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Research article

Periodic and fixed points for F-type contractions in b-gauge spaces

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Abstract: In this paper, we introduce $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in b-gauge spaces $(U, Q_{s;\Omega})$. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness and construct an F-type contraction $T : U \to U$. Furthermore, we develop novel periodic and fixed point results for these mappings in the setting of b-gauge spaces using $\mathcal{J}_{s;\Omega}$-families on U, which generalize and improve some of the results in the corresponding literature. The validity and importance of our theorems are shown through an application via an existence solution of an integral equation.

Keywords: b-gauge space; generalized pseudo-b-distance; F-contraction; fixed point; periodic point
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1. Introduction

For a nonempty set U, let $T : U \to U$ be a single valued map. The set of its fixed points is denoted by Fix(T) and is defined by Fix$(T) = \{u \in U : u = T(u)\}$. The set of all periodic points of T denoted by Per(T) is defined by Per$(T) = \{u \in U : u = T^{[k]}(u)$ for some k in $\mathbb{N}\}$, where $T^{[k]} = T \circ T \circ T \circ \cdots \circ T$ (k-times). Also, for each $z^0 \in U$, a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at z^0 such that $z^m = T^{[m]}(z^0)$, for all $m \in \{0\} \cup \mathbb{N}$ is called a Picard iteration.

The metric fixed point theory is originated from the concept of Picard successive approximations by Picard (one result in this direction can be found in [1]). The famous mathematician Banach placed the underlying idea into an abstract framework, hence presented his most eminent research as the Banach contraction principle. Since then, numerous researchers expanded the Banach contraction principle in different directions by generalizing the metric spaces and the contraction conditions for single valued as well as for multi valued maps. For instance, see [2–4]. In general, fixed point theory remained successful in solving various problems and has participated significantly to many real world problems, such as optimization theory [5], image processing [6] and game theory [7, 8].

A useful generalization of Banach contraction is an F-contraction presented by Wardowski [30]. Due to the simplicity and effectiveness of F-contractions numerous research papers have been published in this direction which can be seen in [9–16].

In 1966, Dugundji [19] initiated the idea of gauge spaces which generalizes metric spaces (or more generally, pseudo-metric spaces). Gauge spaces have the characteristic that even the distance between two distinct points of the space may be zero. This simple characterization has been the center of interest for many researchers world wide. For further facts on gauge spaces, we recommend the readers to Agarwal et al. [22], Frigon [20], Chis and Precup [21], Chifu and Petrusel [23], Lazara and Petrusel [26], Cherichi et al. [24, 25], Jleli et al. [27] and Branga [28].

In 2013, Wlodarczyk and Plebaniak [31] have given the notion of left (right) $\mathcal{J}$-families of generalized quasi-pseudo distances in quasi-gauge spaces that generalizes the structure of a quasi gauge and provides powerful and useful tools to obtain more general results with weaker assumptions.

The aim of this paper is to introduce $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in b-gauge spaces $(U, Q_{s;\Omega})$. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness and construct F-type contractions $T : U \to U$. Furthermore, we develop novel periodic and fixed point results for these mappings in the setting of b-gauge spaces using $\mathcal{J}_{s;\Omega}$-families on U. The obtained results generalize and improve the existing ones in the literature of fixed point theory. The validity and importance of our theorems are shown through an application by ensuring the existence of a solution of an integral type equation.

2. Preliminaries

The core reason behind to add this section is to recollect some essential concepts and results which are valuable throughout this paper.

In view of generalizing the concept of Banach contraction, Wardowski [30] succeeded to generalize Banach contraction condition by introducing the notions of F-contractions. He introduced the family $\mathfrak{F}$ of all functions $F : (0, \infty) \to \mathbb{R}$ which satisfies the following three conditions:

(F_1) F is a strictly increasing function, i.e., for any $a, b \in (0, \infty)$ with $a < b$ we have $F(a) < F(b)$.
(F_2) For any sequence $(b_n : n \in \mathbb{N})$ in $\mathbb{R}_+$, we have $\lim_{n\to\infty} b_n = 0$ iff $\lim_{n\to\infty} F(b_n) = -\infty$.
(F_3) There exists $p \in (0, 1)$ such that $\lim_{b\to 0^+} b^p F(b) = 0$.

According to Wardowski [30], a mapping $T : U \to U$ on the metric space (U, d) is called an F-contraction if there exists $\tau > 0$, such that

$$d(Tu, Tv) > 0 \Rightarrow \tau + F(d(Tu, Tv)) \leq F(d(u, v)), \quad \text{for all } u, v \in U. \tag{2.1}$$

The implication (2.1) covers various types of contractions. For instance, the case $F_y = \ln y$ corresponds to the Banach contraction.

Wardowski [30] stated a fixed point theorem involving F-contraction mappings.

Theorem 2.1. Let (U, d) be a complete metric space and let $T : U \to U$ be an F-contraction mapping. Then T has a unique fixed point $z \in U$ and for any $z^0 \in U$, the sequence $(z^m = T^{[m]}(z^0) : m \in \mathbb{N})$ converges to the fixed point z.

Minak et al. [15] generalized the above result in the following way:

Theorem 2.2. Let (U, d) be a complete metric space and let $T : U \to U$ be an F-contraction such that

$$\tau + F(d(Tu, Tv)) \leq F\Big(\max\Big\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv) + d(v, Tu)}{2}\Big\}\Big)$$

for all $u, v \in U$, with $d(Tu, Tv) > 0$. Then T has a unique fixed point, whenever T or F is continuous.

A generalized F-contraction of Hardy-Rogers-type is as follows:

Theorem 2.3. Let (U, d) be a complete metric space and let $T : U \to U$ be a generalized F-contraction of Hardy-Rogers-type, i.e., there exist $\tau > 0$ and $F \in \mathfrak{F}$ such that

$$\tau + F(d(Tu, Tv)) \leq F\Big(ad(u, v) + bd(u, Tu) + cd(v, Tv) + ed(u, Tv) + Ld(v, Tu)\Big)$$

for all $u, v \in U$, with $d(Tu, Tv) > 0$ where $a, b, c, d, L \geq 0$, $c \neq 1$, then T has a fixed point. Further, if $a + d + L \leq 1$, such a fixed is unique.

Cosentino et al. [14] introduced the family $\mathfrak{F}_s$ of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following four conditions:

(F_1) F is a strictly increasing function, i.e., for any $a, b \in (0, \infty)$ with $a < b$ we have $F(a) < F(b)$.
(F_2) For any sequence $(b_n : n \in \mathbb{N})$ in $\mathbb{R}_+$, we have $\lim_{n\to\infty} b_n = 0$ iff $\lim_{n\to\infty} F(b_n) = -\infty$.
(F_3) For any sequence $(b_n : n \in \mathbb{N})$ in $\mathbb{R}_+$, we have $\lim_{n\to\infty} b_n = 0$, there exists $p \in (0, 1)$ such that $\lim_{n\to\infty} b_n^p F(b_n) = 0$.
(F_4) For any sequence $(b_n : n \in \mathbb{N})$ in $\mathbb{R}_+$ such that $\tau + F(sb_n) \leq F(b_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n b_n) \leq F(s^{n-1} b_{n-1})$.

Some examples of functions belonging to $\mathfrak{F}_s$ are given below:

(i) $F_x = x + \ln x$ for any $x \in (0, \infty)$.
(ii) $F_y = \ln y$ for any $y \in (0, \infty)$.

Recently, Ali et al. [18] introduced the notion of b-gauge spaces, thus extended the idea of gauge spaces in the setting of b-metric spaces. We note down the following definitions of their work.

Definition 2.4. A map $q : U \times U \to [0, \infty)$ is called as a b-pseudo metric, if for all $x, y, z \in U$, there exists $s \geq 1$ satisfying the following conditions:

(a) $q(x, x) = 0$;
(b) $q(x, y) = q(y, x)$;
(c) $q(x, z) \leq s\{q(x, y) + q(y, z)\}$.

The pair (U, q) is called a b-pseudo metric space.

Definition 2.5. Each family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ of b-pseudo metrics $q_\beta : U \times U \to [0, \infty)$, is called as a b-gauge on U.

Definition 2.6. The family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ is called to be separating if for every pair (x, y) where $x \neq y$, there exists $q_\beta \in Q_{s;\Omega}$ such that $q_\beta(x, y) > 0$.

Definition 2.7. Let the family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ be a b-gauge on U. The topology $\mathcal{T}(Q_{s;\Omega})$ on U whose subbase is defined by the family $\mathcal{B}(Q_{s;\Omega}) = \{B(x, \epsilon_\beta) : x \in U, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(x, \varepsilon_\beta) = \{y \in U : q_\beta(x, y) < \epsilon_\beta\}$, is called the topology induced by $Q_{s;\Omega}$. The pair $(U, \mathcal{T}(Q_{s;\Omega}))$ is called to be a b-gauge space and is Hausdorff if $Q_{s;\Omega}$ is separating.

The following examples shows that a b-pseudo metric space (in fact, a b-gauge space) is the generalization of a metric space, a pseudo metric space (in fact, a gauge space) and a b-metric space.

Example 2.8. [18] Let $U = C([0, \infty), \mathbb{R})$. Describe $q : U \times U \to [0, \infty)$ by

$$q(x(t), y(t)) = \sup_{t\in[0,1]} |x(t) - y(t)|^2.$$

Then q is a b-pseudo metric, but neither a metric, nor a pseudo metric, nor a b-metric.

In this regard, consider $x, y, z \in U$ defined by

$$x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ t - 1 & \text{if } t > 1, \end{cases}$$

$y(t) = 3$ for each $t \geq 0$ and $z(t) = -3$ for each $t \geq 0$. We note that $d(y, z) = 36 \nleq 18 = d(y, x) + d(x, z)$.

Therefore, q is neither a metric, nor a pseudo metric on U. Also, if $u, v \in U$ are defined by

$$u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ t - 1 & \text{if } t > 1, \end{cases}$$

and

$$v(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ 2t - 2 & \text{if } t > 1, \end{cases}$$

then $u \neq v$, but $q(u, v) = 0$. Therefore, q is not a b-metric on U.

Example 2.9. [18] Let $U = C([0, \infty), \mathbb{R})$. Define the family of b-pseudo metrics as $q_m(x(t), y(t)) = \sup_{t\in[0,m]} |x(t) - y(t)|^2$, $m \in \mathbb{N}$. Obviously, $Q_{s;\Omega} = \{q_m : m \in \mathbb{N}\}$ defines a b-gauge on U. Thus $(U, Q_{s;\Omega})$ is a b-gauge space.

Note that $(U, Q_{s;\Omega})$ is not a gauge space, and hence it is not a metric space (as explained in Example 2.8).

3. Main results

In this section, we introduce $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in the b-gauge space $(U, Q_{s;\Omega})$. The new structure determined by these families of distances is a generalization of b-gauges and gives valuable and important tools for inquiring periodic points and fixed points of maps in b-gauge spaces. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness which generalizes the usual $Q_{s;\Omega}$-sequential completeness. Furthermore, we develop novel periodic and fixed point results for F-type contractions in the setting of b-gauge spaces using $\mathcal{J}_{s;\Omega}$-families on U, which generalize and improve all the results in [17] and some of the results in [29].

We now introduce the notion of $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U ($\mathcal{J}_{s;\Omega}$-family on U is the generalization of b-gauges).

Definition 3.1. Let $(U, Q_{s;\Omega})$ be a b-gauge space. The family $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$, is said to be the $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U (for short, $\mathcal{J}_{s;\Omega}$-family on U) if the following statements hold for all $\beta \in \Omega$ and for all $u, v, w \in U$:

($\mathcal{J}1$) $J_\beta(u, w) \leq s_\beta\{J_\beta(u, v) + J_\beta(v, w)\}$; and
($\mathcal{J}2$) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{3.1}$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{3.2}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m, u_m) = 0. \tag{3.3}$$

Take

$$\mathbb{J}_{(U,Q_{s;\Omega})} = \{\mathcal{J}_{s;\Omega} : \mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}\}.$$

Also, we denote

$$U^0_{\mathcal{J}_{s;\Omega}} = \{u \in U : J_\beta(u, u) = 0\}, \ \text{ for all } \ \beta \in \Omega$$

and

$$U^+_{\mathcal{J}_{s;\Omega}} = \{u \in U : J_\beta(u, u) > 0\}, \ \text{ for all } \ \beta \in \Omega.$$

Then, of course $U = U^0_{\mathcal{J}_{s;\Omega}} \cup U^+_{\mathcal{J}_{s;\Omega}}$.

Example 3.2. Let $(U, Q_{s;\Omega})$ be a b-gauge space, where U contains at least two distinct elements and suppose $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ the family of pseudo-b-metrics is a b-gauge on U.

Let the set $F \subset U$ contain at least two distinct elements, but arbitrary and fixed. Let $d_\beta \in (0, \infty)$ satisfy $\delta_\beta(F) < d_\beta$, where $\delta_\beta(F) = \sup\{q_\beta(e, f) : e, f \in F\}$, for all $\beta \in \Omega$. Let $J_\beta : U \times U \to [0, \infty)$ for all $e, f \in U$ and for all $\beta \in \Omega$ be defined by

$$J_\beta(e, f) = \begin{cases} q_\beta(e, f) & \text{if } F \cap \{e, f\} = \{e, f\} \\ d_\beta & \text{if } F \cap \{e, f\} \neq \{e, f\}. \end{cases} \tag{3.4}$$

Then $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}_{(U,Q)}$.

We observe that $J_\beta(e, g) \leq s_\beta\{J_\beta(e, f) + J_\beta(f, g)\}$, for all $e, f, g \in U$, thus condition ($\mathcal{J}_1$) holds. Indeed, condition ($\mathcal{J}_1$) will not hold in case if there exist some $e, f, g \in U$ such that $J_\beta(e, g) = d_\beta$, $J_\beta(e, f) = q_\beta(e, f)$, $J_\beta(f, g) = q_\beta(f, g)$ and $s_\beta\{q_\beta(e, f) + q_\beta(f, g)\} \leq d_\beta$. However, this implies the existence of $h \in \{e, g\}$ such that $h \notin F$ and on other hand, $e, f, g \in F$, which is impossible.

Now, suppose that (3.1) and (3.2) are satisfied by the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in U. Then (3.2) yields that for all $0 < \epsilon < d_\beta$, for all $\beta \in \Omega$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \epsilon \text{ for all } m \geq m_1, \text{ for all } \beta \in \Omega. \tag{3.5}$$

By (3.5) and (3.4), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$F \cap \{v_m, u_m\} = \{v_m, u_m\}, \text{ for all } m \geq m_2$$

and

$$q_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \epsilon.$$

Thus, (3.3) is satisfied by the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$. Therefore, $\mathcal{J}_{s;\Omega}$ is a $\mathcal{J}_{s;\Omega}$-family on U.

Example 3.3. Let $U = [0, 1]$ and $B = \{\frac{1}{2^m} : m \in \mathbb{N}\}$.

Let $Q_{s;\Omega} = \{q\}$, where $q : U \times U \to [0, \infty)$ is a pseudo-b-metric on U defined for all $x, y \in U$ by

$$q(x, y) = \begin{cases} |x - y|^2 & \text{if } x = y \text{ or } \{x, y\} \cap B = \{x, y\}, \\ |x - y|^2 + 1 & \text{if } x \neq y \text{ and } \{x, y\} \cap B \neq \{x, y\}. \end{cases} \tag{3.6}$$

Then $(U, Q_{s;\Omega})$ is a b-gauge space.

Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $x, y \in U$ be defined by

$$J(x, y) = \begin{cases} q(x, y) & \text{if } F \cap \{x, y\} = \{x, y\}, \\ 4 & \text{if } F \cap \{x, y\} \neq \{x, y\}. \end{cases} \tag{3.7}$$

Then $\mathcal{J}_{s;\Omega} = \{J\}$ is a $\mathcal{J}_{s;\Omega}$-family on U (see Example 3.2).

We mention here some trivial properties of $\mathcal{J}_{s;\Omega}$-families in the following remark.

Remark 3.4. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Then the following hold:

(a) $Q_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$.
(b) Let $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$. If $J_\beta(v, v) = 0$ and $J_\beta(u, v) = J_\beta(v, u)$ for all $\beta \in \Omega$ and for all $u, v \in U$ then for each $\beta \in \Omega$, J_β is a pseudo-b metric.
(c) There exist examples of $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$ which show that the maps J_β, $\beta \in \Omega$ are not pseudo-b metrics.

Proposition 3.5. Let $(U, Q_{s;\Omega})$ be a Hausdorff b-gauge space and $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U. Then for each $e, f \in U$, there exists $\beta \in \Omega$ such that

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0.$$

Proof. Let there be $e, f \in U$ where $e \neq f$ such that $J_\beta(e, f) = 0 = J_\beta(f, e)$ for all $\beta \in \Omega$. Then by using property $(\mathcal{J}1)$, we have $J_\beta(e, e) = 0$, for all $\beta \in \Omega$.

Defining sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in U by $u_m = f$ and $v_m = e$, we see that conditions (3.1) and (3.2) of property $(\mathcal{J}2)$ are satisfied, and therefore condition (3.3) holds, which implies that $q_\beta(e, f) = 0$, for all $e, f \in U$ and for all $\beta \in \Omega$. It is a contradiction to the fact that $(U, Q_{s;\Omega})$ is a Hausdorff b-gauge space. Therefore, our supposition is wrong and there exists $\beta \in \Omega$ such that for all $e, f \in U$

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0.$$

□

Now, using $\mathcal{J}_{s;\Omega}$-families on U, we establish the following concept of $\mathcal{J}_{s;\Omega}$-completeness in the b-gauge space $(U, Q_{s;\Omega})$ which generalizes the usual $Q_{s;\Omega}$-sequential completeness.

Definition 3.6. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. A sequence $(u_m : m \in \mathbb{N})$ is a $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U if

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0, \quad \text{for all } \beta \in \Omega.$$

Definition 3.7. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. The sequence $(u_m : m \in \mathbb{N})$ is called to be $\mathcal{J}_{s;\Omega}$-convergent to $u \in U$ if $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} u_m = u$, where

$$\overset{\mathcal{J}_{s;\Omega}}{\lim_{m\to\infty}} u_m = u \Leftrightarrow \lim_{m\to\infty} J_\beta(u, u_m) = 0 = \lim_{m\to\infty} J_\beta(u_m, u), \quad \text{for all } \beta \in \Omega.$$

Definition 3.8. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. If $S^{\mathcal{J}_{s;\Omega}}_{(u_m:m\in\mathbb{N})} \neq \emptyset$, where

$$S^{\mathcal{J}_{s;\Omega}}_{(u_m:m\in\mathbb{N})} = \{u \in U : \overset{\mathcal{J}_{s;\Omega}}{\lim_{m\to\infty}} u_m = u\}.$$

Then the sequence $(u_m : m \in \mathbb{N})$ in U is $\mathcal{J}_{s;\Omega}$-convergent in U.

Definition 3.9. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. The space $(U, Q_{s;\Omega})$ is called $\mathcal{J}_{s;\Omega}$-sequentially complete, if every $\mathcal{J}_{s;\Omega}$-Cauchy in U is a $\mathcal{J}_{s;\Omega}$-convergent in U.

Example 3.10. Let U, $Q_{s;\Omega} = \{q\}$, F and $\mathcal{J}_{s;\Omega} = \{J\}$ be as in Example 3.3.

First, we show that $(U, Q_{s;\Omega})$ is not $Q_{s;\Omega}$-sequential complete.

For this, let $\{v_m\} = \{\frac{1}{2^m} : m \in \mathbb{N}\}$, then by (3.6) for all $\epsilon > 0$ and for all $n, m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that

$$q(v_m, v_n) = \left|\frac{1}{2^m} - \frac{1}{2^n}\right|^2 < \epsilon, \quad \text{for all } n \geq m \geq k_0.$$

Thus, $\{v_m : m \in \mathbb{N}\}$ is a $Q_{s;\Omega}$-Cauchy sequence. However, this sequence is not $Q_{s;\Omega}$-convergent in U. Otherwise, suppose that $\lim_{m\to\infty} v_m = v$, for some $v \in U$. We may suppose without loosing generality that for all $0 < \epsilon < 1$, there exists $k_0 \in \mathbb{N}$ such that

$$q(v, v_m) < \epsilon < 1, \quad \text{for all } m \geq k_0. \tag{3.8}$$

We have the following two cases:

(i) If $v \notin B$, then using (3.6) we can write

$$q(v, v_m) = |v - v_m|^2 + 1 < \epsilon < 1, \quad \text{for all } \; m \geq k_0.$$

It is not possible.

(ii) If $v \in B$, then let $v = \frac{1}{2^{m_1}}$, for some $m_1 \in \mathbb{N}$ and using (3.6), we can write

$$q(v, v_m) = |v - v_m|^2 = \left| \frac{1}{2^{m_1}} - \frac{1}{2^m} \right|^2.$$

Taking limit interior as $m \to \infty$, we get

$$\lim_{m \to \infty} q(v, v_m) = \frac{1}{2^{m_2}}, \quad \text{where } \; m_2 = 2m_1.$$

By (3.8), it is impossible.

Thus, we conclude that $(U, Q_{s;\Omega})$ is not $Q_{s;\Omega}$-sequential complete. Next, we show that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.

Let $\{v_m : m \in \mathbb{N}\}$ be a $\mathcal{J}_{s;\Omega}$-Cauchy sequence. Without loosing generality, we may assume that for $0 < \epsilon < 1$, there exists $k_0 \in \mathbb{N}$ such that

$$J(v_m, v_n) < \epsilon < 1, \quad \text{for all } \; n \geq m \geq k_0. \tag{3.9}$$

Then by (3.7), (3.6) and (3.9), we obtain

$$J(v_m, v_n) = q(v_m, v_n) = |v_m - v_n|^2 < \epsilon < 1, \quad \text{for all } \; n \geq m \geq k_0, \tag{3.10}$$

$$v_m \in F = \left[\frac{1}{8}, 1\right], \quad \text{for all } \; m \geq k_0, \tag{3.11}$$

and

$$v_m = v_{m_0}, \quad \text{for all, } \; m_0 \geq k_0 \; \text{ or } \; v_m \in B, \quad \text{for all } \; m \geq k_0. \tag{3.12}$$

From (3.12), we have two cases:

(i) If $v_m = v_{m_0}$ for all $m_0 \geq k_0$, then $\{v_m : m \in \mathbb{N}\}$ represents a constant sequence and by (3.11), (3.7), (3.6) and (3.12) the sequence $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-convergent to v_{m_0}.

(ii) If $v_m \in B$, for all $m_0 \geq k_0$, let $v_{k_0+s} \in B$ for all $s \in \mathbb{N}$. This together with (3.10)–(3.12) imply that $v_{k_0+s} = \frac{1}{2}$ for all $s \in \mathbb{N}$ or $v_{k_0+s} = \frac{1}{4}$ for all $s \in \mathbb{N}$ or $v_{k_0+s} = \frac{1}{8}$ for all $s \in \mathbb{N}$. Therefore, the sequence $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-convergent to the point $\frac{1}{2}$ or $\frac{1}{4}$ or $\frac{1}{8}$, respectively.

Thus, we conclude that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.

Remark 3.11. Let $(U, Q_{s;\Omega})$ be a b-gauge space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U.

(i) Example 3.10 indicates that there exists a b-gauge spaces $(U, Q_{s;\Omega})$ and $\mathcal{J}_{s;\Omega}$-family on U with $\mathcal{J}_{s;\Omega} \neq Q_{s;\Omega}$ such that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete, but not $Q_{s;\Omega}$-sequential complete.

(ii) For each subsequence $(v_m : m \in \mathbb{N})$ of $(u_m : m \in \mathbb{N})$, where $(u_m : m \in \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-convergent in U, we have $S^{\mathcal{J}_{s;\Omega}}_{(u_m : m \in \mathbb{N})} \subset S^{\mathcal{J}_{s;\Omega}}_{(v_m : m \in \mathbb{N})}$.

Definition 3.12. Let $(U, Q_{s;\Omega})$ be a b-gauge space. The map $T^{[k]} : U \to U$ (where $k \in \mathbb{N}$) is called to be a $Q_{s;\Omega}$-closed map on U if for each sequence $(x_m : m \in \mathbb{N})$ in $T^{[k]}(U)$, which is $Q_{s;\Omega}$-converging in U, i.e., $S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})} \neq \emptyset$ and its subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfy $v_m = T^{[k]}(u_m)$, for all $m \in \mathbb{N}$ has the property that there exists $w \in S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}$ such that $w \in T^{[k]}(w)$.

Now, we present some fixed and periodic point theorems in the b-gauge space $(U, Q_{s;\Omega})$, using $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances by incorporating the idea of Cosentino for the family $\mathfrak{F}_{\mathfrak{s}}$ of all functions $F : (0, \infty) \to \mathbb{R}$ in the setting of b-metric spaces and F-contraction of Hardy-Rogers type.

Theorem 3.13. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let $T : U \to U$ be a mapping such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and we have $F \in \mathfrak{F}_{\mathfrak{s}}$ and $\tau > 0$ such that:

$$\begin{aligned} \alpha(u,v) \geq 1 \Rightarrow \tau + F(s_\beta J_\beta(Tu,Tv)) \ &\leq F\Big(a_\beta J_\beta(u,v) + b_\beta J_\beta(u,Tu) + c_\beta J_\beta(v,Tv) \\ &\quad + e_\beta J_\beta(u,Tv) + L_\beta J_\beta(v,Tu)\Big) \end{aligned} \tag{3.13}$$

for all $\beta \in \Omega$ and for any $u, v \in U$, whenever $J_\beta(Tu, Tv) \neq 0$.

Further, $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ are such that $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$ for each $\beta \in \Omega$. Moreover, assume that the following conditions hold:

(i) There exists $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$.
(ii) If $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$.
(iii) If a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \geq 1$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z^m = z$, then $\alpha(z^m, z) \geq 1$ and $\alpha(z, z^m) \geq 1$.

Then the following statements hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is $Q_{s;\Omega}$-closed map on U and $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, for each $\beta \in \Omega$. Then
   - (a_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
   - (a_2) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and
   - (a_3) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.
(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{s;\Omega})$ is a Hausdorff space. Then
   - (b_1) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;
   - (b_2) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and
   - (b_3) for all $z \in \mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is a $\mathcal{J}_{s;\Omega}$-cauchy sequence in U.

Using assumption (i), there exists $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$. Now, for each $\beta \in \Omega$, using (3.13) we can write

$$\begin{aligned} \tau + F(s_\beta J_\beta(z^1, z^2)) \ &= \ \tau + F(s_\beta J_\beta(Tz^0, Tz^1)) \\ &\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, Tz^0) + c_\beta J_\beta(z^1, Tz^1) \end{aligned}$$

$$
\begin{aligned}
&+e_\beta J_\beta(z^0, Tz^1) + L_\beta J_\beta(z^1, Tz^0)\Big) \\
&\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, z^1) + c_\beta J_\beta(z^1, z^2) \\
&\quad +e_\beta J_\beta(z^0, z^2) + L_\beta.0\Big) \\
&\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, z^1) + c_\beta J_\beta(z^1, z^2) \\
&\quad +e_\beta s_\beta(J_\beta(z^0, z^1) + J_\beta(z^1, z^2))\Big) \\
&= F\Big((a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^0, z^1) + (c_\beta + e_\beta s_\beta)J_\beta(z^1, z^2)\Big). \qquad (3.14)
\end{aligned}
$$

As F is strictly increasing, we can write from above that

$$s_\beta J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^0, z^1) + (c_\beta + e_\beta s_\beta)J_\beta(z^1, z^2), \quad \text{for all } \beta \in \Omega.$$

It is written as

$$(s_\beta - c_\beta - e_\beta s_\beta)J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^0, z^1), \quad \text{for all } \beta \in \Omega.$$

That is,

$$(1 - \frac{c_\beta}{s_\beta} - e_\beta)s_\beta J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^0, z^1), \quad \text{for all } \beta \in \Omega.$$

Since $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$, we get

$$1 - \frac{c_\beta}{s_\beta} - e_\beta \geq 1 - c_\beta - e_\beta > a_\beta + b_\beta + s_\beta e_\beta \geq 0,$$

hence

$$s_\beta J_\beta(z^1, z^2) < J_\beta(z^0, z^1), \quad \text{for all } \beta \in \Omega.$$

Now, using (3.14), we can write

$$\tau + F(s_\beta J_\beta(z^1, z^2)) < F(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega.$$

Using assumption (ii), we have $\alpha(Tz^0, Tz^1) = \alpha(z^1, z^2) \geq 1$. For each $\beta \in \Omega$, using (3.13) we can write

$$
\begin{aligned}
\tau + F(s_\beta J_\beta(z^2, z^3)) &= \tau + F(s_\beta J_\beta(Tz^1, Tz^2)) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, Tz^1) + c_\beta J_\beta(z^2, Tz^2) \\
&\quad +e_\beta J_\beta(z^1, Tz^2) + L_\beta J_\beta(z^2, Tz^1)\Big) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, z^2) + c_\beta J_\beta(z^2, z^3) \\
&\quad +e_\beta J_\beta(z^1, z^3) + L_\beta.0\Big) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, z^2) + c_\beta J_\beta(z^2, z^3) \\
&\quad +e_\beta s_\beta(J_\beta(z^1, z^2) + J_\beta(z^2, z^3))\Big) \\
&= F\Big((a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^1, z^2) + (c_\beta + e_\beta s_\beta)J_\beta(z^2, z^3)\Big). \qquad (3.15)
\end{aligned}
$$

As F is strictly increasing, we can write from above that

$$s_\beta J_\beta(z^2, z^3) < (a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^1, z^2) + (c_\beta + e_\beta s_\beta)J_\beta(z^2, z^3), \quad \text{for all } \beta \in \Omega.$$

We can also write it as

$$(s_\beta - c_\beta - e_\beta s_\beta))J_\beta(z^2, z^3) < (a_\beta + b_\beta + e_\beta s_\beta)J_\beta(z^1, z^2), \quad \text{for all } \beta \in \Omega.$$

Since $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$, we get

$$1 - \frac{c_\beta}{s_\beta} - e_\beta \geq 1 - c_\beta - e_\beta > a_\beta + b_\beta + s_\beta e_\beta \geq 0,$$

$$s_\beta J_\beta(z^2, z^3) < J_\beta(z^1, z^2), \quad \text{for all } \beta \in \Omega.$$

Now, using (3.15), we can write

$$\tau + F(s_\beta J_\beta(z^2, z^3)) < F(J_\beta(z^1, z^2)), \quad \text{for all } \beta \in \Omega.$$

Proceeding in the above manner, we get a sequence $(z^m : m \in \{0\} \cup \mathbb{N}) \subset U$ such that $z^m = Tz^{m-1}, z^{m-1} \neq z^m$ and $\alpha(z^{m-1}, z^m) \geq 1$, for each $m \in \mathbb{N}$. Furthermore,

$$\tau + F(s_\beta J_\beta(z^m, z^{m+1})) < F(J_\beta(z^{m-1}, z^m)), \quad \text{for all } \beta \in \Omega.$$

Using property (F_4), for all $m \in \mathbb{N}$, we can write

$$\tau + F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(s_\beta^{m-1} J_\beta(z^{m-1}, z^m)), \quad \text{for all } \beta \in \Omega.$$

Thus,

$$F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(J_\beta(z^0, z^1)) - m\tau, \quad \text{for all } \beta \in \Omega \text{ and } m \in \mathbb{N}. \tag{3.16}$$

Letting $m \to \infty$, from (3.16) we get $\lim_{m\to\infty} F(s_\beta^m J_\beta(z^m, z^{m+1})) = -\infty$ for all $\beta \in \Omega$. Hence, using property (F_2) we get $\lim_{m\to\infty} s_\beta^m J_\beta(z^m, z^{m+1}) = 0$. Let $(J_\beta)_m = J_\beta(z^m, z^{m+1})$ for all $\beta \in \Omega$ and $m \in \mathbb{N}$. From (F_3), there exists $p \in (0, 1)$ such that

$$\lim_{m\to\infty} (s_\beta^m (J_\beta)_m)^p F(s_\beta^m (J_\beta)_m) = 0, \quad \text{for all } \beta \in \Omega.$$

From (3.16), for all $\beta \in \Omega$ and $m \in \mathbb{N}$, we can write

$$(s_\beta^m (J_\beta)_m)^p F((s_\beta^m J_\beta)_m) - (s_\beta^m (J_\beta)_m)^p F((J_\beta)_0) \leq -(s_\beta^m (J_\beta)_m)^p m\tau \leq 0. \tag{3.17}$$

Applying $m \to \infty$, we have

$$\lim_{m\to\infty} m(s_\beta^m (J_\beta)_m)^p = 0, \quad \text{for all } \beta \in \Omega. \tag{3.18}$$

This implies there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that $m(s_\beta^m (J_\beta)_m)^p \leq 1$ for each $m \geq m_1$ and for all $\beta \in \Omega$. Hence, we can write

$$s_\beta^m (J_\beta)_m \leq \frac{1}{m^{\frac{1}{p}}}, \quad \text{for all } m \geq m_1 \text{ and } \beta \in \Omega\,. \tag{3.19}$$

Now, by repeated use of ($\mathcal{J}1$) and (3.19) for all $m, n \in \mathbb{N}$ such that $n > m > m_1$ and for all $\beta \in \Omega$, we get

$$J_\beta(z^m, z^n) \leq \sum_{i=m}^{n-1} s_\beta^i (J_\beta)_i \leq \sum_{i=m}^{\infty} s_\beta^i (J_\beta)_i \leq \sum_{i=m}^{\infty} \frac{1}{i^{\frac{1}{p}}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{p}}}$ is a convergent series, we have

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.20}$$

Since $(U, Q_{s;\Omega})$ is a $\mathcal{J}_{s;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, we can write

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.21}$$

Thus, from (3.20) and (3.21), fixing $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying ($\mathcal{J}2$) to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = 0, \quad \text{for all } \beta \in \Omega.$$

This implies $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$, we have

$$z^{(m+1)k} = T^{[k]}(z^{mk}), \qquad \text{for } m \in \{0\} \cup \mathbb{N}.$$

Thus, defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we can write

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset.$$

Also,

$$(y_m = z^{(m+1)k} : m \in \mathbb{N}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk} : m \in \mathbb{N}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are $Q_{s;\Omega}$-convergent to each point $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, using the fact below $S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(y_m:m\in\mathbb{N})}$ and $S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}$ and the supposition that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U, there exists $z \in S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$. Thus, (a_1) holds.

The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

To prove (a_3), on contrary suppose that $J_\beta(z, Tz) > 0$ for some $\beta \in \Omega$, there exists $m_0 \in \mathbb{N}$ such that $J_\beta(z^m, Tz) > 0$ for each $m \geq m_0$. Hence, for each $m \geq m_0$, use triangular inequality and inequality (3.13) to obtain

$$\begin{aligned}
J_\beta(z, Tz) &\leq s_\beta\{J_\beta(z, z^{m+1}) + J_\beta(z^{m+1}, Tz)\}\\
&= s_\beta\{J_\beta(z, z^{m+1}) + J_\beta(Tz^m, Tz)\}\\
&\leq s_\beta\{J_\beta(z, z^{m+1}) + a_\beta J_\beta(z^m, z) + b_\beta J_\beta(z^m, Tz^m) + c_\beta J_\beta(z, Tz)\\
&\quad + e_\beta J_\beta(z^m, Tz) + L_\beta J_\beta(z, Tz^m)\}\\
&\leq s_\beta\{J_\beta(z, z^{m+1}) + a_\beta J_\beta(z^m, z) + b_\beta J_\beta(z^m, z^{m+1}) + c_\beta J_\beta(z, Tz)\\
&\quad + e_\beta s_\beta\{J_\beta(z^m, z) + J_\beta(z, Tz)\} + L_\beta J_\beta(z, z^{m+1})\}.
\end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(z, Tz) \leq s_\beta\{c_\beta + e_\beta s_\beta\}J_\beta(z, Tz), \qquad \forall\ \beta \in \Omega.$$

We have assumed that $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, so

$$J_\beta(z, Tz) \leq s_\beta\{c_\beta + e_\beta s_\beta\}J_\beta(z, Tz) < J_\beta(z, Tz), \qquad \forall\ \beta \in \Omega.$$

It is absurd, thus $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$.

Next, we prove that $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. On contrary suppose that $J_\beta(Tz, z) > 0$ for some $\beta \in \Omega$, there exists $m_0 \in \mathbb{N}$ such that $J_\beta(Tz, z^m) > 0$ for each $m \geq m_0$. Hence, for each $m \geq m_0$, use triangular inequality and inequality (3.13) to obtain

$$\begin{aligned}
J_\beta(Tz, z) &\leq s_\beta\{J_\beta(Tz, z^{m+1}) + J_\beta(z^{m+1}, z)\}\\
&= s_\beta\{J_\beta(Tz, Tz^m) + J_\beta(z^{m+1}, z)\}\\
&\leq s_\beta\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, Tz^m) + e_\beta J_\beta(z, Tz^m)\\
&\quad + L_\beta J_\beta(z^m, Tz) + J_\beta(z^{m+1}, z)\}\\
&\leq s_\beta\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, z^{m+1}) + e_\beta J_\beta(z, z^{m+1})\\
&\quad + L_\beta s_\beta\{J_\beta(z^m, z) + J_\beta(z, Tz)\} + J_\beta(z^{m+1}, z)\}.
\end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(Tz, z) \leq s_\beta\{b_\beta + L_\beta s_\beta\}J_\beta(z, Tz), \qquad \forall\ \beta \in \Omega.$$

We have proved that $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$, so $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. Hence, the assertion (a_3) holds.

(III) Since $(U, Q_{s;\Omega})$ is a Hausdorff space, using Proposition (3.5), assertion (a_3) suggests that for $z \in \text{Fix}(T^{[k]})$, we have $z = T(z)$. This gives $z \in \text{Fix}(T)$. Hence, (b_1) is true.

Assertions (a_2) and (b_1) imply (b_2) . To prove assertion (b_3), consider $(\mathcal{J}1)$ and use (a_3) and (b_1) to have for all $z \in \text{Fix}(T^{[k]}) = \text{Fix}(T)$,

$$J_\beta(z, z) \leq s_\beta\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0, \ \text{ for all } \beta \in \Omega.$$

□

Theorem 3.14. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let $T : U \to U$ be a mapping such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and we have $F \in \mathfrak{F}_s$ and $\tau > 0$, so that

$$\begin{aligned}\alpha(u, v) \geq 1 \Rightarrow \tau + F(s_\beta J_\beta(Tu, Tv))\\ \leq F\Big(\max\Big\{J_\beta(u, v), J_\beta(u, Tu), J_\beta(v, Tv), \frac{J_\beta(u, Tv) + J_\beta(v, Tu)}{2s_\beta}\Big\} + L_\beta J_\beta(v, Tu)\Big)\end{aligned} \tag{3.22}$$

for all $\beta \in \Omega$ and for any $u, v \in U$, whenever $J_\beta(Tu, Tv) \neq 0$. Also, $L_\beta \geq 0$.

Assume, moreover that, the following conditions hold:

(i) There exists $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$.
(ii) If $\alpha(u, v) \geq 1$, then $\alpha(Tu, Tv) \geq 1$.

Then the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U, thus $S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U. Then
  (a$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
  (a$_2$) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$.
(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and T be continuous. Then
  (b$_1$) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;
  (b$_2$) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$; and
  (b$_3$) for all $z \in \mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-cauchy sequence in U.

Using assumption (i) there exists $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$. For each $\beta \in \Omega$, using (3.22) we can write

$$\begin{aligned}\tau + F(s_\beta J_\beta(z^1, z^2)) &= \tau + F(s_\beta J_\beta(Tz^0, Tz^1))\\ &\leq F\Big(\max\Big\{J_\beta(z^0, z^1), J_\beta(z^0, Tz^0), J_\beta(z^1, Tz^1),\\ &\qquad \frac{J_\beta(z^0, Tz^1) + J_\beta(z^1, Tz^0)}{2s_\beta}\Big\} + L_\beta J_\beta(z^1, Tz^0)\Big)\\ &= F\Big(\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\}\Big).\end{aligned}$$

We observe a contradiction if we choose $\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\} = J_\beta(z^1, z^2)$. Hence, choosing $\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\} = J_\beta(z^0, z^1)$ for all $\beta \in \Omega$, we get

$$\tau + F(s_\beta J_\beta(z^1, z^2)) < F(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega.$$

Using assumption (ii), we have $\alpha(Tz^0, Tz^1) = \alpha(z^1, z^2) \geq 1$. For each $\beta \in \Omega$, using (3.22) we can write

$$\begin{aligned}\tau + F(s_\beta J_\beta(z^2, z^3)) &= \tau + F(s_\beta J_\beta(Tz^1, Tz^2))\\ &\leq F\Big(\max\Big\{J_\beta(z^1, z^2), J_\beta(z^1, Tz^1), J_\beta(z^2, Tz^2),\\ &\qquad \frac{J_\beta(z^1, Tz^2) + J_\beta(z^2, Tz^1)}{2s_\beta}\Big\} + L_\beta J_\beta(z^2, Tz^1)\Big)\\ &= F\Big(\max\{J_\beta(z^1, z^2), J_\beta(z^2, z^3)\}\Big).\end{aligned}$$

We observe a contradiction if we choose $\max\{J_\beta(z^1,z^2), J_\beta(z^2,z^3)\} = J_\beta(z^2,z^3)$. Hence, choosing $\max\{J_\beta(z^1,z^2), J_\beta(z^2,z^3)\} = J_\beta(z^1,z^2)$ for all $\beta \in \Omega$, we get

$$\tau + F(s_\beta J_\beta(z^2,z^3)) < F(J_\beta(z^1,z^2)), \quad \text{for all } \beta \in \Omega.$$

Proceeding in the above manner, we get a sequence $(z^m : m \in \{0\}\cup\mathbb{N}) \subset U$ such that $z^m = Tz^{m-1}, z^{m-1} \neq z^m$ and $\alpha(z^{m-1}, z^m) \geq 1$, for each $m \in \mathbb{N}$. Furthermore,

$$\tau + F(s_\beta J_\beta(z^m,z^{m+1})) < F(J_\beta(z^{m-1},z^m)), \quad \text{for all } \beta \in \Omega.$$

Using property (F_4), for all $m \in \mathbb{N}$, we get

$$\tau + F(s_\beta^m J_\beta(z^m,z^{m+1})) < F(s_\beta^{m-1} J_\beta(z^{m-1},z^m)), \quad \text{for all } \beta \in \Omega.$$

Furthermore,

$$F(s_\beta^m J_\beta(z^m,z^{m+1})) < F(J_\beta(z^0,z^1)) - m\tau, \quad \text{for all } \beta \in \Omega \text{ and } m \in \mathbb{N}. \tag{3.23}$$

Now, letting $m \to \infty$, from (3.23) we get $\lim_{m\to\infty} F(s_\beta^m J_\beta(z^m,z^{m+1})) = -\infty$ for all $\beta \in \Omega$. Hence, using property (F_2) we get $\lim_{m\to\infty} s_\beta^m J_\beta(z^m,z^{m+1}) = 0$. Let $(J_\beta)_m = J_\beta(z^m,z^{m+1})$ for all $\beta \in \Omega$ and $m \in \mathbb{N}$. From (F_3), there exists $p \in (0,1)$ such that

$$\lim_{m\to\infty} (s_\beta^m (J_\beta)_m)^p F(s_\beta^m (J_\beta)_m) = 0, \quad \text{for all } \beta \in \Omega.$$

From (3.23), we can write

$$(s_\beta^m (J_\beta)_m)^p F((s_\beta^m J_\beta)_m) - (s_\beta^m (J_\beta)_m)^p F((J_\beta)_0) \leq -(s_\beta^m (J_\beta)_m)^p m\tau \leq 0, \quad \text{for all } \beta \in \Omega \text{ and } m \in \mathbb{N}. \tag{3.24}$$

Applying $m \to \infty$, we have

$$\lim_{m\to\infty} m(s_\beta^m (J_\beta)_m)^p = 0, \quad \text{for all } \beta \in \Omega. \tag{3.25}$$

This implies there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that $m(s_\beta^m (J_\beta)_m)^p \leq 1$ for each $m \geq m_1$ and for all $\beta \in \Omega$. Hence, we can write

$$s_\beta^m (J_\beta)_m \leq \frac{1}{m^{\frac{1}{p}}}, \quad \text{for all } m \geq m_1 \text{ and } \beta \in \Omega\ . \tag{3.26}$$

Now, by repeated use of $(\mathcal{J}1)$ and (3.26) for all $m, n \in \mathbb{N}$ such that $n > m > m_1$ and for all $\beta \in \Omega$, we get

$$J_\beta(z^m,z^n) \leq \sum_{i=m}^{n-1} s_\beta^i (J_\beta)_i \leq \sum_{i=m}^{\infty} s_\beta^i (J_\beta)_i \leq \sum_{i=m}^{\infty} \frac{1}{i^{\frac{1}{p}}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{p}}}$ is a convergent series, we have

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m,z^n) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.27}$$

Now, since $(U, Q_{s;\Omega})$ is a $\mathcal{J}_{s;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\}\cup\mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-convergent in U. Thus for all $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$, we can write

$$\lim_{m\to\infty} J_\beta(z,z^m) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.28}$$

Thus, from (3.27) and (3.28), fixing $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = 0, \quad \text{for all } \beta \in \Omega.$$

This implies that $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$, and

$$z^{(m+1)k} = T^{[k]}(z^{mk}), \qquad \text{for } m \in \{0\} \cup \mathbb{N}$$

defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we can write

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset.$$

Also,

$$(y_m = z^{(m+1)k} : m \in \mathbb{N}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk} : m \in \mathbb{N}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are $Q_{s;\Omega}$-convergent to each point $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, using the fact below $S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(y_m:m\in\mathbb{N})}$, $S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}$ and the supposition that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U, there exists $z \in S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$. Thus, (a_1) holds.

The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(III) By (a_2), for all $z^0 \in U$, there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, and so we have $\lim_{m\to\infty} z^m = z$.

Now, if T is continuous, then $z = \lim_{m\to\infty} z^{m+1} = \lim_{m\to\infty} Tz^m = T(\lim_{m\to\infty} z^m) = T(z)$. This gives $z \in \text{Fix}(T)$. Hence, (b_1) is true. Assertions (a_2) and (b_1) imply (b_2). To prove assertion (b_3), since $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$, this implies that $z = T(z) \in U^0_{\mathcal{J}_{s;\Omega}}$.

Therefore, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$. □

Example 3.15. Let $U = [0, 1]$ and $B = \{\frac{1}{2^m} : m \in \mathbb{N}\}$.

Let $Q_{s;\Omega} = \{q\}$, where $q : U \times U \to [0, \infty)$ is a pseudo-b-metric on U defined for all $x, y \in U$ by

$$q(x, y) = \begin{cases} |x - y|^2 & \text{if } x = y \text{ or } \{x, y\} \cap B = \{x, y\}, \\ |x - y|^2 + 1 & \text{if } x \neq y \text{ and } \{x, y\} \cap B \neq \{x, y\}. \end{cases} \tag{3.29}$$

Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $x, y \in U$ be defined by

$$J(x, y) = \begin{cases} q(x, y) & \text{if } F \cap \{x, y\} = \{x, y\}, \\ 4 & \text{if } F \cap \{x, y\} \neq \{x, y\}. \end{cases} \tag{3.30}$$

Define $\alpha : U \times U \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 5 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

The single-valued map T is defined by

$$T(x) = \frac{x+1}{5}, \quad \text{for all } x \in U. \tag{3.31}$$

Note that $T(U) = [\frac{1}{5}, \frac{2}{5}] \subset U^0_{\mathcal{J}_{s;\Omega}} = [\frac{1}{8}, 1]$. Also, take $F(x) = \ln(x)$, then $F \in \mathfrak{F}_s$.

(I.1) $(U, Q_{s;\Omega})$ is a b-gauge space, which is also Hausdorff.
(I.2) The family $\mathcal{J}_{s;\Omega} = \{J\}$ is $\mathcal{J}_{s;\Omega}$-family on U (see Example 3.2).
(I.3) $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete (follows from Example 3.10).
(I.4) Next, applying $F(x) = \ln(x)$ to condition (3.13), we show that T satisfies the following condition.

$$\alpha(x, y) \geq 1 \Rightarrow J(Tx, Ty) \leq aJ(x, y) + bJ(x, Tx) + cJ(y, Ty) + eJ(x, Ty) + LJ(y, Tx)$$

for any $x, y \in U$ whenever $J(Tx, Ty) \neq 0$. It is obvious that above condition holds for $a = b = c = \frac{1}{5}$ and $e = L = 0$.
(I.5) Assumptions (i)–(iii) of Theorem 3.13 hold. For $z_0 = 0$ and $z_1 = Tz_0 = \frac{1}{5}$, we have $\alpha(z_0, Tz_0) > 1$. Also, $\alpha(Tx, Ty) > 1$ if $\alpha(x, y) > 1$. Finally, if a sequence $(z_m : m \in \mathbb{N})$ in U is such that $\alpha(z_m, z_{m+1}) \geq 1$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z$, then $\alpha(z_m, z) \geq 1$ and $\alpha(z, z_m) \geq 1$.
(I.6) Finally, we show that T is a $Q_{s;\Omega}$-closed map on U. For this, let $(z_m : m \in \mathbb{N})$ be a sequence in $T(U) = [\frac{1}{5}, \frac{2}{5}]$ which is $Q_{s;\Omega}$-convergent to each point of $S^{Q_{s;\Omega}}_{(z_m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$. Let the subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfy $v_m = T(u_m)$, for all $m \in \mathbb{N}$.
Let $z \in S^{Q_{s;\Omega}}_{(z_m : m\in\{0\}\cup\mathbb{N})}$, then without loosing generality we may assume that for all $0 < \epsilon_1 < 1$ there exists $k \in \mathbb{N}$ such that

$$q(z, z_m) = |z - z_m|^2 < \epsilon_1 < 1, \quad \text{for all } m \geq k.$$

As a result, for $\epsilon = \sqrt{\epsilon_1}$, we can also write for all $0 < \epsilon < 1$ there exists $k \in \mathbb{N}$ such that

$$[| z - z_m |< \epsilon] \wedge [| z - u_m |< \epsilon] \wedge [| z - v_m |< \epsilon] \wedge [v_m = T(u_m)], \quad \text{for all } m \geq k.$$

In particular, this implies that

$$| z - u_m |=| z - 5v_m + 1 |=| 5z - 4z - 5v_m + 1 |=| 4(\frac{1}{4} - z) - 5(v_m - z) |< \epsilon$$

and we obtain

$$4 \,| \frac{1}{4} - z |< \epsilon + 5 \,| v_m - z |, \quad \text{for all } m \geq k.$$

Since $| z - v_m |\to 0$, when $m \to \infty$, we get $| \frac{1}{4} - z |< \epsilon_2$ where $\epsilon_2 = \frac{\epsilon}{4} < \frac{1}{4}$. This gives $S^{Q_{s;\Omega}}_{(z_m : m\in\mathbb{N})} = \{\frac{1}{4}\}$ and so there exists $z = \frac{1}{4} \in S^{Q_{s;\Omega}}_{(z_m : m\in\mathbb{N})}$ such that $\frac{1}{4} = T(\frac{1}{4})$. Hence, T is a$Q_{s;\Omega}$-closed map on U.

(I.7) As all the assumptions of Theorem 3.13 hold, we have

$$\mathrm{Fix}(T) = \Big\{\frac{1}{4}\Big\},$$

$$\lim_{m\to\infty}^{Q_{s;\Omega}} z^m = \frac{1}{4},$$

and

$$J(\frac{1}{4}, \frac{1}{4}) = 0.$$

Let $(U, Q_{s;\Omega})$ be a b-gauge space and $G = (V, E)$ be a directed graph such that set of vertices V is equal to U and set of edges E includes $\{(u, u) : u \in U\}$, but G includes no parallel edges. We obtain the following corollaries from our theorems by defining $\alpha : U \times U \to [0, \infty)$ for some $\kappa \geq 1$ in the following way.

$$\alpha(u, v) = \begin{cases} \kappa & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{3.32}$$

Corollary 3.16. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let $T : U \to U$ be a mapping such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\begin{aligned}(u, v) \in E \Rightarrow \tau + F(s_\beta J_\beta(Tu, Tv)) \leq & F(a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) \\ & + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu))\end{aligned} \tag{3.33}$$

for all $\beta \in \Omega$ and for any $u, v \in U$ whenever $J_\beta(Tu, Tv) \neq 0$.

Further, $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ are such that $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$ for each $\beta \in \Omega$. Assume, moreover that, the following conditions hold:

(i) There exists $z^0 \in U$ such that $(z^0, z^1) \in E$.
(ii) If $(u, v) \in E$, then $(Tu, Tv) \in E$.
(iii) If a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim_{m\to\infty}^{\mathcal{J}_{s;\Omega}} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.

Then the following statements hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U and $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, for each $\beta \in \Omega$. Then
   (a_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
   (a_2) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$; and
   (a_3) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.
(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{s;\Omega})$ is a Hausdorff space. Then
   (b_1) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;

(b$_2$) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and
(b$_3$) for all $z \in \mathrm{Fix}(T) = \mathrm{Fix}(T^{[k]})$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Corollary 3.17. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, is the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let $T : U \to U$ be a mapping such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$(u, v) \in E \Rightarrow \tau + F(s_\beta J_\beta(Tu, Tv)) \leq F\Big(\max\Big\{J_\beta(u, v), J_\beta(u, Tu), J_\beta(v, Tv), \frac{J_\beta(u, Tv) + J_\beta(v, Tu)}{2s_\beta}\Big\} + L_\beta J_\beta(v, Tu)\Big) \tag{3.34}$$

for all $\beta \in \Omega$ and for any $u, v \in U$, whenever $J_\beta(Tu, Tv) \neq 0$. Also, $L_\beta \geq 0$.

Assume, moreover that, the following conditions hold:

(i) There exists $z^0 \in U$ such that $(z^0, z^1) \in E$.
(ii) If $(u, v) \in E$, then $(Tu, Tv) \in E$.

Then the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U. Then
  (a$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
  (a$_2$) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.
(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and T be continuous. Then
  (b$_1$) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;
  (b$_2$) for all $z^0 \in U$, there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and
  (b$_3$) for all $z \in \mathrm{Fix}(T) = \mathrm{Fix}(T^{[k]})$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

4. Application

A volterra integral equation

$$u(t) = f(t) + \int_0^{g(t)} K(t, s)u(s)ds \quad t, s \in [0, \infty) \tag{4.1}$$

is the integral equation located in the space $C[0, \infty)$ of all continuous functions defined on the interval $[0, \infty)$, where $K(t, s) : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and $f, g : [0, \infty) \to \mathbb{R}$ are continuous functions so that $g(t) \geq 0$ for all $t \in [0, \infty)$. Let $U = (C[0, \infty), \mathbb{R})$. Define the family of b-pseudo metrics by

$$q_m(u, v) = \max_{t\in[0,m]}\{|u(t) - v(t)|^2 e^{-|\tau t|}\}.$$

Obviously, $Q_{s;\Omega} = \{q_m : m \in \mathbb{N}\}$ defines a complete Hausdorff b-gauge structure on U. Here, in particular we consider the case when $Q_{s;\Omega} = \mathcal{J}_{s;\Omega} = \{q_m : m \in \mathbb{N}\}$. Define the map $\alpha : U \times U \to [0, \infty)$ for some $\kappa \geq 1$ in the following way:

$$\alpha(u, v) = \begin{cases} \kappa & \text{if } u \neq v \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. Define the operator $T : C[0,\infty) \to C[0,\infty)$ as follows:

$$Tu(t) = f(t) + \int_0^{g(t)} K(t,s)u(s)ds \quad t,s \in [0,\infty) \tag{4.2}$$

where $K(t,s) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $f, g : [0,\infty) \to \mathbb{R}$ are continuous functions so that $g(t) \geq 0$ for all $t \in [0,\infty)$.

Assume, moreover there exist $\gamma : U \to (0,\infty)$ and $\alpha : U \times U \to (0,\infty)$ such that the following statements hold:

(i) There is $\tau > 0$ such that

$$|K(t,s)u(s) - K(t,s)v(s)| \leq \sqrt{\frac{e^{-\tau}}{\gamma(u+v)} q_m(u,v)}$$

for each $t,s \in [0,\infty)$ and $u,v \in U$. Also,

$$\Big| \int_0^{g(t)} \frac{1}{\sqrt{\gamma(u(s)+v(s))}} \, ds \Big|^2 \leq e^{|\tau t|}.$$

(ii) There exists $z^0 \in U$ such that $\alpha(z^0, Tz^0) \geq 1$.
(iii) For $x, y \in U$ with $\alpha(x,y) \geq 1$ we have $\alpha(Tx, Ty) \geq 1$.
(iv) If a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \geq 1$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z^m = z$, then $\alpha(z^m, z) \geq 1$ and $\alpha(z, z^m) \geq 1$.
(v) T is $Q_{s;\Omega}$-closed map.

Then there exists at least one solution of the integral equation (4.1).

Proof. We first prove that T satisfies condition (3.13). For any $u, v \in U$ with $\alpha(u,v) \geq 1$, we have

$$\begin{aligned}
|Tu(t) - Tv(t)|^2 &= \Big| f(t) + \int_0^{g(t)} K(t,s)u(s)ds - (f(t) + \int_0^{g(t)} K(t,s)v(s)ds) \Big|^2 \\
&= \Big| \int_0^{g(t)} K(t,s)u(s)ds - \int_0^{g(t)} K(t,s)v(s)ds \Big|^2 \\
&\leq \Big(\int_0^{g(t)} \Big| K(t,s)u(s)ds - K(t,s)v(s) \Big| ds \Big)^2 \\
&\leq e^{-\tau} q_m(u,v) \Big(\int_0^{g(t)} \frac{1}{\sqrt{\gamma(u(s)+v(s))}} \, ds \Big)^2 \\
&\leq e^{|\tau t|} e^{-\tau} q_m(u,v).
\end{aligned}$$

From here we can write

$$|Tu(t) - Tv(t)|^2 e^{-|\tau t|} \leq e^{-\tau} q_m(u,v).$$

This can be written as

$$q_m(Tu - Tv) \leq e^{-\tau} q_m(u,v).$$

Obviously, natural logarithm belong to the family $\mathfrak{F}_s$, therefore, taking logarithm on both sides, we have

$$\ln(q_m(Tu - Tv)) \leq \ln(e^{-\tau} q_m(u, v)).$$

A simplification leads to the following

$$\tau + \ln(q_m(Tu - Tv)) \leq \ln(q_m(u, v)).$$

This implies that (3.13) holds for $a_m = 1$ and $b_m = c_m = e_m = L_m = 0$, for all $m \in \mathbb{N}$ and $F(u) = \ln u$. Hence, Theorem 3.13 ensures the existence of a fixed point of the operator T, thus, there is at least one solution of the integral equation (4.1). □

5. Concluding remarks

Remark 5.1. The fixed point results concerning F-type-contractions in a gauge space in [17] require the completeness of the space (U, d). Therefore, our theorems and corollaries for F-type-contractions in the b-gauge space are new generalizations of the results in [17] in which assumptions are weaker and assertions are stronger.

Remark 5.2. Our results for F-type-contractions in b-gauge spaces deal with about periodic points as well. Hence, they improve the results in [17].

Remark 5.3. Theorems 3.13 and 3.14 generalize Theorems 4.2 and 5.2, respectively in [29].

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Conflict of interest

The authors declare that they have no conflicts of interest.

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