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Research article

On irresolute multifunctions and related topological games

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Abstract: In this paper, we introduce and study α-irresolute multifunctions, and some of their properties are studied. The properties of α-compactness and α-normality under upper α-irresolute multifunctions are topological properties. Also, we prove that the composition of two upper and lower α-irresolute multifunctions is α-irresolute. We apply the results of α-irresolute multifunctions to topological games. Upper and lower topological games are introduced. The set of places for player ONE in upper topological games may guarantee a gain is semi-closed. Finally, some optimal strategies for topological games are defined and studied.

Keywords: multifunctions; winning strategy; topological games; upper and lower irresolute multifunctions
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1. Introduction and preliminaries

Topological games (TGs, for short) with perfect information were introduced and studied by Berge [3]. Many authors have them to solve some topological problems (e.g., [35, 36]). For further details, see [3, 4, 23, 27]. TGs have been extended to topological spaces [9, 10, 12, 13] and their applications. The continuity on multifunctions is studied in [14]. Pears [32] has defined and studied TGs for continuous multifunctions. Recently, topological spaces have been used in applications to study graphs in [15–18, 22, 24] which are used in physics [8, 11, 19, 20] and smart cities [2].

Topologically, $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces (TSs, for short). A multifunction of $\mathcal{X}$ into $\mathcal{Y}$ is defined as a function $F : \mathcal{X} \to 2^{\mathcal{Y}}$, where $2^{\mathcal{Y}}$ is the power set of $\mathcal{Y}$. Additionally, $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$ [25] if $\exists$ an open set $\mathcal{U}$ of $\mathcal{X}$ s.t. $\mathcal{U} \subseteq \mathfrak{A} \subseteq Cl(\mathcal{U})$, where $Cl(\mathcal{U})$ is the closure of $\mathcal{U}$ w.r. to $\mathcal{X}$. Ewert and Lipski [21] introduced the concept of irresolute multifunctions. Papa and Noiri [33] further studied irresolute multifunctions. For $\mathfrak{A} \subseteq \mathcal{X}$, the interior of $\mathfrak{A}$ will be denoted by $Int(\mathfrak{A})$. Multifunctionally, the upper and lower inverses of $F : \mathcal{X} \to \mathcal{Y}$ are $F^{+}(\mathfrak{B}) = \{\mathfrak{x} \in \mathcal{X} : F(\mathfrak{x}) \subseteq \mathfrak{B}\}$ and $F^{-}(\mathfrak{B}) = \{\mathfrak{x} \in \mathcal{X} : F(\mathfrak{x}) \cap \mathfrak{B} \neq \phi\}$, respectively.

Throughout the present paper, some new properties of upper (lower) α-irresolute multifunctions due to Neubrunn [30] and Noiri and Nasef [31] are modified and studied. Also, we apply the results to introduce and study new types of TGs for irresolute multifunctions, such as locally finite games, upper and lower TGs and optimal strategies for TGs.

Here, the class of semi-open sets of $\mathcal{X}$ is named $\mathcal{SO}(\mathcal{X})$, and $\mathcal{SO}(\mathcal{X}, \mathfrak{x})$ is all semi-open sets of $\mathcal{X}$ containing $\mathfrak{x} \in \mathcal{X}$. Its complement is called semi-closed [5] and named $\mathcal{SC}(\mathcal{X})$. $\mathfrak{A} \subseteq \mathcal{X}$ is α-open [29] if $\mathfrak{A} \subseteq Int(Cl(Int(\mathfrak{A})))$. The class of all α-open sets is denoted by $\alpha O(\mathcal{X})$. Its complement is α-closed and is denoted by $\alpha C(\mathcal{X})$.

Definition 1.1. *[21] A multifunction $F : X \to \mathcal{Y}$ is*

(a) upper irresolute (resp. lower irresolute) at $\mathfrak{x} \in \mathcal{X}$ if $\forall\, \mathcal{V} \in \mathcal{SO}(\mathcal{Y})$ s.t. $F(\mathfrak{x}) \subseteq \mathcal{V}$ (resp. $F(\mathfrak{x}) \cap \mathcal{V} \neq \phi$), $\exists\, \mathcal{U} \in \mathcal{SO}(\mathcal{X}, \mathfrak{x})$ s.t. $F(u) \subseteq \mathcal{V}$ (resp. $F(u) \cap \mathcal{V} \neq \phi$), $\forall\, u \in \mathcal{U}$.

(b) upper irresolute (resp. lower irresolute) if it is upper irresolute (resp. lower irresolute) at all $\mathfrak{x} \in \mathcal{X}$.

Definition 1.2. *[6] A subset $\mathfrak{A}$ of $(\mathcal{X}, \tau)$ is semi-comp (s-comp, for short) if every cover of $\mathfrak{A}$ by $\mathcal{SO}(\mathcal{X}, \tau)$ has a finite subcover.*

Lemma 1.1. *[1] For a subset $\mathfrak{A}$ of X, $\alpha Cl(\mathfrak{A}) = \mathfrak{A} \cup \tau - Cl(\tau - Int(\tau - Cl(\mathfrak{A})))$.*

2. α-irresolute multifunctions and their properties

Numerous characterizations of upper (resp. lower) α-irresolute functions have been published in the literature [4, 30, 31], and we add a few more.

Definition 2.1. *A multifunction $\mathfrak{F} : (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is called α-irresolute at $\mathfrak{x} \in \mathcal{X}$ if $\forall$ pairs $\mathfrak{W}_i \in \alpha O(\mathcal{Y}, \sigma)$, $i = 1, 2$, s.t. $\mathfrak{F}(\mathfrak{x}) \subseteq \mathfrak{W}_1$ and $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$, $\exists\, \mathfrak{H} \in \alpha(\mathcal{X}, \mathfrak{x})$ with $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{W}_1$ s.t. $\mathfrak{F}(h) \cap \mathfrak{W}_2 \neq \phi$ $\forall\, h \in \mathfrak{H}$.*

Thus, $\mathfrak{F} : (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is α-irresolute if it exhibits the aforementioned quality at each $\mathfrak{x} \in \mathcal{X}$.

Theorem 2.1. *The following are equivalent:*

(i) $\mathfrak{F}$ is α-irresolute at $\mathfrak{x} \in \mathcal{X}$;

(ii) for any $\mathfrak{W}_1, \mathfrak{W}_2 \in \alpha O(\mathcal{Y}, \sigma)$ s.t. $\mathfrak{F}(\mathfrak{x}) \subseteq \mathfrak{W}_1$ and $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$, we get $\mathfrak{x} \in \tau - Int(\tau - Cl(\tau - Int[\mathfrak{F}^{+}(\mathfrak{W}_1) \cap \mathfrak{F}^{-}(\mathfrak{W}_2)]))$;

(iii) $\forall\, \mathfrak{W}_1, \mathfrak{W}_2 \in \alpha O(\mathcal{Y}, \sigma)$ with $\mathfrak{F}(\mathfrak{x}) \subseteq \mathfrak{W}_1$, $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$ and for any open set $\mathcal{U} \subseteq \mathcal{X}$ having $\mathfrak{x}$, $\exists$ a nonempty open set $\mathcal{G}$ of $\mathcal{X}$ with $\mathcal{G} \subseteq \mathcal{U}$, $\mathfrak{F}(\mathcal{G}) \subset \mathfrak{W}_1$ and $\mathfrak{F}(g) \cap \mathfrak{W}_2 \neq \phi$ $\forall\, g \in \mathcal{G}$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{W}_i \in \alpha O(\mathcal{Y}, \sigma)$, $i = 1, 2$, s.t $\mathfrak{F}(\mathfrak{x}) \subseteq \mathfrak{W}_1$ and $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$. By assumption, $\exists$ $\mathfrak{H} \in \alpha O(\mathcal{Y}, \mathfrak{x})$ s.t. $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{W}_1$ and $\mathfrak{F}(h) \cap \mathfrak{W}_2 \neq \phi$ $\forall\, h \in \mathfrak{H}$. So, $\mathfrak{x} \in \mathfrak{H} \subseteq \mathfrak{F}^{+}(\mathfrak{W}_1)$ and, $\mathfrak{x} \in \mathfrak{H} \subseteq \mathfrak{F}^{-}(\mathfrak{W}_2) \neq \phi$. Hence, $\mathfrak{x} \in \mathfrak{H} \subseteq \mathfrak{F}^{+}(\mathfrak{W}_1) \cap \mathfrak{F}^{-}(\mathfrak{W}_2)$. Since $\mathfrak{H}$ is α-open in $\mathcal{X}$, $\mathfrak{x} \in \mathfrak{H} \subseteq \tau - Int(\tau - Cl(\tau - Int(\mathfrak{H}))) \subseteq \tau - Int(\tau - Cl(\tau - Int[\mathfrak{F}^{+}(\mathfrak{W}_1) \cap \mathfrak{F}^{-}(\mathfrak{W}_2)]))$.

(ii) $\Rightarrow$ (iii): Let $\mathfrak{W}_i \in \alpha O(\mathcal{Y}, \sigma)$, $\mathfrak{F}(\mathfrak{x}) \subseteq \mathfrak{W}_1$, and $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$. However, *(ii)* gives $\mathfrak{x} \in \tau - Int(\tau - Cl(\tau - Int[\mathfrak{F}^{+}(\mathfrak{W}_1) \cap \mathfrak{F}^{-}(\mathfrak{W}_2)]))$. Also, let $\mathcal{U} \neq \phi$ containing x. Then, $\mathcal{U} \cap [\tau - Int[\mathfrak{F}^{+}(\mathfrak{W}_1) \cap \mathfrak{F}^{-}(\mathfrak{W}_2)]] \subseteq \mathcal{U} \cap [\tau - Int\mathfrak{F}^{+}(\mathfrak{W}_1) \cap \tau - Int\mathfrak{F}^{-}(\mathfrak{W}_2)] = \mathcal{G}$, which is open, and $\mathcal{G} \subseteq \tau - Int\mathfrak{F}^{+}(\mathfrak{W}_1)$. Also, $\mathcal{G} \subseteq \tau - Int\mathfrak{F}^{-}(\mathfrak{W}_2) \subseteq \mathfrak{F}^{-}(\mathfrak{W}_2)$, so $\mathfrak{F}(\mathcal{G}) \subseteq \mathfrak{W}_1$ and $\mathfrak{F}(g) \cap \mathfrak{W}_2 \neq \phi$ $\forall\, g \in \mathcal{G}$.

(iii) $\Rightarrow$ (i): This follows immediately from the observation $\tau(\mathfrak{x}) \subseteq \alpha(\mathcal{X}, \mathfrak{x})$.

Theorem 2.2. *The following are equivalent:*

(i) $\mathfrak{F}$ *is* α*-irresolute;*

(ii) for any $\mathfrak{W}_1, \mathfrak{W}_2 \in \alpha O(\mathcal{Y}, \sigma)$*,* $\mathfrak{F}^+(\mathfrak{W}_1) \cap \mathfrak{F}^-(\mathfrak{W}_2) \in \alpha O(\mathcal{X}, \tau)$*;*

(iii) $\forall$ α*-closed sets* $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{Y}$*,* $\mathfrak{F}^-(\mathcal{K}_1) \cup \mathfrak{F}^+(\mathcal{K}_2)$ *is* α*-closed;*

(iv) $\forall$ $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathcal{Y}$, $\tau - Cl(\tau - Int(\tau - Cl[\mathfrak{F}^-(\mathfrak{B}_1) \cup \mathfrak{B}^+(\mathfrak{B}_2)])) \subseteq \mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))$*;*

(v) $\alpha Cl[\mathfrak{F}^-(\mathfrak{B}_1) \cup \mathfrak{F}^+(\mathfrak{B}_2)] \subseteq \mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2)$ *for any* $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathcal{Y}$*;*

(vi) $\mathfrak{F}^-(\alpha Int(\mathfrak{B}_1)) \cap \mathfrak{F}^+(\alpha Int(\mathfrak{B}_2)) \subseteq \alpha Int[\mathfrak{F}^-(\mathfrak{B}_1) \cap \mathfrak{F}^+(\mathfrak{B}_2)]$ *for any* $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathcal{Y}$*;*

(vii) for $\mathfrak{x} \in \mathcal{X}$ *and* $\forall$ α*-nbd* $\mathfrak{N}$ *of* $\mathfrak{F}(\mathfrak{x})$*, then for every* $\mathfrak{W} \in \alpha O(\mathcal{Y}, \sigma)$ *s.t.* $\mathfrak{W} \cap \mathfrak{F}(\mathfrak{x}) \neq \phi$, $\mathfrak{F}^+(\mathfrak{N}) \cap \mathfrak{F}^-(\mathfrak{W})$ *is an* α*-nbd of* $\mathfrak{x}$*;*

(viii) Let for any $\mathfrak{x} \in \mathcal{X}$ *and* $\forall$ α*-nbd* $\mathfrak{N}$ *of* $\mathfrak{F}(\mathfrak{x})$*. Then, for every* $\mathfrak{W} \in \alpha O(\mathcal{Y}, \sigma)$ *s.t.* $\mathfrak{W} \cap \mathfrak{F}(x) \neq \phi$, $\exists$ α*-nbd* $\mathcal{U}$ *of* $\mathfrak{x}$ *s.t.* $\mathfrak{F}(\mathcal{U}) \subseteq \mathfrak{N}$ *and* $\mathfrak{F}(u) \cap \mathfrak{W} \neq \phi$ $\forall$ $u \in \mathcal{U}$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{x} \in \mathfrak{F}^+(\mathfrak{W}_1) \cap \mathfrak{F}^-(\mathfrak{W}_2)$ for any $\mathfrak{W}_1, \mathfrak{W}_2 \in \alpha O(\mathcal{Y}, \sigma)$. Then, $\mathfrak{F}(\mathfrak{x}) \in \mathfrak{W}_1$ and $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W}_2 \neq \phi$. Since $\mathfrak{F}$ is α-irresolute, by Theorem 2.1, $\mathfrak{x} \in \tau - Int(\tau - Cl(\tau - Int[\mathfrak{F}^+(\mathfrak{W}_1) \cap \mathfrak{F}^-(\mathfrak{W}_2)]))$.

(ii) $\Rightarrow$ (iii): Immediately from that, if $\mathcal{V} \subseteq \mathcal{Y}$, then $\mathfrak{F}^-(\mathcal{Y} - \mathcal{V}) = \mathcal{X} - \mathfrak{F}^+(\mathcal{V})$, and $\mathfrak{F}^+(\mathcal{Y} - \mathcal{V}) = \mathcal{X} - \mathfrak{F}^-(\mathcal{V})$.

(iii) $\Rightarrow$ (iv): Let $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathcal{Y}$. Then, $\alpha Cl(B_i) \in \alpha C(\mathcal{Y})$, $\forall$ $i = 1, 2$, where $\alpha C(\mathcal{Y})$ will denote to the class of α closed sets of $\mathcal{Y}$. By *(iii)*, $\mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2)) \in \alpha C(\mathcal{X}, \tau)$, i.e., $\tau - Cl(\tau - Int(\tau - Cl[\mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))])) \subseteq \mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))$, since $\mathfrak{F}^+(\mathfrak{B}_2) \subseteq \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))$ and $\mathfrak{F}^-(\mathfrak{B}_1) \subseteq \mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1))$. Consequently, $\tau - Cl(\tau - Int(\tau - Cl[\mathfrak{F}^-(\mathfrak{B}_1) \cup \mathfrak{F}^+(\mathfrak{B}_2)])) \subseteq \tau - Cl(\tau - Int(\tau - Cl[\mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))])) \subseteq \mathfrak{F}^-(\alpha Cl(\mathfrak{B}_1)) \cup \mathfrak{F}^+(\alpha Cl(\mathfrak{B}_2))$.

(iv) $\Rightarrow$ (v): Directly by Lemma 1.1.

(v) $\Rightarrow$ (vi): $\mathcal{X} - \alpha Int[\mathfrak{F}^-(\mathfrak{B}_1) \cap \mathfrak{F}^+(\mathfrak{B}_2)] \subseteq \alpha Cl[\mathcal{X} - (\mathfrak{F}^-((\mathfrak{B}_1) \cap \mathfrak{F}^+(\mathfrak{B}_2)] = \alpha Cl[(\mathcal{X} - \mathfrak{F}^-(\mathfrak{B}_1)) \cup (\mathcal{X} - \mathfrak{F}^+(\mathfrak{B}_2))] = \alpha Cl[\mathfrak{F}^+(\mathcal{Y} - \mathfrak{B}_1) \cup \mathfrak{F}^-(\mathcal{Y} - \mathfrak{B}_2) \subseteq \mathfrak{F}^+(\alpha Cl(\mathcal{Y} - \mathfrak{B}_1)) \cup \mathfrak{F}^-(\alpha Cl(\mathcal{Y} - \mathfrak{B}_2)) = \mathfrak{F}^+(\mathcal{Y} - \alpha Int(\mathfrak{B}_1)) \cup \mathfrak{F}^-(\mathcal{Y} - \alpha Int(\mathfrak{B}_2)) = (\mathcal{X} - \mathfrak{F}^-(\alpha Int(\mathfrak{B}_1)) \cup (\mathcal{X} - \mathfrak{F}^+(\alpha Int(\mathfrak{B}_2)) = \mathcal{X} - [\mathfrak{F}^-(\alpha Int(\mathfrak{B}_1)) \cap \mathfrak{F}^+(\alpha Int(\mathfrak{B}_2))]$. Therefore, $\alpha Int[\mathfrak{F}^-(\mathfrak{B}_1) \cap \mathfrak{F}^+(\mathfrak{B}_2)] \supseteq [\mathfrak{F}^-(\alpha Int(\mathfrak{B}_1) \cap \mathfrak{F}^+(\alpha Int(\mathfrak{B}_2)]$.

(vi) $\Rightarrow$ (vii): Let $\mathfrak{x} \in \mathcal{X}$, $\mathfrak{N}$ be an α-nbd of $\mathfrak{F}(\mathfrak{x})$, and $\mathfrak{W} \in \alpha O(\mathcal{Y})$ with $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W} \neq \phi$. Then, $\exists$ $\mathcal{U}_1, \mathcal{U}_2 \in \alpha O(\mathcal{Y})$ s.t. $\mathcal{U}_1 \subseteq \mathfrak{N}$, $\mathcal{U}_2 \subseteq \mathfrak{W}$, $\mathfrak{F}(\mathfrak{x}) \subseteq \mathcal{U}_1$ and $\mathfrak{F}(\mathfrak{x}) \cap \mathcal{U}_2 \neq \phi$. Thus, $\mathfrak{x} \in \mathfrak{F}^+(\mathcal{U}_1) \cap \mathfrak{F}^-(\mathcal{U}_2)$. By assumption, $\mathfrak{x} \in \mathfrak{F}^+(\mathcal{U}_1) \cap \mathfrak{F}^-(\mathcal{U}_2) = F^+(\alpha Int(\mathcal{U}_1)) \cap \mathfrak{F}^-(\alpha Int(\mathcal{U}_2)) \subseteq \alpha Int[\mathfrak{F}^+(\mathcal{U}_1) \cap \mathfrak{F}^-(\mathcal{U}_2)] \subseteq \alpha Int[\mathfrak{F}^+(\mathfrak{N}) \cap \mathfrak{F}^-(\mathfrak{W})] \subseteq \mathfrak{F}^+(\mathfrak{N}) \cap \mathfrak{F}^-(\mathfrak{W})$. It follows that $\mathfrak{F}^+(\mathfrak{N}) \cap \mathfrak{F}^-(\mathfrak{W})$ is an α-nbd of $\mathfrak{x}$.

(vii) $\Rightarrow$ (viii): Let $\mathfrak{x} \in \mathcal{X}$, $\mathfrak{N}$ be an α-nbd of $\mathfrak{F}(\mathfrak{x})$ and $\mathfrak{W} \in \alpha O(\mathcal{Y}, \sigma)$ with $\mathfrak{F}(x) \cap \mathfrak{W} \neq \phi$. Then, $\mathcal{U} = \mathfrak{F}^+(\mathfrak{N}) \cap \mathfrak{F}^-(\mathfrak{W})$ is an α-nbd of $\mathfrak{x}$, $\mathfrak{F}(\mathcal{U}) \subseteq \mathfrak{N}$, and $\mathfrak{F}(u) \cap \mathfrak{W} \neq \phi$ $\forall$ $u \in \mathcal{U}$.

(viii) $\Rightarrow$ (i): Clear by given hypothesis.

Noiri and Nasef [31] provided the following definitions of upper and lower α-irresoluteness.

Theorem 2.3. *The following are equivalent:*

(1) $\mathfrak{F}$ *is upper (resp. lower)* α*-irresolute;*

(2) $\mathfrak{F}^+(\mathfrak{W})$ *(resp.* $\mathfrak{F}^-(\mathfrak{W})$*)* $\in \alpha O(\mathcal{X}, \tau)$, $\forall$ $\mathfrak{W} \in \alpha O(\mathcal{Y}, \sigma)$*;*

(3) $\mathfrak{F}^{-}(\mathfrak{K})$ *(resp.* $\mathfrak{F}^{+}(\mathfrak{K})$*)* $\in \alpha C(\mathcal{X},\tau)\ \forall\ K \in \alpha C(\mathcal{Y},\sigma)$*;*

(4) $sInt(Cl(\mathfrak{F}^{-}(\mathfrak{B}))) \subseteq \mathfrak{F}^{-}(\alpha Cl(\mathfrak{B}))$ *(resp.* $sInt(Cl(\mathfrak{F}^{+}(\mathfrak{B}))) \subseteq \mathfrak{F}^{+}(\alpha Cl(\mathfrak{B}))\ \forall\ \mathfrak{B} \subseteq \mathcal{Y}$*;*

(5) $\alpha Cl(\mathfrak{F}^{-}(\mathfrak{B})) \subseteq F^{-}(\alpha Cl(\mathfrak{B}))$ *(resp.* $\alpha Cl(\mathfrak{F}^{+}(\mathfrak{B})) \subseteq \mathfrak{F}^{+}(\alpha Cl(\mathfrak{B}))\ \forall\ \mathfrak{B} \subseteq \mathcal{Y}$*.*

Theorem 2.4. *The following are equivalent:*

(1) $\mathfrak{F}$ *is lower* α*-irresolute;*

(2) $\mathfrak{F}(\tau - Cl(\tau - Int(\tau - Cl(\mathfrak{H})))) \subseteq \mathfrak{F}(\mathfrak{H})\ \forall\ \mathfrak{H} \in \alpha O(\mathcal{X},\tau)$*;*

(3) $\mathfrak{F}(\alpha Cl(\mathfrak{H})) \subseteq \mathfrak{F}(\mathfrak{H})\ \forall\ \mathfrak{H} \in \alpha O(\mathcal{X},\tau)$*.*

Proof. (1) $\Leftrightarrow$ (2): By comparison with Theorem 2.3 and $\mathfrak{W} = \mathfrak{F}(\mathfrak{H})$, the proof is followed.

(2) $\Rightarrow$ (3): Follows using Lemma 1.1.

(3) $\Rightarrow$ (1): Let $\mathfrak{x} \in \mathcal{X}$ and $\mathfrak{W} \in \alpha O(\mathcal{Y})$ with $\mathfrak{F}(\mathfrak{x}) \cap \mathfrak{W} \neq \phi$. Then, $\mathfrak{x} \in \mathfrak{F}^{-}(\mathfrak{W})$. By (iii), $\mathfrak{F}(\alpha Cl(\mathfrak{F}^{+}(\mathcal{Y}-\mathfrak{W}))) \subseteq \mathfrak{F}(\mathfrak{F}^{+}(\mathcal{Y} - \mathfrak{W})) \subseteq \mathcal{Y} - \mathfrak{W}$. So, $\alpha Cl(\mathfrak{F}^{+}(\mathcal{Y} - \mathfrak{W})) \subseteq \mathfrak{F}^{+}(\mathcal{Y} - \mathfrak{W})$. Hence, $\mathfrak{F}^{+}(\mathcal{Y} - \mathfrak{W}) \in \alpha C(\mathcal{X},\tau)$, and then $\mathfrak{F}^{-}(\mathfrak{W}) \in \alpha O(\mathcal{X})$. Set $\mathfrak{H} = \mathfrak{F}^{-}(\mathfrak{W})$, $\mathfrak{H} \in \alpha(\mathcal{X},\mathfrak{x})$, and $\mathfrak{F}(h) \cap \mathfrak{W} \neq \phi\ \forall\ h \in \mathfrak{H}$. Therefore, $\mathfrak{F}$ is lower α-irresolute.

Lemma 2.1. *[33] For all* $\mathcal{V} \in O(\mathcal{Y})$*,* $(\alpha Cl\mathfrak{F})^{-}(\mathcal{V}) = \mathfrak{F}^{-}(\mathcal{V})$*.*

Proof. Let $\mathcal{V} \in O(\mathcal{Y})$ and $\mathfrak{x} \in (\alpha Cl\mathfrak{F})^{-}(\mathcal{V})$. Then, $(\alpha Cl\mathfrak{F})(\mathfrak{x}) \cap \mathcal{V} = \alpha Cl(\mathfrak{F}(\mathfrak{x})) \cap \mathcal{V} \neq \phi$, and so $\mathfrak{F}(\mathfrak{x}) \cap \mathcal{V} = \neq \phi$. By openness of $\mathcal{V}$, $\mathfrak{x} \in \mathfrak{F}^{-}(\mathcal{V})$, and so $(\alpha Cl\mathfrak{F})^{-}(\mathcal{V}) \subseteq \mathfrak{F}^{-}(\mathcal{V})$. On the other side, let $\mathfrak{x} \in \mathfrak{F}^{-}(\mathcal{V})$. Then, $\phi \neq \mathfrak{F}(\mathfrak{x}) \cap \mathcal{V} \subseteq (\alpha Cl\mathfrak{F})(\mathfrak{x}) \cap \mathcal{V}$, and so $\mathfrak{x} \in (\alpha Cl\mathfrak{F})^{-}(\mathcal{V})$. Thus, we get $\mathfrak{F}^{-}(\mathcal{V}) \subseteq (\alpha Cl\mathfrak{F})^{-}(\mathcal{V})$. Therefore, $(\alpha Cl\mathfrak{F})^{-}(\mathcal{V}) = \mathfrak{F}^{-}(\mathcal{V})$.

Theorem 2.5. $\mathfrak{F} : \mathcal{X} \to \mathcal{Y}$ *is lower* α*-irresolute iff* $\alpha Cl\mathfrak{F} : \mathcal{X} \to \mathcal{Y}$ *is so.*

Proof. "$\Rightarrow$", let $\mathfrak{F}$ be lower α-irresolute, $\mathcal{V} \in O(\mathcal{Y})$ s.t. $(\alpha Cl\mathfrak{F})(\mathfrak{x}) \cap \mathcal{V} \neq \phi$, for $\mathfrak{x} \in \mathcal{X}$. By Lemma 2.1, $\mathfrak{x} \in (\alpha Cl\mathfrak{F})^{-}(\mathcal{V}) = \mathfrak{F}^{-}(\mathcal{V})$, and so $\mathfrak{F}(\mathfrak{x}) \cap \mathcal{V} \neq \phi$. By assumption of $\mathfrak{F}$, $\exists\ \mathcal{U} \in \alpha(\mathcal{X},\mathfrak{x})$ s.t. $\mathcal{U} \subseteq \mathfrak{F}^{-}(\mathcal{V}) = (\alpha Cl\mathfrak{F})^{-}(\mathcal{V})$. Hence, $\alpha Cl\mathfrak{F}$ is lower α-irresolute. "$\Leftarrow$", let αClF be lower α-irresolute, $\mathfrak{x} \in \mathcal{X}$, and $\mathcal{V} \in O(\mathcal{Y})$ with $\mathfrak{F}(\mathfrak{x}) \cap \mathcal{V} \neq \phi$. By Lemma 2.1, $\mathfrak{x} \in \mathfrak{F}^{-}(\mathcal{V}) = (\alpha Cl\mathfrak{F})^{-}(\mathcal{V})$. By assumption, $\exists\ \mathcal{U} \in \alpha(\mathcal{X},\mathfrak{x})$ s.t. $\mathcal{U} \subseteq (\alpha Cl\mathfrak{F})^{-}(\mathcal{V}) = \mathfrak{F}^{-}(\mathcal{V})$.

3. Some miscellaneous results

The following lemma was shown by Mashhour [26] and Rielly and Vamanamurthly [34]. A subset $\mathfrak{A}$ is γ-open [7] if $\mathfrak{A} \subseteq Int(Cl(\mathfrak{A})) \cup Cl(Int(\mathfrak{A}))$. The class of all γ-open sets is denoted by $\gamma O(\mathcal{X})$. It is noted that $\mathcal{SO}(X) \cup \mathcal{PO}(X) \subseteq \gamma O(\mathcal{X})$.

Lemma 3.1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be subsets of a TS* $(\mathcal{X},\tau)$*. Then*

(i) If $\mathfrak{A} \in \gamma O(\mathcal{X})$ *and* $\mathfrak{B} \in \alpha O(\mathcal{X})$*, then* $\mathfrak{A} \cap \mathfrak{B} \in \alpha O(\mathfrak{A})$*.*

(ii) If $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathcal{X}$*,* $\mathfrak{A} \in \alpha O(\mathfrak{B})$*, and* $\mathfrak{B} \in \alpha O(\mathcal{X})$*, then* $\mathfrak{A} \in \alpha O(\mathcal{X})$*.*

Theorem 3.1. *Let* $\mathfrak{F} : (\mathcal{X},\tau) \to (\mathcal{Y},\sigma)$ *be upper (resp. lower)* α*-irresolute, and* $\mathcal{X}_0 \in \gamma O(X,\tau)$*. Then, the restriction* $\mathfrak{F} \mid \mathcal{X}_0 : (\mathcal{X}_0, \tau \mid \mathcal{X}_0) \to (\mathcal{Y},\sigma)$ *is upper (resp. lower)* α*-irresolute.*

Proof. Let $\mathfrak{x} \in \mathcal{X}_0$ and $V \in \alpha O(\mathcal{Y})$ s.t. $(\mathfrak{F} \mid \mathcal{X}_0)(\mathfrak{x}) \subseteq \mathcal{V}$. By upper α-irresoluteness of $\mathfrak{F}$, $(\mathfrak{F} \mid \mathcal{X}_0)(\mathfrak{x}) = \mathfrak{F}(\mathfrak{x})$, and $\exists\, \mathcal{U} \in \alpha O(\mathcal{X})$ having $\mathfrak{x}$ s.t. $\mathfrak{F}(\mathcal{U}) \subseteq \mathcal{V}$. Take $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{X}_0$, and then by Lemma 3.1, we get $\mathfrak{x} \in \mathcal{U}_0 \in \alpha O(\mathcal{X}_0)$ and $(\mathfrak{F} \mid \mathcal{X}_0)(\mathcal{U}_0) \subseteq \mathcal{V}$. Hence, $\mathfrak{F} \mid \mathcal{X}_0$ is upper α-irresolute. Lower α-irresoluteness is analogous.

Theorem 3.2. *$\mathfrak{F} : (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is upper (resp. lower) α-irresolute if $\forall\, \mathfrak{x} \in \mathcal{X}$, $\exists\, \mathcal{X}_0 \in \alpha O(\mathcal{X})$ having $\mathfrak{x}$ s.t. the restriction $\mathfrak{F} \mid \mathcal{X}_0 : (\mathcal{X}_0, \tau \mid \mathcal{X}_0) \to (\mathcal{Y}, \sigma)$ is upper (resp. lower) α-irresolute.*

Proof. Let $\mathfrak{x} \in \mathcal{X}$ and $\mathcal{V} \in \alpha O(\mathcal{Y})$ s.t. $\mathfrak{F}(\mathfrak{x}) \subseteq \mathcal{V}$. $\exists\, \mathcal{X}_0 \in \alpha(\mathcal{X}, \mathfrak{x})$ s.t. $\mathfrak{F} \mid \mathcal{X}_0$ is upper α-irresolute. Therefore, $\exists\, \mathcal{U}_0 \in \alpha O(\mathcal{X}_0)$ having $\mathfrak{x}$ s.t. $(\mathfrak{F} \mid \mathcal{X}_0)(\mathcal{U}_0) \subseteq \mathcal{V}$. By Lemma 1.1, $\mathcal{U}_0 \in \alpha O(\mathcal{X})$ and $\mathfrak{F}(u) = (\mathfrak{F} \mid \mathcal{X}_0)(u)\ \forall\, u \in \mathcal{U}_0$. Hence, $\mathfrak{F}$ is upper α-irresolute. Lower α-irresoluteness is analogous.

Corollary 3.1. *Let $\mathcal{X} = \bigcup\limits_{\lambda \in \nabla} \mathcal{U}_\lambda$, $\mathcal{U}_\lambda \in \alpha O(\mathcal{X})$. $\mathfrak{F} : (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is upper (resp. lower) α-irresolute iff the restriction $\mathfrak{F} \mid \mathcal{U}_\lambda : (\mathcal{U}_\lambda, \tau \mid \mathcal{U}_\lambda) \to (\mathcal{Y}, \sigma)$ is upper (resp. lower) α-irresolute for $\lambda \in \nabla$.*

Proof. Immediate consequence from Theorems 3.1 and 3.2.

$\mathfrak{A} \subseteq \mathcal{X}$ is α-compact (α-comp, for short) if $\mathfrak{A} \subseteq \bigcup\limits_{i=1}^{\infty} \mathcal{U}_i$, $\mathcal{U}_i \in \alpha O(\mathcal{X})$, then $\mathfrak{A} = \bigcup\limits_{i=1}^{n} \mathcal{U}_i$, where n is finite. In other words, $\mathcal{X}$ is α-comp [28] iff $\mathcal{X}$ is α-comp of itself. Moreover, $\mathfrak{A}$ is α-comp iff $\mathfrak{A}$ is comp w.r. to τ^α.

Theorem 3.3. *Let $\mathfrak{F}$ be upper α-irresolute, and $\mathfrak{F}(\mathfrak{x})$ is α-comp w.r. to $\tau^\alpha_{\mathcal{Y}}$ $\forall\, \mathfrak{x} \in \mathcal{X}$. If $\mathfrak{A}$ is an α-comp w.r. to $\mathcal{X}$, then $\mathfrak{F}(\mathfrak{A})$ is an α-comp w.r. to $\mathcal{Y}$.*

Proof. Let $\mathfrak{F}(A) = \bigcup\limits_{\lambda \in \nabla} \{\mathcal{V}_\lambda : \mathcal{V}_\lambda \in \alpha O(\mathcal{Y})\}$. $\forall\, \mathfrak{x} \in \mathfrak{A}$, $\exists$ a finite $\nabla(\mathfrak{x}) \subset \nabla$ s.t. $\mathfrak{F}(\mathfrak{x}) \subseteq \bigcup\limits_{\lambda \in \nabla(\mathfrak{x})} \{\mathcal{V}_\lambda : \mathcal{V}_\lambda \in \alpha O(\mathcal{Y})\}$. Set $\mathcal{V}(\mathfrak{x}) = \bigcup\limits_{\lambda \in \nabla(\mathfrak{x})} \{\mathcal{V}_\lambda : \mathcal{V}_\lambda \in \alpha O(\mathcal{Y})\}$. Then, $\mathfrak{F}(\mathfrak{x}) \subseteq \mathcal{V}(\mathfrak{x}) \in \alpha O(\mathcal{Y})$, and $\exists\, \mathcal{U}(\mathfrak{x}) \in \alpha(\mathcal{X}, \mathfrak{x})$ s.t. $\mathfrak{F}(\mathcal{U}(\mathfrak{x})) \subseteq \mathcal{V}(\mathfrak{x})$. Since $\{\mathcal{U}(\mathfrak{x}) : \mathfrak{x} \in \mathfrak{A}\}$ is an α-open cover of $\mathfrak{A}$, $\exists$ a finite number of $\mathfrak{A}$, say, $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$ s.t. $\mathfrak{A} \subseteq \bigcup\{\mathcal{U}(x_i) : i = 1, 2, \cdots, n\}$. Therefore, we get $\mathfrak{F}(\mathfrak{A}) \subseteq \mathfrak{F}(\bigcup\limits_{i=1}^{n} \mathcal{U}(\mathfrak{x}_i)) \subseteq \bigcup\limits_{i=1}^{n} \mathcal{V}(\mathfrak{x}_i) \subseteq \bigcup\limits_{i=1}^{n} \bigcup\limits_{\lambda \in \nabla(\mathfrak{x})} \mathcal{V}_\lambda$. Hence, $\mathfrak{F}(\mathfrak{A})$ is α-comp w.r. to $\mathcal{Y}$.

Corollary 3.2. *Let $\mathfrak{F}$ be α-irresolute, and $\mathfrak{F}(\mathfrak{x})$ is α-comp w.r. to $\mathcal{Y}$, $\forall\, \mathfrak{x} \in \mathcal{X}$. If $\mathcal{X}$ is an α-comp, then $\mathcal{Y}$ is so.*

Recall that $\mathcal{X}$ is α-normal if for $\mathfrak{A}, \mathfrak{B} \in C(\mathcal{X})$ s.t. $\mathfrak{A} \cap \mathfrak{B} = \phi$, $\exists\, \mathcal{U}, \mathcal{V} \in \alpha O(\mathcal{X})$ s.t. $\mathcal{U} \cap \mathcal{V} = \phi$, and $\mathfrak{A} \subseteq \mathcal{U}$ and $\mathfrak{B} \subseteq \mathcal{V}$.

Theorem 3.4. *Let $\mathcal{Y}$ be α-normal, and $\mathfrak{F}_i : \mathcal{X}_i \to \mathcal{Y}$ is upper α-irresolute s.t. $\mathfrak{F}_i$ is closed, $\forall\, i = 1, 2$. Then, $\{(\mathfrak{x}_1, \mathfrak{x}_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : \mathfrak{F}_1(\mathfrak{x}_1) \cap \mathfrak{F}_2(\mathfrak{x}_2) \neq \phi\} \in \alpha C(\mathcal{X}_1 \times \mathcal{X}_2)$.*

Proof. Let $\mathfrak{A} = \{(\mathfrak{x}_1, \mathfrak{x}_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : \mathfrak{F}_1(\mathfrak{x}_1) \cap \mathfrak{F}_2(\mathfrak{x}_2) \neq \phi\}$, and $(\mathfrak{x}_1, \mathfrak{x}_2) \notin \mathfrak{A}$. Then, $\mathfrak{F}_1(\mathfrak{x}_1) \cap \mathfrak{F}_2(\mathfrak{x}_2) = \phi$. Since $\mathcal{Y}$ is α-normal, and $\mathfrak{F}_i$ is closed for $i = 1, 2$, $\exists$ disjoint $\mathcal{V}_1, \mathcal{V}_2 \in \alpha O(\mathcal{X})$ s.t. $\mathfrak{F}_i(\mathfrak{x}_i) \subseteq \mathcal{V}_i$ for $i = 1, 2$. By assumption, $\mathfrak{F}_i^+(\mathcal{V}_i) \in \alpha O(\mathcal{X}_i, \mathfrak{x}_i)$ for $i = 1, 2$. Set $\mathcal{U} = \mathfrak{F}_1^+(\mathcal{V}_1) \times \mathfrak{F}_2^+(\mathcal{V}_2)$. Then, $\mathcal{U} \in \alpha O(\mathcal{X}_1 \times \mathcal{X}_2)$, and $(\mathfrak{x}_1, \mathfrak{x}_2) \in \mathcal{U} \subseteq (\mathcal{X}_1 \times \mathcal{X}_2) - \mathfrak{A}$. Hence, $(\mathcal{X}_1 \times \mathcal{X}_2) - \mathfrak{A} \in \alpha O(\mathcal{X}_1 \times \mathcal{X}_2)$.

For a multifunction $\mathfrak{F} : \mathcal{X} \to \mathcal{Y}$, $\mathcal{G}(\mathfrak{F})$ is $\mathcal{G}(\mathfrak{F}) = \{(\mathfrak{x}, \mathfrak{y}) \in \mathcal{X} \times \mathcal{Y} : \mathfrak{x} \in \mathcal{X} \text{ and } \mathfrak{y} \in \mathfrak{F}(\mathfrak{x})\}$.

Theorem 3.5. *Let $\mathcal{Y}$ be a Hausdorff space, and $F : \mathcal{X} \to \mathcal{Y}$ is upper α-irresolute s.t. $\mathfrak{F}(\mathfrak{x})$ is comp, $\forall$ $\mathfrak{x} \in \mathcal{X}$. Then, $\mathcal{G}(\mathfrak{F}) \in \alpha C(\mathcal{X} \times \mathcal{Y})$.*

Proof. Let $(\mathfrak{x}, \mathfrak{y}) \in (\mathcal{X} \times \mathcal{Y}) - \mathcal{G}(\mathfrak{F})$. Then, $\mathfrak{y} \in \mathcal{Y} - \mathfrak{F}(\mathfrak{x})$. $\forall$ $\mathfrak{z} \in \mathfrak{F}(\mathfrak{x})$, $\exists$ disjoint $\mathcal{V}(\mathfrak{z}), \mathfrak{W}(\mathfrak{z}) \in O(\mathcal{Y})$ s.t. $\mathfrak{z} \in \mathcal{V}(\mathfrak{z})$ and $\mathfrak{y} \in \mathfrak{W}(\mathfrak{z})$. $\mathfrak{F}(\mathfrak{x}) = \bigcup\limits_{\mathfrak{z} \in \mathfrak{F}(\mathfrak{x})} \mathcal{V}(\mathfrak{z})$, and $\exists$ a finite number in $\mathfrak{F}(\mathfrak{x})$, say, $\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_n$ s.t. $\mathfrak{F}(\mathfrak{x}) \subseteq \bigcup\{\mathcal{V}(\mathfrak{z}_i) : 1 \leq i \leq n\}$ and $\mathfrak{W} = \bigcap\{\mathfrak{W}(\mathfrak{z}_i) : 1 \leq i \leq n\}$. By the upper α-irresoluteness of $\mathfrak{F}$, and $\mathfrak{F}(\mathfrak{x}) \subseteq \mathcal{V}$, $\exists$ $\mathcal{U} \in \alpha(\mathcal{X}, \mathfrak{x})$ s.t. $\mathfrak{F}(\mathcal{U}) \subseteq \mathcal{V}$. Therefore, $\mathfrak{F}(\mathcal{U}) \cap \mathfrak{W} = \phi$, and so $(\mathcal{U} \times \mathfrak{W}) \cap \mathcal{G}(\mathfrak{F}) = \phi$. Since $\mathcal{U} \times \mathfrak{W} \in \alpha O(\mathcal{X} \times \mathcal{Y})$, and $(\mathfrak{x}, \mathfrak{y}) \in \mathcal{U} \times \mathfrak{W} \subseteq (\mathcal{X} \times \mathcal{Y}) - \mathcal{G}(\mathfrak{F})$, $(\mathcal{X} \times \mathcal{Y}) - \mathcal{G}(\mathfrak{F}) \in \alpha O(\mathcal{X} \times \mathcal{Y})$.

Theorem 3.6. *If $\mathfrak{F} : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{G} : \mathcal{Y} \to \mathcal{Z}$ are lower (resp. upper) α-irresolute, then $\mathcal{G} \circ \mathfrak{F} : \mathcal{X} \to \mathcal{Z}$ is so.*

Proof. Let $\mathcal{V} \in \alpha O(\mathcal{Z})$. Since $(\mathcal{G} \circ \mathfrak{F})^-(\mathcal{V}) = \mathfrak{F}^-(\mathcal{G}^-)(\mathcal{V})$ and by lower α-irresoluteness of $\mathfrak{F}$ and $\mathcal{G}$, we get $(\mathcal{G} \circ \mathfrak{F})^-(\mathcal{V}) \in \alpha O(\mathcal{X})$. Thus, $\mathcal{G} \circ \mathfrak{F}$ is lower α-irresolute. Similarly, the upper is satisfied.

4. Applications: Games and winning strategy

Due to these applications, consider each $\mathcal{X}_i$ to be a topological structure, $\forall$ $i = 1, 2, \cdots, n$, and topologically $\mathcal{X} = \bigoplus\limits_{i \in I} \mathcal{X}_i$.

4.1. Upper and lower topological games

Definition 4.1. *A game $\mathfrak{G}$ on $\mathcal{X}$ for the players $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_n$ consists of the following*

(i) $\{\mathfrak{N}^+, \mathfrak{N}^-\}$ from $\mathfrak{N}$ is a partition of players.

(ii) $\{\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_n\}$ from $\mathcal{X}$ is a partition of sets.

(iii) An irresolute multifunction F of X onto itself s.t. $F(\mathcal{X}_i) \cap \mathcal{X}_i = \phi$ for $i = 1, \cdots, n$.

(iv) n-bounded real valued functions $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \cdots, \mathfrak{L}_n$ on $\mathcal{X}$.

The procedures of $\mathfrak{G}$ are as follows:

The locations are represented by the components of $\mathcal{X}$, and play begins at any point in $\mathcal{X}$. $\mathfrak{x} \in \mathcal{X}_i$ denotes the location of player $\mathcal{I}_i$ at $\mathfrak{x}$. If $\mathfrak{x}_0$ is the starting location, the following sequence occurs: Player $\mathcal{I}_i$ selects $\mathfrak{x}_1 \in F(\mathfrak{x}_0)$ for $\mathfrak{x} \in \mathcal{X}_i$. If $\mathfrak{x}_1 \in \mathcal{X}_j$, player $\mathcal{I}_j$ selects $\mathfrak{x}_2 \in F(\mathfrak{x}_1)$, and so on. If $F(\mathfrak{x}) = \phi$, the play ends at $\mathfrak{x}$. In other terms, a play is a sequence consisting of the elements $< \mathfrak{x}_0, F(\mathfrak{x}_0), \mathfrak{x}_1, F(\mathfrak{x}_1), \cdots >$ s.t. $\mathfrak{x}_0 \in F(\mathfrak{x}_0)$, $\mathfrak{x}_1 \in F(\mathfrak{x}_1)$ and so on.

Definition 4.2. *For a sequence of a play $< \mathfrak{x}_0, F(\mathfrak{x}_0), \mathfrak{x}_1, F(\mathfrak{x}_1), \cdots, \mathfrak{x}_k, F(\mathfrak{x}_k) >$ with $k + 1$ points, the length of it is k. Here, the $k^{\underline{th}}$ element satisfies $F(\mathfrak{x}_k) = \phi$.*

Definition 4.3. *$\mathfrak{G}$ is locally finite (LF, for short) if each play length is finite. If $\mathcal{S}$ is the set of locations in a play, the payoff to $\mathcal{I}_i$ is either $\sup\{\mathfrak{L}_i(\mathfrak{x}) : \mathfrak{x} \in \mathcal{S}\}$ or $\inf\{\mathfrak{L}_i(\mathfrak{x}) : \mathfrak{x} \in \mathcal{S}\}$, depending on whether $\mathcal{I}_i \in \mathfrak{N}^+$ or $\mathcal{I}_i \in \mathfrak{N}^-$. Each player's objective is to maximize their payoff.*

Definition 4.4. *If player $\mathcal{I}_i$ can make sure that Payoff($\mathcal{I}_i$)$\geq \xi$, no matter what other players do, $\forall$ plays beginning with $\mathfrak{x}$, no matter what other players do. If Payoff($\mathcal{I}_i$)$> \xi$, he is rigorously guaranteeing ξ from $\mathfrak{x}$.*

Lemma 4.1. *[1] If $\mathcal{G} \in O(\mathcal{X})$ and $\mathfrak{A} \subseteq \mathcal{X}$, then $\mathcal{G} \cap Cl(\mathfrak{A}) \subseteq Cl(\mathcal{G} \cap \mathfrak{A})$.*

Proposition 4.1. *If $\mathcal{X} = \bigoplus_{i\in I} \mathcal{X}_i$ and $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$, then $\mathfrak{A} \cap \mathcal{X}_i \in \mathcal{SO}(\mathcal{X}_i)$ $\forall$ $i \in I$. The converse holds only if $\mathfrak{A} \in O(\mathcal{X})$.*

Proof. Let $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$. Then, $\mathfrak{A} \cap \mathcal{X}_i \subseteq Cl(Int(\mathfrak{A})) \cap \mathcal{X}_i$. Since $\mathcal{X}_i \in O(\mathcal{X})$, $\forall$ i, by Lemma 4.1, $\mathfrak{A} \cap \mathcal{X}_i \subset Cl(Int(\mathfrak{A})) \cap \mathcal{X}_i = Cl(Int(\mathfrak{A})) \cap \mathcal{X}_i$. Then, $\mathfrak{A} \cap \mathcal{X}_i \subset (Cl(Int(\mathfrak{A})) \cap \mathcal{X}_i) \cap \mathcal{X}_i = (Cl_{\mathcal{X}_i}(Int(\mathfrak{A})) \cap \mathcal{X}_i) \cap \mathcal{X}_i$. Since $\mathcal{X}_i$ is a subspace of $\mathcal{X}$, $\forall$ $i \in I$, then $(Int(\mathfrak{A}) \cap \mathcal{X}_i) \cap \mathcal{X}_i \in O(\mathcal{X}_i)$. Therefore, $\mathfrak{A} \cap \mathcal{X}_i \subseteq Cl_{\mathcal{X}_i}(Int_{\mathcal{X}_i}(Int(\mathfrak{A} \cap \mathcal{X}_i))) \cap \mathcal{X}_i = Cl_{\mathcal{X}_i}(Int_{\mathcal{X}_i}(Int(\mathfrak{A} \cap \mathcal{X}_i))) \subseteq Cl_{\mathcal{X}_i}(Int_{\mathcal{X}_i}(\mathfrak{A} \cap \mathcal{X}_i))$. So, $\mathfrak{A} \cap \mathcal{X}_i \in \mathcal{SO}(\mathcal{X}_i)$, $\forall$ $i \in I$. On the other hand, let $\mathfrak{A} \in O(\mathcal{X})$, and $\mathfrak{A} \in \mathcal{SO}(\mathcal{X}_i)$. Then, $\mathfrak{A} \cap \mathcal{X}_i \subseteq Cl_{\mathcal{X}_i} Int_{\mathcal{X}_i}(\mathfrak{A} \cap \mathcal{X}_i) \subseteq Cl_{\mathcal{X}_i}(\mathfrak{A} \cap \mathcal{X}_i) = \mathcal{X}_i \cap Cl(\mathfrak{A} \cap \mathcal{X}_i) \subseteq Cl(\mathfrak{A}) \cap \mathcal{X}_i$ implies $\mathfrak{A} \subseteq Cl(\mathfrak{A})$. By openness of $\mathfrak{A}$, $\mathfrak{A} \subseteq Cl(Int\mathfrak{A})$, and $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$.

Definition 4.5. *For a TS $\mathcal{X}$, $\mathfrak{L} : \mathcal{X} \to \mathbb{R}$ is upper and lower $\mathfrak{s}$-continuous if $\forall$ $r \in \mathbb{R}$, $\{\mathfrak{L}(\mathfrak{x}) < r, \forall\ \mathfrak{x} \in \mathcal{X}\}$ and $\{\mathfrak{L}(\mathfrak{x}) > r, \forall\ \mathfrak{x} \in \mathcal{X}\}$, $\exists\ \mathcal{U} \in SO(\mathcal{X})$ s.t. $\{\mathfrak{L}(x') < r, \forall\ x' \in \mathcal{U}\}$ and $\{\mathfrak{L}(x') > r, \forall\ x' \in \mathcal{U}\}$, respectively.*

Definition 4.6. *$\mathfrak{G}$ is called*

(i) upper topological (UT, for short) for $\mathcal{I}_i$ if $\mathfrak{L}_i$ is upper $\mathfrak{s}$-continuous.

(ii) lower topological (LT, for short) for $\mathcal{I}_i$ if $\mathfrak{L}_i$ is lower $\mathfrak{s}$-continuous.

Theorem 4.1. *If $\mathfrak{G}$ is lower for $\mathcal{I}_1 \in \mathfrak{N}^+$, then all locations that satisfy $\mathcal{I}_1$ is strictly guarantee a gain ξ is in $\mathcal{SO}(\mathcal{X})$.*

Proof. (By transfinite induction). Consider the set of starting locations $\mathfrak{A}_\xi$ s.t $\mathcal{I}_1$ is a rigorous guarantee of ξ. Then, $(\mathcal{X}_1 \cap F^-(\mathfrak{A}_\xi)) \cup (\bigcup_{j=1}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{A}_\xi))) \subseteq \mathfrak{A}_\xi$. Note that $F^+(\mathfrak{A}_\xi) = \{\mathfrak{x} \in \mathcal{X} : F(\mathfrak{x}) \subseteq \mathfrak{A}_\xi\}$, and $F^-(\mathfrak{A}_\xi) = \{\mathfrak{x} \in \mathcal{X} : F(\mathfrak{x}) \cap \mathfrak{A}_\xi \neq \phi\}$. Construct $\mathfrak{A}(\Delta) \in \mathcal{SO}(\mathcal{X})$ s.t. $\mathfrak{A}(\Delta) \subseteq \mathfrak{A}_\xi$, $\forall$ ordinal Δ as follows: Let $\mathfrak{A}(0) = \{\mathfrak{x} \in \mathcal{X} : \mathfrak{L}_1(\mathfrak{x}) > \xi\}$. By assumption, we get that $\mathfrak{L}_1$ is lower $\mathfrak{s}$-continuous. Thus, $\mathfrak{A}(0) \in \mathcal{SO}(\mathcal{X})$, and $\mathfrak{A}(0) \subseteq \mathfrak{A}_\xi$. Define $\mathfrak{A}(\beta) \in \mathcal{SO}(\mathcal{X}) \subseteq \mathfrak{A}_\xi$, $\forall$ $\nabla < \Delta$. Let Δ be a limit and $\mathfrak{A}(\Delta) = \bigcup_{\nabla<\Delta} \mathfrak{A}(\nabla)$. Then, $\mathfrak{A}(\Delta) \in \mathcal{SO}(\mathcal{X})$, and $\mathfrak{A}(\Delta) \subseteq \mathfrak{A}_\xi$. If Δ is not a limit ordinal, then $\Delta = \Delta' + 1$. Let $\mathfrak{A}(\Delta) = \mathfrak{A}(\Delta')$ $\cup$ $(\mathcal{X}_1 \cap F^-(\mathfrak{A}(\Delta'))) \cup \bigcup_{n=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{A}(\Delta')))$ by hypothesis, $\mathfrak{A}(\Delta') \in \mathcal{SO}(\mathcal{X})$ and by Proposition 4.1, $\mathcal{X}_1 \cap F^-(\mathfrak{A}(\Delta'))$ and $\mathcal{X}_j \cap F^+(\mathfrak{A}(\Delta')) \in \mathcal{SO}(\mathcal{X})$, $\forall$ $j = 2, 3, \cdots, n$. Since $\mathcal{X} = \bigoplus_{i\in I} \mathcal{X}_i$ and F is irresolute, then $\mathfrak{A}(\Delta) \in \mathcal{SO}(\mathcal{X})$ and $\mathfrak{A}(\Delta) \subseteq \mathfrak{A}_\xi$, for $\mathfrak{A}(\Delta') \subseteq \mathfrak{A}_\xi$ and $(\mathcal{X}_1 \cap F^-(\mathfrak{A}(\Delta'))) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{A}(\Delta'))) \subseteq (\mathcal{X}_1 \cap F^-(\mathfrak{A}_\xi)) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{A}_\xi)) \subseteq \mathfrak{A}_\xi$. Hence, $\forall$ ordinal Δ, $\mathfrak{A}(\Delta) \in \mathcal{SO}(\mathcal{X})$, and $\mathfrak{A}(\Delta) \subseteq \mathfrak{A}_\xi$. Since the sequence $\{\mathfrak{A}(\Delta)\}$ is increasing and cannot be constant, $\mathfrak{A}(\Delta_0) = \mathfrak{A}(\Delta_0 + 1) = \cdots$, for some Δ_0. Let $\mathfrak{A}' = \mathcal{X} - \mathfrak{A}(\Delta_0)$. If $\mathfrak{x} \in \mathfrak{A}' \cap \mathcal{X}_1$, then $F(\mathfrak{x}) \subseteq \mathfrak{A}'$, while if $\mathfrak{x} \in \mathfrak{A}' \cap \mathcal{X}_j$, where $j \neq 1$, then $F(\mathfrak{x})$ $\cap \mathfrak{A}' \neq \phi$. Therefore, if a play begins from a point in $\mathfrak{A}'$, for instance, $\mathcal{I}_1$ does, players $\mathcal{I}_2, \mathcal{I}_3, \cdots, \mathcal{I}_n$ can ascertain that a location in $\mathfrak{A}(\Delta_0)$ is never achieved. $\mathfrak{A}(\Delta_0) \supseteq \mathfrak{A}(0) = \{\mathfrak{x} : \mathfrak{L}_1(\mathfrak{x}) > \xi\}$. Thus, if $\mathfrak{x} \in \mathfrak{A}'$, then $\mathfrak{x} \notin \mathfrak{A}(\Delta_0)$, and so $\mathfrak{x} \notin \mathfrak{A}_\xi$, and so $\mathfrak{A}_\xi \subseteq \mathfrak{A}(\Delta_0)$. $\mathfrak{A}(\Delta_0) \subseteq \mathfrak{A}_\xi$ by construction, and so $\mathfrak{A}_\xi = \mathfrak{A}(\Delta_0)$. Hence, $\mathfrak{A}_\xi \in \mathcal{SO}(\mathcal{X})$.

Remark 4.1. *Although the complement of $\mathcal{SO}(\mathcal{X})$ is $\mathcal{SC}(\mathcal{X})$, and F is irresolute, $F^+(\mathcal{X}) \in \mathcal{SO}(\mathcal{X})$. Then, $\mathcal{X}_0 = \mathcal{X} - F^+(\mathcal{X}) \in \mathcal{SC}(\mathcal{X})$ and the complement of criteria in Theorem 4.1 does not hold that the set of places from which $\mathcal{I}_1$ may ensure a benefit for ξ is semi-closed. Theorem 4.2 specifies additional requirements for the semi-closed nature of this set.*

Theorem 4.2. *Let $\mathfrak{G}$ be UT for $\mathcal{I}_1 \in \mathfrak{R}^+$, LF, and $\mathcal{X}_0 = \{\mathfrak{x} : F(\mathfrak{x}) = \phi\} \in \mathcal{SO}(\mathcal{X})$. Then, the set of places s.t. $\mathcal{I}_1$ may guarantee a gain to the holder is in $\mathcal{SO}(\mathcal{X})$.*

Proof. (By transfinite induction). Define $\mathcal{X}(\Delta) \in \mathcal{SO}(\mathcal{X})$ $\forall$ ordinal Δ. Let $\mathcal{X}(0) = \mathcal{X}_0 = \{\mathfrak{x} : F(\mathfrak{x}) = \phi\}$. Then, $\mathcal{X}(0) \in \mathcal{SO}(\mathcal{X})$. Construct $\mathcal{X}(\nabla) \in \mathcal{SO}(\mathcal{X})$ $\forall$ ordinals $\nabla < \Delta$. If Δ is limit, and $\mathcal{X}(\Delta) = (\bigcup_{\nabla<\Delta} \mathcal{X}(\nabla)) \in \mathcal{SO}(\mathcal{X})$. If Δ has a precursor Δ' i.e. $\Delta = \Delta' + 1$, take $\mathcal{X}(\Delta) = \mathcal{X}(\Delta') \cup F^+(\mathcal{X}(\Delta'))$. By F irresoluteness, $\mathcal{X}(\Delta) \in \mathcal{SO}(\mathcal{X})$. Thus, $\forall$ ordinal Δ, by transfinite induction, $\mathcal{X}(\Delta) \in \mathcal{SO}(\mathcal{X})$. If $\nabla < \Delta$, $\mathcal{X}(\nabla) < \mathcal{X}(\Delta)$. $\mathfrak{H}_\xi$ is defined as the collection of places from which $\mathcal{I}_1$ may guarantee ξ. Then, $(\mathcal{X}_1 \cap F^-(\mathfrak{H}_\xi)) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{H}_\xi)) \subseteq \mathfrak{H}_\xi$. Define a set $\mathfrak{H}(\Delta)$, $\forall$ Δ s.t.

(i) $\mathfrak{H}(\Delta) \subseteq \mathfrak{H}_\xi$;

(ii) if $\nabla < \Delta$, $\mathfrak{H}(\nabla) \subseteq \mathfrak{H}(\Delta)$;

(iii) if $\nabla < \Delta$, $\mathfrak{H}(\Delta) \cap \mathcal{X}(\nabla) = \mathfrak{H}(\nabla) \cap \mathcal{X}(\nabla)$; and

(iv) $\mathfrak{H}(\Delta) \cap \mathcal{X}(\Delta) \in \mathcal{SC}(\mathcal{X})$ in $\mathcal{X}(\Delta)$.

The conditions (i)-(iv) can be satisfied in three claims.

Claim I. Let $\mathfrak{H}(0) = \{\mathfrak{x} : \mathfrak{L}_1(\mathfrak{x}) \geq \xi\}$. Since $\mathfrak{L}_1$ is upper $\mathfrak{s}$-continuous, $\mathfrak{H}(0) \in \mathcal{SC}(\mathcal{X})$ in $\mathcal{X}$. Also, $\mathfrak{H}(0) \subseteq \mathfrak{H}_\xi$, and so $(\mathfrak{H}(0) \cap \mathcal{X}(0)) \in \mathcal{SC}(\mathcal{X})$ in $\mathcal{X}(0)$. Consider $\mathfrak{H}(\nabla)$ that satisfies conditions (i)-(iv) is constructed, $\forall$ $\nabla < \Delta$.

Claim II. If Δ is a limit ordinal, take $\mathfrak{H}(\Delta) = \bigcup_{\nabla<\Delta} \mathfrak{H}(\nabla)$. Then, $\mathfrak{H}(\Delta) \subseteq \mathfrak{H}_\xi$, $\forall$ $\nabla < \Delta$. Also, if $\nabla < \Delta$, then $\mathfrak{H}(\nabla) \subseteq \mathfrak{H}(\Delta)$, and if $\nabla' < \Delta$, $\mathfrak{H}(\Delta) \cap \mathcal{X}(\nabla') = (\bigcup_{\nabla<\Delta} \mathfrak{H}(\nabla)) \cap \mathcal{X}(\nabla') = \bigcup_{\nabla<\Delta}(\mathfrak{H}(\nabla)) \cap \mathcal{X}(\nabla')$. If $\nabla < \Delta'$, then $\mathfrak{H}(\nabla) \cap \mathcal{X}(\nabla') \subseteq \mathfrak{H}(\nabla') \cap \mathcal{X}(\beta')$; and if $\nabla' \leq \nabla < \Delta$, $\mathfrak{H}(\nabla) \cap \mathcal{X}(\nabla') = \mathfrak{H}(\nabla') \cap \mathcal{X}(\nabla')$. Hence, $\mathfrak{H}(\nabla) \cap \mathcal{X}(\nabla') = \mathfrak{H}(\nabla') \cap \mathcal{X}(\nabla')$, and (iii) is satisfied. If $\mathfrak{x} \in \mathcal{X}(\Delta)$, and $\mathfrak{x} \notin \mathfrak{H}(\Delta)$, then $\mathfrak{x} \in \mathcal{X}(\nabla)$ for some $\nabla < \Delta$, and $\mathfrak{x} \notin \mathfrak{H}(\nabla)$. Now, $\mathfrak{H}(\nabla) \cap \mathcal{X}(\nabla) \in \mathcal{SC}(\mathcal{X})$ in $\mathcal{X}(\nabla)$, and so $\exists$ a semi-open nbd $\mathfrak{A}$ of $\mathfrak{x}$ in $\mathcal{X}(\nabla)$ s.t. $\mathfrak{A} \cap \mathfrak{H}(\nabla) = \phi$. Now, since $\mathcal{X}(\nabla)$, and $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$ and $\mathfrak{A} \subseteq \mathcal{X}(\nabla) \subseteq \mathcal{X}(\Delta)$, $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$ in $\mathcal{X}(\Delta)$. By (iii), $\mathcal{X}(\nabla) \cap \mathfrak{H}(\Delta) = \mathcal{X}(\nabla) \cap \mathfrak{H}(\nabla)$, and so $\mathfrak{A}$ is a semi-open nbd of $\mathfrak{x}$ in $\mathcal{X}(\Delta)$ s.t. $\mathfrak{A} \cap \mathfrak{H}(\Delta) = \phi$. Thus, (iv) is satisfied.

Claim III. If Δ has a predecessor Δ'. This means that $\Delta = \Delta' + 1$. Take $\mathfrak{H}(\Delta) = \mathfrak{H}(\Delta') \cup (\mathcal{X}_1 \cap F^-(\mathfrak{H}(\Delta'))) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{H}(\Delta')))$. Since $\mathfrak{H}(\Delta') \subseteq \mathfrak{H}_\xi$, and $\mathcal{X}_1 \cap F^-(\mathfrak{H}(\Delta')) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{H}(\Delta'))) \subseteq (\mathcal{X}_1 \cap F^-(\mathfrak{H}_\xi)) \cup \bigcup_{j=2}^{n}(\mathcal{X}_j \cap F^+(\mathfrak{H}_\xi)) \subseteq \mathfrak{H}_\xi$, (i) is satisfied, and (ii) is clear. Suppose $\nabla' < \Delta$. If $\mathfrak{x} \in \mathcal{X}(\nabla')$, and $F(\mathfrak{x}) \neq \phi$, then $F(\mathfrak{x}) \subseteq \mathcal{X}(\nabla)$ for some $\nabla < \nabla'$. Thus, if $\mathfrak{x} \in \mathcal{X}(\nabla') \cap (\mathcal{X}_1 \cap F^-(\mathfrak{H}(\Delta')))$, $F(\mathfrak{x}) \cap \{\mathcal{X}(\nabla) \cap \mathfrak{H}(\Delta')\} \neq \phi$ for $\nabla < \nabla' \leq \Delta'$. Thus, $\mathfrak{x} \in \mathcal{X}(\nabla') \cap (\mathcal{X}_1 \cap F^-(\mathfrak{H}(\nabla))) \subset \mathcal{X}(\nabla') \cap \mathfrak{H}(\nabla + 1) \subset \mathcal{X}(\nabla') \cap \mathfrak{H}(\nabla')$. In the same manner, if $j \neq 1$, $\mathcal{X}(\nabla') \cap (\mathcal{X}_j \cap F^+(\mathfrak{H}(\Delta'))) \subseteq \mathcal{X}(\nabla') \cap \mathfrak{H}(\nabla')$. Also, $\mathcal{X}(\nabla') \cap \mathfrak{H}(\Delta') = \mathcal{X}(\nabla') \cap \mathfrak{H}(\nabla')$. Thus, $\mathcal{X}(\nabla') \cap \mathcal{X}(\Delta) = \mathcal{X}(\nabla') \cap [\mathfrak{H}(\Delta') \cup (\mathcal{X}_1 \cup F^-(\mathfrak{H}(\Delta'))) \cup \bigcup_{j=2}^{n} \mathcal{X}_j \cap F^+(\mathfrak{H}(\Delta'))] = \mathcal{X}(\nabla') \cap \mathfrak{H}(\nabla')$, and so (iii) is satisfied. Finally, for (iv), suppose that $\mathfrak{x} \in \mathcal{X}(\Delta)$, and $\mathfrak{x} \notin \mathfrak{H}(\Delta)$. If $\mathfrak{x} \in \mathcal{X}(\Delta')$, then $\mathfrak{x} \notin \mathfrak{H}(\Delta')$, and since $(\mathfrak{H}(\Delta') \cap \mathcal{X}(\Delta')) \in \mathcal{SC}(\mathcal{X})$ in $\mathcal{X}(\Delta')$, $\exists$ a semi-open nbd A of $\mathfrak{x}$ in $\mathcal{X}(\Delta')$ s.t. $A \cap H(\Delta') = \phi$. Since $\mathcal{X}(\Delta') \in \mathcal{SO}(\mathcal{X})$, and $\mathfrak{A} \subseteq \mathcal{X}(\Delta') \subseteq \mathcal{X}(\Delta)$, $\mathfrak{A}$ is a semi-open nbd of $\mathfrak{x}$ in $\mathcal{X}(\Delta)$; and since $\mathcal{X}(\Delta') \cap \mathfrak{H}(\Delta) = \mathcal{X}(\Delta')\mathfrak{H}(\Delta')$, by (iii), $\mathfrak{A} \cap \mathfrak{H}(\Delta) = \phi$. If $\mathfrak{x} \in (\mathcal{X}(\Delta) - \mathcal{X}(\Delta')) \cap \mathcal{X}_1$, then $F(\mathfrak{x}) \subseteq (\mathcal{X}(\Delta') - \mathfrak{H}(\Delta'))$. $(\mathcal{X}(\Delta') - \mathfrak{H}(\Delta')) \in \mathcal{SO}(\mathcal{X})$ in $\mathcal{X}(\Delta')$ and so is semi-open in $\mathcal{X}$. $\mathcal{X}_1 \cap F^+(\mathcal{X}(\Delta') - \mathfrak{H}(\Delta'))$ is a semi-open nbd of $\mathfrak{x}$ s.t. $[\mathcal{X}_1 \cap F^+(\mathcal{X}(\Delta') - \mathfrak{H}(\Delta'))] \cap \mathfrak{H}(\Delta) = \phi$. If $\mathfrak{x} \in (\mathcal{X}(\Delta) - \mathcal{X}(\Delta')) \cap \mathcal{X}_j$, then $F(\mathfrak{x}) \cap (\mathcal{X}(\Delta) - \mathfrak{H}(\Delta')) \neq \phi$, and $\mathcal{X}_j \cap F^-(\mathcal{X}(\Delta') - \mathfrak{H}(\Delta'))$ is a semi-open nbd of $\mathfrak{x}$ s.t. $[\mathcal{X}_j \cap F^-(\mathcal{X}(\Delta') - \mathfrak{H}(\Delta'))] \cap \mathfrak{H}(\Delta) = \phi$. In either case, if $\mathfrak{x} \in \mathcal{X}(\Delta)$ and $\mathfrak{x} \notin \mathfrak{H}(\Delta)$, $\exists$ a semi-open nbd of $\mathfrak{x}$ in $\mathcal{X}(\Delta)$ which does not intersect with $\mathfrak{H}(\Delta)$. Therefore, $(\mathfrak{H}(\Delta) \cap \mathcal{X}(\Delta)) \in \mathcal{SO}(\mathcal{X})$ in $\mathcal{X}(\Delta)$, and (iv) is satisfied. Thus, construct $\mathfrak{H}(\Delta)$, $\forall$ ordinal Δ s.t. (i)-(iv) are satisfied. By Berge [3], since $\mathfrak{G}$ is locally finite, $\mathcal{X} = \mathcal{X}(\Delta_0)$ for some ordinal Δ_0. Thus, $\mathfrak{H}(\Delta_0) \in \mathcal{SO}(\mathcal{X})$; and if $\Delta > \Delta_0$, $\mathfrak{H}(\Delta) = \mathfrak{H}(\Delta) \cap \mathfrak{H}(\Delta_0) = \mathfrak{H}(\Delta_0)$. Let $\mathfrak{H}' = \mathcal{X} - \mathfrak{H}(\Delta_0)$. If $\mathfrak{x} \in \mathfrak{H}' \cap \mathcal{X}_1$, then $F(\mathfrak{x}) \subseteq \mathfrak{H}'$, and if $\mathfrak{x} \in \mathfrak{H}' \cap \mathcal{X}_j$, where $j \neq 1$, then $F(\mathcal{X}) \cap \mathfrak{H}' \neq \phi$. Thus, if a play starts with a location in $\mathfrak{H}'$, whatever $\mathcal{I}_1$ does, players $\mathcal{I}_2, \mathcal{I}_3, \cdots, \mathcal{I}_1$ can prevent a location in $\mathfrak{H}(\Delta_0)$ from ever being reached. However, $\mathfrak{H}(\Delta_0) \supseteq \mathfrak{H}(0) = \{\mathfrak{x} : \mathfrak{L}_1(\mathfrak{x}) \geq \xi\}$, and so $\mathfrak{H}_\xi \subseteq \mathfrak{H}(\Delta_0)$. However, $\mathfrak{H}(\Delta_0) \subseteq \mathfrak{H}_\xi$ by construction, and so $\mathfrak{H}(\Delta_0) = \mathfrak{H}_\xi$. Thus, $\mathfrak{H}_\xi \in \mathcal{SC}(\mathcal{X})$.

The assumption of Theorem 4.2 cannot be weakened. As seen in Example 4.1, if $\mathcal{X}_0 \notin \mathcal{SO}(\mathcal{X})$, the conclusion of Theorem 4.2 is false.

Example 4.1. *Players P_1 and P_2 played on the topological sum of $\mathcal{X}_1$ and $\mathcal{X}_2$ on a segment $(-1, m]$ of $\mathbb{R}$. Let $(\mathfrak{x}; i)$ be the point $\mathfrak{x} \in \mathcal{X}_i$ and consider*

$$F(\mathfrak{x}; i) = \begin{cases} (\mathfrak{x} - 1, j) \quad i \neq j & : \quad \mathfrak{x} > 0, \\ \phi & : \quad \mathfrak{x} \leq 0. \end{cases}$$

Suppose that $\mathcal{I}_1 \in \mathfrak{N}^+$ and

$$\mathfrak{L}_1(\mathfrak{x}) = \begin{cases} 1 & : \quad \mathfrak{x} \in \mathcal{X}_2 \quad \text{and} \quad \mathfrak{x} \leq 0, \\ 0 & : \quad \text{otherwise}. \end{cases}$$

Due to the fact that $\mathfrak{L}_1$ is upper $\mathfrak{s}$-continuous, $\mathfrak{G}$ is UT for $\mathcal{I}_1 \in \mathfrak{N}^+$. The starting locations s.t. $\mathcal{I}_1$ may ensure unit gain are $\{(\mathfrak{x}; 1) : 0 < \mathfrak{x} \leq 1, 2 < \mathfrak{x} \leq 3, \cdots\} \cup \{(\mathfrak{x}; 2) : 1 < \mathfrak{x} \leq 2, 3 < \mathfrak{x} \leq 4, \cdots\} \notin \mathcal{SC}(\mathcal{X})$.

Example 4.2 shows that the conclusion of Theorem 4.2 may be true, in general.

Example 4.2. *Consider $\mathcal{X} = \mathcal{X}_1 \bigoplus \mathcal{X}_2$, where $\mathcal{X}_1 = \mathbb{R}$ and $\mathcal{X}_2 = \mathcal{Y} \bigoplus \mathcal{Z}$, where $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}$. Consider $(\mathfrak{x}; 1)$, $(\mathfrak{x}; 2)$ and $(\mathfrak{x}; 0)$ are denoted by $\mathcal{X}_1$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Let*

$$\begin{aligned} F(\mathfrak{x}; 1) &= \{(\mathfrak{x}; 0)\} \cup \{(\mathfrak{y}; 2) :\mid \mathfrak{x} - \mathfrak{y} \mid \leq 3 \mid \mathfrak{x} \mid\}, \\ F(\mathfrak{x}; 2) &= \{(\mathfrak{y}; 1) :\mid \mathfrak{x} - \mathfrak{y} \mid \leq 1/2 \mid \mathfrak{x} \mid\}, \\ F(\mathfrak{x}; 0) &= \phi. \quad Then, \quad \mathcal{X}_0 = \mathcal{Z} \end{aligned}$$

Consider $\mathcal{I}_1 \in \mathfrak{N}^+$ and $\mathfrak{L}_1(\mathfrak{x}; 0) = 1$ at $\mid \mathfrak{x} \mid$, and $f1 = 0$, otherwise. Although $\mathfrak{G}$ is upper TG for $\mathcal{I}_1 \in \mathfrak{N}^+$ and $X_0 \in \mathcal{SO}(X)$, it is not LF. The locations defined in X_1 from which $\mathcal{I}_1$ may ensure unit gain are as follows: $\bigcup_{n=0}^{\infty} \{(\mathfrak{x}; 1) :\mid x \mid \geq 1/2^n\} = \{(\mathfrak{x}; 1) :\mid \mathfrak{x} \mid > 0\} \notin \mathcal{SC}(\mathcal{X})$. However, $\mathcal{X} = \mathcal{X}_1 \bigoplus \mathcal{X}_2$, and hence the set of beginning locations from which $\mathcal{I}_1$ may ensure unit gain is not included in $\mathcal{SC}(\mathcal{X})$.

Corollary 4.1. *If $\mathfrak{G}$ is UT for $\mathcal{I}_1 \in N^-$, then the set of locations where $\mathcal{I}_1$ can guarantee ξ which is semi-closed is UT.*

Proof. Let $\mathfrak{A}_\xi$ be the set of start locations s.t. $\mathcal{I}_1$ cannot guarantee ξ. Similar to Theorem 4.1, construct $\mathfrak{A} \in \mathcal{SO}(\mathcal{X})$ s.t. $\{\mathfrak{x} : \mathfrak{L}_1(x) < \xi\} \subseteq \mathfrak{A} \subset \mathfrak{A}_\xi$. Then, $\mathfrak{A}^c \in \mathcal{SC}(\mathcal{X})$, where $\mathfrak{A}^c = \mathcal{X} - \mathfrak{A}$. If $\mathfrak{x} \in \mathfrak{A}^c \cap \mathcal{X}_1$, then $F(\mathfrak{x}) \cap \mathfrak{A}^c \neq \phi$, and if $\mathfrak{x} \in \mathfrak{A}^c \cap \mathcal{X}_j$ s.t. $j \neq 1$, then $F(\mathfrak{x}) \subseteq \mathfrak{A}^c$. So, if a play starts with a location in $\mathfrak{A}^c$, $\mathcal{I}_1$ can ascertain that a location in $\mathfrak{H}$ is never gained. Thus, if $\mathfrak{H}_\xi$ is the set of start locations s.t $\mathcal{I}_1$ may guarantee ξ, $\mathfrak{H}_\xi \supseteq \mathfrak{A}^c$. However, $\mathfrak{A}_\xi \supseteq \mathfrak{A}$ and $\mathfrak{H}_\xi \cap \mathfrak{A}_\xi = \phi$ and so $\mathfrak{H}_\xi = \mathfrak{A}^c$. Therefore, $\mathfrak{H}_\xi \in \mathcal{SC}(\mathcal{X})$.

Corollary 4.2. *Let $\mathfrak{G}$ be LF and has a LT dimension for $\mathcal{I}_1 \in \mathfrak{R}^-$, and $\mathcal{X}_0 = \{\mathfrak{x} : F(x) = \phi\} \in \mathcal{SO}(\mathcal{X})$. Then, the set of locations s.t. $\mathcal{I}_1$ may be used to strictly guarantee a gain ξ is semi-closed.*

Proof. Let $\mathfrak{K}_\xi$ be the start locations s.t. $\mathcal{I}_1$ may not strictly guarantee ξ. By a modification in the proof of Theorem 4.2, $\mathfrak{K}_\xi \in \mathcal{SC}(\mathcal{X})$. However, if $\mathfrak{A}_\xi$ is the start locations s.t. $\mathcal{I}_1$ can strictly guarantee ξ, $\mathfrak{A}_\xi = \mathcal{X} - \mathfrak{K}_\xi$, and so $\mathfrak{A}_\xi \in \mathcal{SO}(\mathcal{X})$.

4.2. Optimal strategies for topological games

Definition 4.7. *Let $\mathcal{X}_0 = \{\mathfrak{x} : F(\mathfrak{x}) = \phi\}$, and a strategy for player $\mathcal{I}_i$ is a function $\wp : (\mathcal{X}_i - \mathcal{X}_0) \to \mathcal{X}$ s.t. $\wp(\mathfrak{x}) \in F(\mathfrak{x})$, $\forall\ \mathfrak{x} \in \mathcal{X}_i - \mathcal{X}_0$. The play of $\mathfrak{G}$ is completely determined by its strategy.*

Definition 4.8. *A strategy $\wp$ for player $\mathcal{I}_1$ is guarantee him with ξ for from a start location $\mathfrak{x}$ if play begins with $\mathfrak{x}$ and $\mathcal{I}_1$ employs a strategy $\wp$. He receives a payoff $\geq \xi$ regardless of the strategies used by other players.*

Definition 4.9. *Let $\Psi(\mathfrak{x}) = \sup\{\xi : \mathfrak{x} \in H_\xi\}$, where $\mathfrak{x} \in \mathcal{X}$, and H_ξ is the locations, for each ξ s.t. $\mathcal{I}_1$ may guarantee ξ. A strategy for player $\mathcal{I}_1$ is optimal if it guarantees $\Psi(\mathfrak{x})$ from the start location $\mathfrak{x}$, $\forall\ \mathfrak{x} \in \mathcal{X}$.*

Now, assume that Υ represents the techniques used by player $\mathcal{I}_1$. Given that each strategy for I_1 is a function between $\mathcal{X}_1 - \mathcal{X}_0$ and $\mathcal{X}$. If $\mathcal{X}^{(\mathcal{X}_1 - \mathcal{X}_0)}$ is denoted by functions from $\mathcal{X}_1 - \mathcal{X}_0$ to $\mathcal{X}$, then $\Upsilon \subseteq \mathcal{X}^{(\mathcal{X}_1 - \mathcal{X}_0)}$. In this case, Υ has a relative product topology.

Theorem 4.3. *Suppose that only one of the following holds:*

(i) $\mathfrak{G}$ is LF and UT for $\mathcal{I}_1 \in \mathfrak{R}^+$ s.t. $\mathcal{X}_0 \in \mathcal{SO}(\mathcal{X})$;

(ii) F an upper TG for $\mathcal{I}_1 \in \mathfrak{R}^-$. If $F(\mathfrak{x})$ is s-comp, $\forall\ \mathfrak{x} \in \mathcal{X}_1 - \mathcal{X}_0$, then the optimal strategies for $\mathcal{I}_1$ is nonempty and $\mathcal{SC}(\mathcal{X})$ in Υ

Proof. Let $F(\mathfrak{x})$ be s-comp. Using Definition 1.2 $\forall\ \mathfrak{x} \in \mathcal{X}_1 - \mathcal{X}_0$, $\Upsilon = \prod_{\mathfrak{x} \in \mathcal{X}_1 - \mathcal{X}_0} F(\mathfrak{x})$ is s-comp. Let $\mathfrak{S}_\xi$ be the strategies s.t. $\mathcal{I}_1$ may guarantee ξ from any start point in $\mathfrak{H}_\xi$. Clearly, $\mathfrak{S}_\xi \neq \phi$ if $\mathfrak{H}_\xi \neq \phi$. It is sufficient to prove that $\mathfrak{S}_\xi \in \mathcal{SC}(\mathcal{X})$ in Υ. Suppose $\wp \in \Upsilon$, $\wp \notin \mathfrak{S}_\xi$. Then, for some $\mathfrak{x} \in \mathcal{X}_1 \cap \mathfrak{H}_\xi$, $\wp(\mathfrak{x}) \notin \mathfrak{H}_\xi$. By assumption (i) and Theorem 4.2 or assumption (b) and Corollary 4.1, we get $\mathfrak{H}_\xi \in \mathcal{SC}(\mathcal{X})$, so $\exists$ a semi-open nbd $\mathfrak{N}$ of $\wp(\mathfrak{x})$ in $\mathcal{X}$ s.t. $\mathfrak{N} \cap \mathfrak{H}_\xi = \phi$. If $\mathfrak{M}(\wp) = \{\delta : \delta \in \Upsilon\}$ and $\delta(\mathfrak{x}) \in \mathfrak{N}$, $\mathfrak{M}(\wp)$ is a semi-open nbd of $\wp$, and $\mathfrak{M}(\wp) \cap \mathfrak{S}_\xi = \phi$. Thus, $\mathfrak{S}_\xi \in \mathcal{SC}(\mathcal{X})$ in Υ. Let $\mathfrak{x}_0 \in \mathcal{X}$, and consider $\{\mathfrak{S}_{\xi:\xi<\Psi(\mathfrak{x}_0)}\}$. Consider $\xi_1 < \xi_2 < \cdots < \xi_n < \Psi(\mathfrak{x}_0)$. Then, $\mathfrak{x}_0 \in \mathfrak{H}_{\xi_i}$, $\forall\ i$, and $\mathfrak{H}_{\xi_1} \supseteq \mathfrak{H}_{\xi_2} \supseteq \cdots \supseteq \mathfrak{H}_{\xi_n}$. Suppose that $\mathfrak{H}_{\xi_k} \cap (\mathcal{X}_1 - \mathcal{X}_0) \neq \phi$ and that $k \leq n$ the greatest integer. Then, for $k < j \leq n$, $F(\mathfrak{H}_{\xi_j}) \cap (\mathcal{X}_1 - \mathcal{X}_0) = \phi$, and $\mathfrak{S}_{\xi_j} = \Upsilon$. Let $\wp_i \in \mathfrak{S}_{\xi_i}$ for $1 \leq i \leq k$, and $\wp \in \Upsilon$, where for $\mathfrak{x} \in \mathcal{X}_1 - \mathcal{X}_0$,

$$\wp(\mathfrak{x}) = \begin{cases} \wp_k(\mathfrak{x}) & : \quad \mathfrak{x} \in \mathfrak{H}_{\xi_k} \\ \wp_i(\mathfrak{x}) & : \quad \mathfrak{x} \in \mathfrak{H}_{\xi_i} - \mathfrak{H}_{\xi_{i+1}}, i = 1, 2, \cdots, k-1 \\ \wp_1(x) & : \quad \mathfrak{x} \in \mathfrak{H}_{\xi_2} \end{cases}$$

Then, $\wp \in \mathfrak{S}_{\xi-1} \cap \mathfrak{S}_{\xi_2} \cap \cdots \cap \mathfrak{S}_{\xi_n}$. Thus, $\forall\ \mathfrak{x}_0 \in \mathcal{X}$, $\{\mathfrak{S}_\xi : \xi < \Psi(\mathfrak{x}_0)\} \subseteq \mathcal{SC}(\mathcal{X})$ with the finite intersection property. Let $\mathfrak{S}(\mathfrak{x}) = \bigcap_{\xi<\psi(\mathfrak{x}_0)} \mathfrak{S}_\xi$. So, $\mathfrak{S}(\mathfrak{x}) \in \mathcal{SC}(\mathcal{X})$. Now, consider $\{\mathfrak{S}(\mathfrak{x}) : \mathfrak{x} \in \mathcal{X}\}$. Suppose that $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n \in \mathcal{X}$. If $\Psi(\mathfrak{x}_m) = \max_{1\leq i\leq n} \Psi(\mathfrak{x}_i)$, $\mathfrak{S}(\mathfrak{x}_m) \subset \mathfrak{S}(\mathfrak{x}_i)$. Thus, $\{\mathfrak{S}(\mathfrak{x}) : \mathfrak{x} \in \mathcal{X}\} \subseteq \mathcal{SC}(\mathcal{X})$ with the finite intersection property. Let $\mathfrak{S} = \bigcap_{\mathfrak{x}\in\mathcal{X}} \mathfrak{S}(\mathfrak{x})$. Thus, $\mathfrak{S}$ is nonempty and semi-closed in Υ. However, $\mathfrak{S}$ is an optimal strategy for $\mathcal{I}_1$. If $\wp \in \mathfrak{S}(\mathfrak{x})\ \forall\ \xi < \Psi(\mathfrak{x})$ and so the guarantee for $\mathcal{I}_1$ that is $\Psi(\mathfrak{x})$ when the play beginning with $\mathfrak{x}$. Thus, if $\wp \in \bigcap_{\mathfrak{x}\in\mathcal{X}} \mathfrak{S}(\mathfrak{x})$, then $\wp$ is optimal. Conversely, if $\wp$ is optimal for $\mathcal{I}_1$, $\wp$ guarantees $\mathcal{I}_1$ if the play starts with $\mathfrak{x}$, and so $\wp \in \bigcap_{\xi<\Psi(\mathfrak{x})} \mathfrak{S}_\xi$. This holds for $\mathfrak{x} \in \mathcal{X}$, and so $\wp \in \mathfrak{S} = \bigcap_{\mathfrak{x}\in\mathcal{X}} \bigcap_{\xi<\Psi(\mathfrak{x})} \mathfrak{S}_\xi$.

5. Conclusions

The representation of multifunctions using α-irresoluteness and topological game theory are investigated and discussed. Moreover, new properties of upper (lower) α-irresoluteness due to Neubrunn [30] and Noiri and Nasef [31] are modified and analyzed. The strategy for the play in topological games is completely determined.

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Conflict of interest

The authors declare that they have no competing interests.

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