2})^\beta|f'(z)| = 0\}.$$

$\mathcal{B}^\alpha_{\log^\beta}$ is a Banach space endowed with the norm $\|f\|_{\mathcal{B}^\alpha_{\log^\beta}} = |f(0)| + \|f\|^\alpha_{\log^\beta}$, which reduces to $\mathcal{B}^\alpha$ if $\beta = 0$.

For $0 < \alpha < \infty$, the classical weighted space is defined as:

$$H^\infty_\alpha = \{f \in H(\mathbb{D}) : \|f\|^\infty_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty\}.$$

The little weighted space is defined as:

$$H^\infty_{\alpha,0} = \{f \in H^\infty_\alpha : \lim_{|z|\to1}(1-|z|^2)^\alpha|f(z)| = 0\}.$$

H^∞_α is a Banach space endowed with the norm $\|f\|_{H^\infty_\alpha} = |f(0)| + \|f\|_\alpha$.

For $g \in H(\mathbb{D})$, two integral-type operators are defined by

$$J_g f(z) = \int_0^z f(t)g'(t)dt$$

and

$$I_g f(z) = \int_0^z f'(t)g(t)dt,$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. Obviously, integration by parts gives

$$M_g f = f(0)g(0) + J_g f + I_g f,$$

which shows the close relation among the integral-type operators J_g, I_g and the multiplication operator M_g. Here, the miltiplication operator M_g is defined by

$$M_g f(z) = g(z)f(z), f \in H(\mathbb{D}), z \in \mathbb{D}.$$

Conveniently, the symbol V_g is used to represent J_g or I_g. Composition, integral type operators and their products from or to the weighted logarithmic Bloch space and the little weighted logarithmic Bloch space have been investigated a lot recently (see, for example, [9, 15, 16]). For more information on the logarithmic Bloch spaces, interested readers can refer to [3, 8, 11, 12, 17–20].

Suppose that $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$. For two composition operators $C_\varphi \in \mathcal{B}(\mathcal{B}^\alpha, \mathcal{B}^\alpha)$, $C_\varphi \in \mathcal{B}(\mathcal{B}^\beta, \mathcal{B}^\beta)$, we concentrate on the compact intertwining relations of C_φ whose intertwining operator is the integral-type operators $V_g \in \mathcal{B}(\mathcal{B}^\alpha, \mathcal{B}^\beta)$. In other word, we will study the properties of the difference operator

$$V[\varphi; g, h] := C_\varphi V_g - V_h C_\varphi. \tag{1.1}$$

By $V[\varphi, \psi; g, h]$ we denote the following expression

$$(C_\varphi : \mathcal{B}^\beta \to \mathcal{B}^\beta)(V_g : \mathcal{B}^\alpha \to \mathcal{B}^\beta) - (V_h : \mathcal{B}^\alpha \to \mathcal{B}^\beta)(C_\psi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha). \tag{1.2}$$

We also say that C_φ and V_g are essentially commutative if

$$V_g(C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha) = (C_\varphi : \mathcal{B}^\beta \to \mathcal{B}^\beta)V_g \quad mod \quad \mathcal{K}(\mathcal{B}^\alpha, \mathcal{B}^\beta).$$

Moreover, the notation $\Omega_{co}^{\alpha,\beta}(V_g)$ is denotes the collection of $g \in H(\mathbb{D})$ such that

- $V_g \in \mathcal{B}(\mathcal{B}^\alpha, \mathcal{B}^\beta)$.
- V_g are essentially commutative with C_φ for all φ such that C_φ is bounded on both $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$.

Here, the lower symbol "co" represents "composition operator".

Some authors in their papers such as [21–23, 25] investigate the compact intertwining relations of the integral-type operators and the composition operators on various spaces of analytic functions on the unit disk.

In this paper, we investigate the compact intertwining relations of integral-type operators V_g from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ and the relevant composition operators C_φ. In Section 2, we present some lemmata to be used later in this paper. In Section 3, we investigate the intertwining relations of integral-type operators and composition operators, in which the equivalent conditions of $V[\varphi, \psi; g, h] = 0$ is given. In Section 4, boundedness and compactness of $V[\varphi; g, h]$ are investigated. In Sections 5 and 6, two questions of the compact intertwining relations of V_g and C_φ are investigated respectively.

For simplification, *the hypotheses* $0 < \alpha, \beta < \infty, \varphi \in S(\mathbb{D}), g, h \in H(\mathbb{D})$ *are available throughout this paper which will not be specified later.*

Specially, for two real numbers A and B, we say $A \lesssim B$ if there exists a constant $C \neq 0$ such that $A \leq CB$.

2. Auxiliary results

In this section, we introduce some basic properties of the Bloch-type spaces and the integral-type operators to be used in this paper.

The following folklore lemma is proved in a standard way (see, e.g., [10]), which also implies that the point evaluation functional is continuous on the Bloch-type space.

Proposition 2.1. *For each $f \in \mathcal{B}^\alpha$ and $z \in \mathbb{D}$, we have*

$$|f(z)| \lesssim \begin{cases} \log \dfrac{2}{1-|z|^2}\|f\|_{\mathcal{B}^\alpha}, & \alpha = 1; \\ (1-|z|^2)^{1-\alpha}\|f\|_{\mathcal{B}^\alpha}, & \alpha > 1; \\ \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1. \end{cases}$$

The following result can be also found in [10].

Lemma 2.2. *The composition operator C_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| < \infty. \tag{2.1}$$

The following lemma was proved, e.g., in [6, 7] even in much more general settings.

Lemma 2.3. *The integral-type operators J_g is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*

$$\begin{cases} g \in \mathcal{B}^\beta & \text{when} \quad 0 \le \alpha < 1; \\ g \in \mathcal{B}^\beta_{\log^1} & \text{when} \quad \alpha = 1; \\ g \in \mathcal{B}^{\beta-\alpha+1} & \text{when} \quad \alpha > 1. \end{cases}$$

Lemma 2.4. *[5] The integral-type operators I_g is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*

$$g \in H^\infty_{\beta-\alpha}. \tag{2.2}$$

The proposition below is a crucial criterion for the compactness of $V[\varphi; g, h]$, which can be proved by a little modification of Proposition 3.11 in [4].

Proposition 2.5. *$V[\varphi; g, h]$ is compact from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if whenever $\{f_n\}$ is bounded in $\mathcal{B}^\alpha$ and $f_n \to 0$ uniformly on any compact subset of the unit disk, then*

$$\lim_{n\to\infty} \|V[\varphi; g, h]f_n\|_{\mathcal{B}^\beta} = 0.$$

3. Intertwining relations of C_φ and V_g

Theorem 3.1. *Assume that $J[\varphi, \psi; g, h]$ is defined as* (1.2)*, then*
$J[\varphi, \psi; g, h] = 0$ if and only if
(a) either $\varphi(0) = 0$ or g is a constant;
(b) $\varphi = \psi$;
(c) $h = g \circ \varphi + C$, where C is an arbitrary constant.

Proof. The sufficiency is easily checked by calculation. To prove the necessity, we only show some different details from what Tong and Zhou did in [22] for the study of the intertwining relations for Volterra operators and composition operators on the Bergman space.

$J[\varphi,\psi;g,h]=0$ implies that

$$\sup_{f\in\mathcal{B}^\alpha,\|f\|\neq 0}\frac{\|(C_\varphi J_g-J_hC_\psi)f\|_{\mathcal{B}^\beta}}{\|f\|_{\mathcal{B}^\alpha}}=0,$$

which further implies that, for each $f\in\mathcal{B}^\alpha$,

$$\begin{aligned}0=\|(C_\varphi J_g-J_hC_\psi)f\|_{\mathcal{B}^\beta}=&\left|\int_0^{\varphi(0)}f(t)g'(t)dt\right|\\&+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(\varphi(z))\varphi'(z)g'(\varphi(z))-f(\psi(z))h'(z)|.\end{aligned}$$

Hence, for each $f\in\mathcal{B}^\alpha$, $|(C_\varphi J_g-J_hC_\psi)f(0)|=0$ and

$$\sup_{z\in\mathbb{D}}|f(\varphi(z))\varphi'(z)g'(\varphi(z))-f(\psi(z))h'(z)|=0$$

hold. And the latter one shows that for each $z\in\mathbb{D}$,

$$|f(\varphi(z))\varphi'(z)g'(\varphi(z))-f(\psi(z))h'(z)|=0.$$

To this end, the remaining part of the theorem is parallel with Proposition 3.1 in [22]. This completes the proof. □

Theorem 3.2. *Assume that $I[\varphi,\psi;g,h]$ is defined as* (1.2)*, then*
$I[\varphi,\psi;g,h]=0$ *if and only if*
(a) either $\varphi(0)=0$ or $g\equiv 0$;
(b) $\varphi=\psi$;
(c) $h=g\circ\varphi$.

Proof. The sufficiency is easily verified by calculation. To prove the necessity, we only show some essential details. $I[\varphi,\psi;g,h]=0$ implies that for each $f\in\mathcal{B}^\alpha$,

$$\begin{aligned}0=\|(C_\varphi I_g-I_hC_\psi)f\|_{\mathcal{B}^\beta}=&\left|\int_0^{\varphi(0)}f'(t)g(t)dt\right|\\&+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(\varphi(z))\varphi'(z)g(\varphi(z))-f'(\psi(z))\psi'(z)h(z)|.\end{aligned}$$

Hence, for each $f\in\mathcal{B}^\alpha$, $|(C_\varphi I_g-I_hC_\psi)f(0)|=0$ and

$$\sup_{z\in\mathbb{D}}|f'(\varphi(z))\varphi'(z)g(\varphi(z))-f'(\psi(z))\psi'(z)h(z)|=0$$

hold. If $\varphi(0)=0$, then the first of the conditions is automatically satisfied. If $\varphi(0)\neq 0$, then we can obtain that $g\equiv 0$ by the same method used in Proposition 3.1 in [22]. Moreover, the second equality

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(\varphi(z))\varphi'(z)g(\varphi(z))-f'(\psi(z))\psi'(z)h(z)|=0$$

implies that for each $z \in \mathbb{D}$,

$$|f'(\varphi(z))\varphi'(z)g(\varphi(z)) - f'(\psi(z))\psi'(z)h(z)| = 0.$$

By choosing $f(z) = z \in \mathcal{B}^\alpha$ and $f(z) = z^2 \in \mathcal{B}^\alpha$, we have that

$$\varphi'(z)g(\varphi(z)) - \psi'(z)h(z) = 0,$$

$$2\varphi(z)\varphi'(z)g(\varphi(z)) - 2\psi(z)\psi'(z)h(z) = 0.$$

Combining these two quantities, we have that

$$2\psi'(z)h(z)(\varphi(z) - \psi(z)) = 0.$$

Hence, $\varphi = \psi$ and $h = g \circ \varphi$. This completes the proof. □

Specially, we consider the operators mapping a Bloch-type space into and onto itself, that is $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha, J_g, J_h, I_g, I_h : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$. Combining Theorems 3.1 and 3.2 and the equivalent conditions of the boundedness of J_g and I_g from Lemmas 2.3 and 2.4, we conclude the statements in the following.

Corollary 3.3. *(a) If $\alpha > 1, g, h \in \mathcal{B}^{\beta-\alpha+1}$, then $J_g \propto J_h(C_\varphi)$ if and only if g is a constant or $\varphi(0) = 0$, $h = g \circ \varphi + C$, where C is a constant.*
(b) If $\alpha = 1, g, h \in \mathcal{B}^\beta_{\log^1}$, then $J_g \propto J_h(C_\varphi)$ if and only if g is a constant or $\varphi(0) = 0$, $h = g \circ \varphi + C$, where C is a constant.
(c) If $0 \le \alpha < 1, g, h \in \mathcal{B}^\beta$, then $J_g \propto J_h(C_\varphi)$ if and only if g is a constant or $\varphi(0) = 0$, $h = g \circ \varphi + C$, where C is a constant.

Corollary 3.4. *If $g, h \in H^\infty_{\beta-\alpha}$, then $I_g \propto I_h(C_\varphi)$ if and only if $g \equiv 0$ or $\varphi(0) = 0$, $h = g \circ \varphi$.*

4. Intertwining relations between the composition operators and the integral-type operators between the Bloch-type spaces

In this section, we characterize the boundedness and the compactness of $V[\varphi; g, h]$ defined as (1.1), in which the compactness is essential for our study in this paper. The method of the proof is basic, which is also parallel with Proposition 3.1 in [22] and Corollary 4.3 in [22].

Theorem 4.1. *$J[\varphi; g, h]$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*
(a)

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha-1}} |(g \circ \varphi - h)'(z)| < \infty$$

when $\alpha > 1$;
(b)

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\beta \log \frac{2}{1 - |\varphi(z)|^2} |(g \circ \varphi - h)'(z)| < \infty$$

when $\alpha = 1$;
(c) $g \circ \varphi - h \in \mathcal{B}^\beta$ when $0 < \alpha < 1$.

Proof. The proof of the theorem is essentially given in [7], but since there are some technical differences, we will give some details. We firstly prove the sufficiency. By Proposition 2.1, we estimate the semi-norm $\|J[\varphi; g, h]\|_\beta$ respectively.
(a) when $\alpha > 1$,

$$\begin{aligned}\|J[\varphi; g, h]\|_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f(\varphi(z))||(g\circ\varphi - h)'(z)| \\ &\lesssim \|f\|_{\mathcal{B}^\alpha} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi - h)'(z)| < \infty;\end{aligned}$$

(b) when $\alpha = 1$,

$$\|J[\varphi; g, h]\|_\beta \lesssim \|f\|_{\mathcal{B}^\alpha} \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \log\frac{2}{1-|\varphi(z)|^2}|(g\circ\varphi - h)'(z)| < \infty;$$

(c) when $0 < \alpha < 1$,

$$\|J[\varphi; g, h]\|_\beta \lesssim \|f\|_{\mathcal{B}^\alpha} \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(g\circ\varphi - h)'(z)| < \infty.$$

Therefore, we conclude that the difference operator $J[\varphi; g, h]$ is bounded by the estimations above and the boundedness of the point evaluation functional at 0.

To prove the necessity, we aim to find a contradiction if we suppose that the hypotheses do not hold. When $\alpha > 1$, there exists a sequence $\{z_n\} \subset \mathbb{D}$ satisfying

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha-1}}|(g\circ\varphi - h)'(z_n)| = \infty.$$

For $n \in \mathbb{N}$ and $z \in \mathbb{D}$, assume that the test function is

$$f_{n,1}(z) = \left(\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^2}\right)^{\alpha-1}. \tag{4.1}$$

An easy estimation shows that $\|f_{n,1}\|_\alpha \leq 4^\alpha(\alpha-1)$ and thus $f_{n,1} \in \mathcal{B}^\alpha$. By the boundedness of $J[\varphi; g, h]$, we have

$$\begin{aligned}\|J[\varphi; g, h]f_{n,1}\|_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f_{n,1}(\varphi(z))||(g\circ\varphi - h)'(z)| \\ &\geq \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha-1}}|(g\circ\varphi - h)'(z_n)| \to \infty\end{aligned}$$

as $n \to \infty$, which contradicts to our hypothesis. Assume that the test function is $f_{n,2}(z) = \log\frac{2(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^2}$ when $\alpha = 1$ and is $f_{n,3}(z) = 1$ when $0 < \alpha < 1$ respectively. Further observe the fact that $\|f_{n,2}\|_\alpha \leq 4$ and $\|f_{n,3}\|_\alpha \leq 2^{\alpha+2}$. To this end, we conclude the results in a similar way shown above. This completes the proof. □

Theorem 4.2. *$J[\varphi; g, h]$ is compact from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if $J[\varphi; g, h]$ is bounded and*
(a)

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi - h)'(z)| = 0$$

when $\alpha > 1$*;*
(b)
$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta \log\frac{2}{1-|\varphi(z)|^2}|(g\circ\varphi-h)'(z)| = 0$$
when $\alpha = 1$*;*
(c)
$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1-\alpha}}|(g\circ\varphi-h)'(z)| = 0$$
when $0 < \alpha < 1$.

Proof. The proof of the theorem is also essentially given in [7], but since there are some technical differences, we will give some details. To the sufficiency of the theorem, we present the proof of the hypothesis $\alpha > 1$. Assume that $\{f_n\}$ is bounded in $\mathcal{B}^\alpha$ and $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$, by Proposition 2.5 we are only supposed to check that
$$\lim_{n\to\infty}\|J[\varphi;g,h]f_n\|_{\mathcal{B}^\beta} = 0.$$
For convenience, suppose that there exists a positive number M_1 such that $\sup_{n\in\mathbb{Z}}\|f_n\|_{\mathcal{B}^\alpha} \leq M_1$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that
$$\sup_{|\varphi(z)|>1-\delta}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-h)'(z)| < \frac{\epsilon}{2M_1}$$
and
$$\sup_{|\varphi(z)|\leq 1-\delta}|f_n(\varphi(z))| < \frac{\epsilon}{2M_1}.$$
Furthermore, there exists another positive number M_2 such that
$$\sup_{|\varphi(z)|\leq 1-\delta}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-h)'(z)| < M_2.$$
Hence,
$$\begin{aligned}
&\|J[\varphi;g,h]f_n\|_\beta \\
&\leq \sup_{|\varphi(z)|\leq 1-\delta}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-h)'(z)|\cdot(1-|\varphi(z)|^2)^{\alpha-1}|f_n(\varphi(z))| \\
&+ \sup_{|\varphi(z)|>1-\delta}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-h)'(z)|\cdot(1-|\varphi(z)|^2)^{\alpha-1}|f_n(\varphi(z))| \\
&\lesssim M_2\frac{\epsilon}{2M_1} + \frac{\epsilon}{2M_1}M_1 \lesssim \epsilon.
\end{aligned}$$
Thus $J[\varphi;g,h]$ is compact from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ when $\alpha > 1$. Similarly, the compactness of $J[\varphi;g,h]$ can be obtained when $\alpha = 1$ and $0 < \alpha < 1$ respectively if the hypotheses hold.

To the necessity of the theorem, we firstly consider the situation when $\alpha > 1$. Like what we do in Theorem 4.1, we aim to find a contradiction if we suppose that the hypotheses do not hold. Thus there exists a sequence $\{z_n\} \subset \mathbb{D}$ such that for any $\epsilon > 0$,
$$\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha-1}}|(g\circ\varphi-h)'(z_n)| > \epsilon$$

whenever $|\varphi(z_n)| \to 1$. For $n \in \mathbb{Z}$ and $z \in \mathbb{D}$, assume that the same test function $f_{n,1}$ defined as (4.1) and it is easy to check that $\{f_{n,1}\} \to 0$ as $n \to \infty$ on any compact subset of $\mathbb{D}$. Then for the given $\epsilon > 0$ above, we have

$$\|J[\varphi; g, h]f_{n,1}\|_\beta \geq \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha-1}}|(g \circ \varphi - h)'(z_n)| > \epsilon$$

as $|\varphi(z_n)| \to 1$, which contradicts to our hypothesis.

When $0 < \alpha < 1$, we still aim to find a contradiction if we suppose that the hypotheses do not hold. Thus there exists a sequence $\{u_n\} \subset \mathbb{D}$ such that for any $\epsilon > 0$,

$$(1-|u_n|^2)^\beta |(g \circ \varphi - h)'(u_n)| > \epsilon$$

whenever $|\varphi(u_n)| \to 1$. For $n \in \mathbb{Z}$, let $\varphi(u_n) = r_n e^{i\theta_n}$. Assume that the test function

$$\tilde{f}_{n,3}(z) = r_n(1 - e^{-i\theta_n} r_n z)^{1-\alpha} - r_n^2(1 - e^{-i\theta_n} r_n^2 z)^{1-\alpha}.$$

Observe that $\|\tilde{f}_{n,3}\|_{\mathcal{B}^\alpha} \leq 4(1-\alpha)$ and

$$\begin{aligned}\|J[\varphi; g, h]\tilde{f}_{n,3}\|_\beta &\geq (1-|u_n|^2)^\beta |\tilde{f}_{n,3}(u_n)||(g \circ \varphi - h)'(u_n)| \\ &\geq |\tilde{f}_{n,3}(0)|(1-|u_n|^2)^\beta |(g \circ \varphi - h)'(u_n)| > r_n \epsilon,\end{aligned}$$

as $|\varphi(z_n)| \to 1$, which contradicts to our hypothesis.

Moreover, The result of $\alpha = 1$ can be obtained in a similar way by choosing the testing function which is different from $f_{n,2}$ in Theorem 4.1

$$\tilde{f}_{n,2}(z) = \frac{3(\log \frac{2}{1-\overline{\varphi(w_n)}z})^2}{\log \frac{2}{1-|\varphi(w_n)|^2}} - \frac{2(\log \frac{2}{1-\overline{\varphi(w_n)}z})^3}{(\log \frac{2}{1-|\varphi(w_n)|^2})^2}$$

where $\{w_n\} \subset \mathbb{D}$ is the sequence such that for any $\epsilon > 0$,

$$(1-|w_n|^2)^\beta \log \frac{2}{1-|\varphi(w_n)|^2} |(g \circ \varphi - h)'(w_n)| > \epsilon$$

whenever $|\varphi(w_n)| \to 1$. This completes the proof. □

Theorem 4.3. *Suppose that $\beta - \alpha \geq 0$, then $I[\varphi; g, h]$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*

$$\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)(g \circ \varphi - h)(z)| < \infty.$$

Proof. This theorem essentially follows from the proofs of some known results such as [5,7], but since there are some technical differences, we will give some details. The sufficiency is obvious. To prove the necessity, we aim to find a contradiction if we suppose that the hypotheses do not hold. Thus there exists a sequence $\{z_n\} \subset \mathbb{D}$ satisfying

$$\lim_{n \to \infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |\varphi'(z_n)(g \circ \varphi - h)(z_n)| = \infty.$$

For $n \in \mathbb{Z}$ and $z \in \mathbb{D}$, assume that the test function is

$$g_n(z) = \frac{1}{(2\alpha - 1)\overline{\varphi(z_n)}} \frac{(1 - |\varphi(z_n)|^2)^\alpha}{(1 - \overline{\varphi(z_n)}z)^{2\alpha-1}}. \tag{4.2}$$

An easy estimation shows that $\|g_n\|_\alpha \le 2^{2\alpha}$ and thus $g_n \in \mathcal{B}^\alpha$. By the boundedness of $I[\varphi; g, h]$, we have

$$\|I[\varphi; g, h]g_n\|_\beta \ge \frac{(1 - |z_n|^2)^\beta}{(1 - |\varphi(z_n)|^2)^\alpha} |\varphi'(z_n)(g \circ \varphi - h)(z_n)| \to \infty$$

as $n \to \infty$, which contradicts to our hypothesis. This completes the proof. □

Remark 4.4. We can observe that Theorem 4.3 holds for each $\alpha > 0$ and $\beta > 0$. However, if $\beta - \alpha < 0$, then $g \equiv 0$ by the maximal modulus principal, which can be simplified in the following.

Corollary 4.5. *Suppose that $\beta - \alpha < 0$, then $I[\varphi; g, h]$ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)h(z)| < \infty.$$

We just present the compactness of $I[\varphi; g, h]$ and omit the proof, which can be similarly proved by the method used in Theorem 4.2 with the same test function used in Theorem 4.3.

Theorem 4.6. *Suppose that $\beta - \alpha \ge 0$, then $I[\varphi; g, h]$ is compact from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if $I[\varphi; g, h]$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)(g \circ \varphi - h)(z)| = 0.$$

Corollary 4.7. *Suppose that $\beta - \alpha < 0$, then $I[\varphi; g, h]$ is compact from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ if and only if $I[\varphi; g, h]$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)h(z)| = 0.$$

5. Essential commutation of C_φ and J_g between the Bloch-type spaces

In this section, we concentrate on two questions of the compactly intertwining relations of J_g and C_φ.

Problem 5.1. What properties does a non-constant $g \in H(\mathbb{D})$ have if V_g essentially commutes with C_φ for all C_φ that are bounded on both $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$?

Problem 5.2. What properties does $\varphi \in S(\mathbb{D})$ have if the bounded V_g essentially commutes with C_φ for all C_φ that are bounded on both $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$?

We firstly answer the first problem. Recall that the notation $\Omega_{co}^{\alpha,\beta}(V_g)$ is denotes the collection of $g \in H(\mathbb{D})$ such that

- $V_g \in \mathcal{B}(\mathcal{B}^\alpha, \mathcal{B}^\beta)$.
- V_g are essentially commutative with C_φ for all φ such that C_φ is bounded on both $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$.

Theorem 5.3. $\Omega_{co}^{\alpha,\beta}(J_g) = \mathcal{B}^0_{\beta-\alpha+1}$ *if $\alpha > 1$ and $\beta - \alpha + 1 \ge 0$.*

Proof. In the proof we use the ideas in Theorem 5.1 in [22]. We firstly prove that $\mathcal{B}^0_{\beta-\alpha+1} \subset \Omega^{\alpha,\beta}_{co}(J_g)$. For any $g \in \mathcal{B}^0_{\beta-\alpha+1}$, obviously, J_g is bounded by Lemma 2.3. Furthermore, by the boundedness of C_φ on $\mathcal{B}^\beta$ (see Lemma 2.2),

$$\begin{aligned}
&\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-g)'(z)| \\
&\le \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|\cdot(1-|\varphi(z)|^2)^{\beta-\alpha+1}|g'(\varphi(z))| \\
&+ \frac{(1-|z|^2)^{\alpha-1}}{(1-|\varphi(z)|^2)^{\alpha-1}}\cdot(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \\
&\le \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|\cdot(1-|\varphi(z)|^2)^{\beta-\alpha+1}|g'(\varphi(z))| \\
&+ 2^{\alpha-1}\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{\alpha-1}\cdot(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \\
&\le C_1\cdot(1-|\varphi(z)|^2)^{\beta-\alpha+1}|g'(\varphi(z))| \\
&+ C_2\cdot(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$. Here, we use the following inequality (see, for example, Corollary 2.40 in [4])

$$\frac{1-|z|}{1-|\varphi(z)|} \le \frac{1+|\varphi(0)|}{1-|\varphi(0)|}, z \in \mathbb{D}.$$

This implies that $\mathcal{B}^0_{\beta-\alpha+1} \subset \Omega^{\alpha,\beta}_{co}(J_g)$ by Theorem 4.2.

Next we prove that $\Omega^{\alpha,\beta}_{co}(J_g) \subset \mathcal{B}^0_{\beta-\alpha+1}$. For any $g \in \Omega^{\alpha,\beta}_{co}(J_g)$, by Theorem 4.2, we have that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-g)'(z)| = 0.$$

Choose specifically $\varphi(z) = e^{i\theta}z \in S(\mathbb{D})$, it follows that,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^{\beta-\alpha+1}|e^{i\theta}g'(e^{i\theta}z) - g'(z)| = 0. \tag{5.1}$$

Note that

$$\begin{aligned}
&(1-|z|^2)^{\beta-\alpha+1}|e^{i\theta}g'(e^{i\theta}z) - g'(z)| \\
&\le (1-|e^{i\theta}z|^2)^{\beta-\alpha+1}|g'(e^{i\theta}z)| + (1-|z|^2)^{\beta-\alpha+1}|g'(z)| \le 2\|g\|_{\mathcal{B}^{\beta-\alpha+1}}.
\end{aligned}$$

If we assume that $g(z) = \sum_{n=0}^\infty a_n z^n$, then by integrating the left side of (5.2) with respect to θ from 0 to 2π, we obtain that

$$\begin{aligned}
0 &= \int_0^{2\pi} \lim_{|\varphi(z)|\to 1}(1-|z|^2)^{\beta-\alpha+1}|e^{i\theta}g'(e^{i\theta}z) - g'(z)|d\theta \\
&= \lim_{|\varphi(z)|\to 1}\int_0^{2\pi}(1-|z|^2)^{\beta-\alpha+1}|\sum_{n=1}^\infty na_n z^{n-1}(e^{in\theta}-1)|d\theta
\end{aligned}$$

$$\geq 2\pi \lim_{|z|\to 1}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|,$$

where the Dominant Convergent Theorem is applied in the second line. This implies that $g \in \mathcal{B}_0^{\beta-\alpha+1}$. This completes the proof. □

Corollary 5.4. $\Omega_{co}^{\alpha,\beta}(J_g) = \mathbb{C}$ *if* $\alpha > 1$ *and* $\beta - \alpha + 1 < 0$.

Proof. Obviously by the maximal modulus principle. □

Theorem 5.5. $\Omega_{co}^{\alpha,\beta}(J_g) = \mathcal{B}^{\beta}_{\log^1,0}$ *if* $\alpha = 1$.

Proof. For any $g \in \mathcal{B}^{\beta}_{\log^1,0}$, obviously, J_g is bounded. Furthermore, by the boundedness of C_φ on $\mathcal{B}^\beta$ and observing that $g \in \mathcal{B}^0_\beta$,

$$\begin{aligned}
&(1-|z|^2)^\beta \log\frac{2}{1-|\varphi(z)|^2}|(g\circ\varphi - g)'(z)|\\
&\leq \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|\cdot(1-|\varphi(z)|^2)^\beta \log\frac{2}{1-|\varphi(z)|^2}|g'(\varphi(z))|\\
&+\log\frac{2(1+|\varphi(0)|)}{1-|\varphi(0)|}(1-|z|^2)^\beta|g'(z)| + (1-|z|^2)^\beta\log\frac{2}{1-|z|^2}|g'(z)| \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$. This implies that $\mathcal{B}^{\beta}_{\log^1,0} \subset \Omega_{co}^{\alpha,\beta}(J_g)$ by Theorem 4.2.

Next we prove that $\Omega_{co}^{\alpha,\beta}(J_g) \subset \mathcal{B}^{\beta}_{\log^1,0}$. For any $g \in \Omega_{co}^{\alpha,\beta}(J_g)$, by Theorem 4.2, we have that

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta \log\frac{2}{1-|\varphi(z)|^2}|(g\circ\varphi-g)'(z)| = 0.$$

Choose specifically $\varphi(z) = e^{i\theta}z \in S(\mathbb{D})$, it follows that,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta \log\frac{2}{1-|z|^2}|e^{i\theta}g'(e^{i\theta}z) - g'(z)| = 0. \tag{5.2}$$

Note that

$$\begin{aligned}
&(1-|z|^2)^\beta \log\frac{2}{1-|z|^2}|e^{i\theta}g'(e^{i\theta}z) - g'(z)|\\
&\leq (1-|e^{i\theta}z|^2)^\beta \log\frac{2}{1-|e^{i\theta}z|^2}|g'(e^{i\theta}z)| + (1-|z|^2)^\beta\log\frac{2}{1-|z|^2}|g'(z)| \leq 2\|g\|_{\mathcal{B}^\beta_{\log^1}}.
\end{aligned}$$

If we assume that $g(z) = \sum_{n=0}^\infty a_n z^n$, then by integrating the left side of (5.2) with respect to θ from 0 to 2π, we obtain that

$$\begin{aligned}
0 &= \int_0^{2\pi}\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\log\frac{2}{1-|z|^2}|e^{i\theta}g'(e^{i\theta}z) - g'(z)|d\theta\\
&= \lim_{|\varphi(z)|\to 1}\int_0^{2\pi}(1-|z|^2)^\beta\log\frac{2}{1-|z|^2}|\sum_{n=1}^\infty na_nz^{n-1}(e^{in\theta}-1)|d\theta
\end{aligned}$$

$$\geq 2\pi \lim_{|z|\to 1}(1-|z|^2)^\beta \log \frac{2}{1-|z|^2}|g'(z)|,$$

where the Dominant Convergent Theorem is applied in the second line. This implies that $\mathcal{B}^\beta_{\log^1,0}$. This completes the proof. □

Theorem 5.6. $\Omega^{\alpha,\beta}_{co}(J_g) = \mathcal{B}^0_{\beta+\alpha-1}$ *when* $0 < \alpha < 1$.

Proof. We firstly prove that $\mathcal{B}^0_{\beta+\alpha-1} \subset \Omega^{\alpha,\beta}_{co}(J_g)$. For any $g \in \mathcal{B}^0_{\beta+\alpha-1}$, obviously, $g \in \mathcal{B}^0_\beta$ and hence J_g is bounded by Lemma 2.3. Furthermore, by the boundedness of C_φ on $\mathcal{B}^\beta$ (see Lemma 2.2), observe that

$$\begin{aligned}
&\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1-\alpha}}|(g\circ\varphi-g)'(z)| \\
&\leq \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|\cdot(1-|\varphi(z)|^2)^{\beta+\alpha-1}|g'(\varphi(z))| \\
&+\frac{(1-|z|^2)^{1-\alpha}}{(1-|\varphi(z)|^2)^{1-\alpha}}\cdot(1-|z|^2)^{\beta+\alpha-1}|g'(z)| \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$. The left part remains to be proved in a similar way from Theorem 5.3. This completes the proof. □

Proof. The proof is similar with Theorem 5.3. □

In the following, we partly answer the second problem. We only prove the first result and the other two results can be proved similarly.

Proposition 5.7. *If* $\alpha > 1$, $g \in \mathcal{B}^{\beta-\alpha+1}$,

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}\frac{1}{(1-\max\{|\varphi(z)|^2,|z|^2\})^{\beta-\alpha+1}} = 0$$

and φ *has finite angular derivative at any point of the unit circle, then* $C_\varphi \propto_K C_\varphi(J_g)$.

Proof. By Theorem 4.2, we only ought to check that

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-g)'(z)| = 0.$$

Since φ has finite angular derivative at any point of the unit circle, it follows that

$$\begin{aligned}
&\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|(g\circ\varphi-g)'(z)| \\
&\leq \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|\varphi'(z)|\frac{\|g\|_{\beta-\alpha+1}}{(1-|\varphi(z)|^2)^{\beta-\alpha+1}} \\
&+\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}\frac{\|g\|_{\beta-\alpha+1}}{(1-|z|^2)^{\beta-\alpha+1}} \\
&\lesssim \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}\frac{1}{(1-\max\{|\varphi(z)|^2,|z|^2\})^{\beta-\alpha+1}} \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$. □

Proposition 5.8. *If* $\alpha = 1$, $g \in \mathcal{B}^{\beta}_{\log^1}$,

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} \log \frac{1}{1-|\varphi(z)|^2}}{\max\{(1-|\varphi(z)|^2)^{\beta} \log \frac{2}{1-|\varphi(z)|^2}, (1-|z|^2)^{\beta} \frac{2}{1-|z|^2}\}} = 0$$

and φ *has finite angular derivative at any point of the unit circle, then* $C_{\varphi} \propto_K C_{\varphi}(J_g)$.

Proposition 5.9. *If* $0 < \alpha < 1$, $g \in \mathcal{B}^0_{\beta}$,

$$\lim_{|\varphi(z)|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$$

and φ *has finite angular derivative at any point of the unit circle, then* $C_{\varphi} \propto_K C_{\varphi}(J_g)$.

6. Essential commutation of C_{φ} and I_g between the Bloch-type spaces

In this section, we answer the two questions of the compactly intertwining relations of I_g and C_{φ} respectively.

Theorem 6.1. $\Omega^{\alpha,\beta}_{co}(I_g) = H^{\infty}_{\beta-\alpha,0}$ *if* $\beta - \alpha \geq 0$.

Proof. We only prove that $H^{\infty}_{\beta-\alpha,0} \subset \Omega^{\alpha,\beta}_{co}(I_g)$. For any $f \in H^{\infty}_{\beta-\alpha,0}$, obviously, I_g is bounded. Furthermore, by the boundedness of C_{φ},

$$\begin{aligned}
&\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha}} |(g\circ\varphi - g)(z)||\varphi'(z)| \\
&\leq \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\beta}} |\varphi'(z)| \cdot (1-|\varphi(z)|^2)^{\beta-\alpha}|g(\varphi(z))| \\
&+ \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha}} |\varphi'(z)| \cdot (1-|z|^2)^{\beta-\alpha}|g(z)| \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$, which implies that $H^{\infty}_{\beta-\alpha,0} \subset \Omega^{\alpha,\beta}_{co}(J_g)$. The left part remains to be proved in a similar way from Theorem 5.3. This completes the proof. □

Corollary 6.2. $\Omega^{\alpha,\beta}_{co}(I_g) = \{0\}$ *if* $\beta - \alpha < 0$.

Proposition 6.3. *If* $\beta - \alpha \geq 0$, $g \in H^{\infty}_{\beta-\alpha}$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha}} |\varphi'(z)| \frac{1}{(1-\max\{|\varphi(z)|^2, |z|^2\})^{\beta-\alpha}} = 0,$$

then $C_{\varphi} \propto_K C_{\varphi}(I_g)$.

Proof. The proof can be completed in a similar way from Proposition 5.7. □

Remark 6.4. Obviously, under the hypothesis of Proposition 6.3, we can further conclude that $C_{\varphi} \in \mathcal{B}(\mathcal{B}^{\alpha}, \mathcal{B}^{\alpha})$ and $C_{\varphi} \in \mathcal{B}(\mathcal{B}^{\beta}, \mathcal{B}^{\beta})$ are both compact linear operators.

Theorem 6.5. *If $\beta - \alpha \geq 0$, $g \in A(\mathbb{D})$, then $C_\varphi \propto_K C_\varphi(I_g)$ if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)(\varphi(z)-z)| = 0,$$

where $A(\mathbb{D})$ denotes the disk algebra.

Proof. The necessity is obvious by setting $g = Id$ in Theorem 4.6, where Id denoted the identity function. Next we prove the sufficiency. Suppose that $h_n(z) = z^n$, it follows that

$$\begin{aligned}
&\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)(h_n \circ \varphi - h_n)(z)| = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)||\varphi(z)^n - z^n| \\
&\leq n \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)(\varphi(z)-z)| \to 0
\end{aligned}$$

as $|\varphi(z)| \to 1$. For each $g \in A(\mathbb{D})$, there exists a subsequence of $\{h_n\}$, denoted by $\{h_n^{[g]}\}$ such that $\lim_{n\to\infty} h_n^{[g]} = g$. Thus,

$$\begin{aligned}
&\|C_\varphi I_g - I_g C_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \leq \|(C_\varphi I_g - I_g C_\varphi) - (C_\varphi I_{h_n^{[g]}} - I_{h_n^{[g]}} C_\varphi)\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&\leq (\|C_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\alpha} + \|C_\varphi\|_{\mathcal{B}^\beta\to\mathcal{B}^\beta})\|I_g - I_{h_n^{[g]}}\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&\leq (\|C_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\alpha} + \|C_\varphi\|_{\mathcal{B}^\beta\to\mathcal{B}^\beta}) \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \left\| \int_0^z f'(t)(g(t) - h_n^{[g]}(t))dt \right\|_{\mathcal{B}^\beta} \\
&\leq (\|C_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\alpha} + \|C_\varphi\|_{\mathcal{B}^\beta\to\mathcal{B}^\beta})\|g - h_n^{[g]}\|_\infty \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

7. Conclusions

Main conclusions are given in the following.

Theorem $\Omega_{co}^{\alpha,\beta}(J_g) = \mathcal{B}^0_{\beta-\alpha+1}$ if $\alpha > 1$ and $\beta - \alpha + 1 \geq 0$.

Theorem $\Omega_{co}^{\alpha,\beta}(J_g) = \mathcal{B}^\beta_{\log^1,0}$ if $\alpha = 1$.

Theorem $\Omega_{co}^{\alpha,\beta}(J_g) = \mathcal{B}^0_{\beta+\alpha-1}$ if $0 < \alpha < 1$.

Theorem $\Omega_{co}^{\alpha,\beta}(I_g) = H^\infty_{\beta-\alpha,0}$ if $\beta - \alpha \geq 0$.

Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. A. Biswas, A. Lambert, S. Petrovic, Extended eigenvalues and the Volterra operator, *Glasg. Math. J.*, **44** (2002), 521–534. https://doi.org/10.1017/S001708950203015X
2. P. S. Bourdon, J. H. Shapiro, Intertwining relations and extended eigenvalues for analytic Toeplitz operators, *Illinois J. Math.*, **52** (2008), 1007–1030. https://doi.org/10.1215/ijm/1254403728

3. R. E. Castillo, D. D. Clahane, J. F. Farías-López, J. C. Ramos-Fernández, Composition operators from logarithmic Bloch spaces to weighted Bloch spaces, *Appl. Math. Comput.*, **219** (2013), 6692–6706. https://doi.org/10.1016/j.amc.2012.11.091

4. C. C. Cowen, B. D. Maccluer, *Composition operators on spaces of analytic functions*, CRC Press, 1995.

5. S. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, **338** (2008), 1282–1295. https://doi.org/10.1016/j.jmaa.2007.06.013

6. S. Li, S. Stević, Riemann-Stieltjes type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Var. Elliptic Equ.*, **52** (2007), 495–517. https://doi.org/10.1080/17476930701235225

7. X. S. Li, S. Stević, Products of integral-type operators and composition operators between Bloch-type spaces, *J. Math. Anal. Appl.*, **349** (2009), 596–610. https://doi.org/10.1016/j.jmaa.2008.09.014

8. Y. Liu, Y. Yu, On a Li-Stević integral-type operator from Bloch-type spaces into logarithmic Bloch spaces, *Integr. Transf. Spec. F.*, **21** (2010), 93–103. https://doi.org/10.1080/10652460903047468

9. Y. X. Liang, Z. H. Zhou, The Products of differentiation and composition operators from logarithmic Bloch spaces to mu-Bloch Spaces, *B. Iran. Math. Soc.*, **46** (2020), 159–176. https://doi.org/10.1007/s41980-019-00248-w

10. S. Ohno, K. Stroethoff, R. H. Zhao, *Weighted composition operators between Bloch type spaces*, *Rocky MT J. Math.*, **33** (2003), 191–215. https://doi.org/10.1216/rmjm/1181069993

11. J. C. Ramos-Fernández, Logarithmic Bloch spaces and their weighted composition operators, *Rend. Circ. Mat. Palerm.*, **65** (2016), 159–174. https://doi.org/10.1007/s12215-015-0226-6

12. S. Stević, R. P. Agarwal, Weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces, *J. Inequal. Appl.*, 2009, 964814. https://doi.org/10.1155/2009/964814

13. S. Stević, On new Bloch-type spaces, *Appl. Math. Comput.*, **215** (2009), 841–849. https://doi.org/10.1016/j.amc.2009.06.009

14. S. Stević, On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Anal.-Theor.*, **71** (2009), 6323–6342. https://doi.org/10.1016/j.na.2009.06.087

15. S. Stević, Norm and essential norm of an integral-type operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, *Math. Method. Appl. Sci.*, 2022, 1–11. https://doi.org/10.1002/mma.8487

16. S. Stević, Z. J. Jiang, Weighted iterated radial composition operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *Math. Method. Appl. Sci.*, **45** (2022), 3083–3097. https://doi.org/10.1002/mma.7978

17. S. Stević, On an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball, *Appl. Math. Comput.*, **215** (2010), 3817–3823. https://doi.org/10.1016/j.amc.2009.11.022

18. S. Stević, On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball, *Appl. Math. Comput.*, **215** (2010), 4248–4255. https://doi.org/10.1016/j.amc.2009.12.048

19. S. Stević, Weighted composition operators from the logarithmic weighted-type space to the weighted Bergman space in $\mathbb{C}^n$, *Appl. Math. Comput.*, **216** (2010), 924–928.

20. S. Stević, Norm of some operators from logarithmic Bloch-type spaces to weighted-type spaces, *Appl. Math. Comput.*, **218** (2012), 11163–11170. https://doi.org/10.1016/j.amc.2012.04.073

21. C. Z. Tong, Z. H. Zhou, Compact intertwining relations for composition operators between the weighted Bergman spaces and the weighted Bloch spaces, *J. Korean Math. Soc.*, **51** (2014), 125–135. https://doi.org/10.4134/JKMS.2014.51.1.125

22. C. Z. Tong, Z. H. Zhou, Intertwining relations for Volterra operators on the Bergman space, *Illinois J. Math.*, **57** (2013), 195–211. https://doi.org/10.1215/ijm/1403534492

23. C. Z. Tong, C. Yuan, Z. H. Zhou, Compact intertwining relations for composition operators on H^∞ and the Bloch spaces, *New York J. Math.*, **24** (2018), 611–629.

24. K. Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics 226, Springer, New York, 2005.

25. Z. H. Zhou, L. Zhang, H. G. Zeng, Essential commutativity of some integral and composition operators, *Bull. Aust. Math. Soc.*, **85** (2012), 143–153. https://doi.org/10.1017/S0004972711002723