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Research article

Study of Sturm-Liouville boundary value problems with p-Laplacian by using generalized form of fractional order derivative

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Abstract: This manuscript is related to deriving some necessary and appropriate conditions for qualitative results about a class of Sturm-Liouville (S-L) boundary value problems (BVPs) with the p-Laplacian operator under a fractional ϑ-Caputo type derivative. For the required results, we use Mönch's fixed point theorem with a measuring of non-compactness. Here, it is important to mention that the aforesaid equations belong to a highly significant class of problems that have many of the same properties and applications to solving various problems of dynamics and wave equations theory. For the demonstration of our theoretical results, we provide an example.

Keywords: FDEs; Sturm-Liouville BVPs; measure of non-compactness; Mönch's fixed point theorem; p-Laplacian operator; Banach spaces
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1. Introduction

In the previous few decades, the subject of fractional calculus has received significant attention from researchers. In fractional calculus, we use an arbitrary real or complex order instead of an integer order. Thus, the said area generalizes the ordinary differential and integral operators from integer to any real or complex order. Usually, the traditional integer order operators are local operator, while the corresponding fractional order operators are considered global. The significant interest of researchers in fractional calculus is due to its pertinent applications in various fields of science and technology. These are because many real world processes and phenomena when formulated in terms of FDEs are better described as compared to classical order, because many evolutionary problems where short memory is involved can be excellently described via fractional order derivatives as compared to integer order. In this regards, plenty of work has been published by many researchers. Here, we refer to some books on the mentioned area like, Kilbas et al. [1], Lakshmikantham et al. [2], Miller and Ross [3], Pudlubny [4], Tarasov [5], etc. In addition to the mentioned applications, recently some more interesting uses of FDEs have been investigated in epidemiology, rheology, porous media, dynamics of quasi-chaotic systems, etc.

Keeping in mind the important applications of FDEs, in recent times researchers have being working regularly in investigating different areas. One of the interesting areas of research in the aforesaid field in the recent time is devoted to the qualitative analysis of solutions to various kinds problems of FDEs, including initial and BVPs. The second one is devoted to numerical and analytical investigations of different types of problems ordinary as well as partial FDEs. In qualitative theory, usually researchers use fixed point theory, various degree theories and different forms of contractive operators to establish sufficient results for existence, uniqueness and stability analysis. In this regards, plenty of research work has been published up to date (we refer to a few like [6–9]) and the references therein).

The area related to p-Laplacian differential operators has been given proper attention during the previous few decades. This is because such problems have numerous applications in various fields, including flow problems, non-Newtonian mechanics, theory of combustion and quantum mechanics. Recently, some useful work in this respect has been published, where applications of the aforesaid area have been presented (see [10–13]). The area related to FDEs involving p-Laplacian operators has also been considered very well. For some remarkable work, we refer to [14–16]. Since the derivative with fractional order has not been uniquely defined yet, historically there are various definitions introduced in the literature about the said derivative. From Reimann-Liouville (R-L) to the recent non-singular type differential operators, there have been given several definitions. For instance, Kilbas and his co-authors have extended traditional R-L fractional differential operators with respect to another function in 2004. In the same line, Almeida [17] extended the usual Caputo operator of derivative to φ-Caputo type, which is more general than the usual derivative of Caputo. Also, the aforementioned author presented some valuable and interesting properties about the extended version in [18, 19]. Further, some authors have established further properties of the φ-fractional derivative. Also, the applications of the Laplace transform have been reported in [20]. This extension has the advantage selecting the operators for a particular process freely to describe real-world problems via mathematical formulations more precisely. Keeping the aforementioned importance, currently researchers have established general analysis including qualitative theory, stability and numerical analysis for various problems, including φ-Caputo kind differential operators. The aforementioned results have been established by the applications of standard fixed point theorems. For recent results for the reader interest, we refer [21–27]. Here we state that Vivek et al. [28] investigated the qualitative results by using φ-Caputo operator for the following BVP of FDEs

$$\begin{cases} {}^{c}\mathcal{D}^{\nu,\varphi}_{a^+}\vartheta(\varrho) = \mathscr{K}(\varrho, \vartheta(\varrho)), \quad \varrho \in \mathcal{J} := [a, T], \\ \quad a\vartheta(0) + b(T) = c, \;\; a + b \neq 0, \end{cases}$$

where ${}^{c}\mathcal{D}^{\nu,\varphi}_{0^+}$ are the φ-Caputo FD of order $\nu \in (0, 1)$, and $\varphi \in C^1(\mathcal{J}, \mathcal{X})$ is an increasing function such that $\varphi^{'}(\varrho) \neq 0$, for all $\varrho \in \mathcal{J}$.

Authors [29] established the aforesaid analysis by using φ-Caputo operators for anti-periodic BVP of FDEs as

$$\begin{cases} {}^{c}\mathcal{D}^{\nu,\varphi}_{a^+}\vartheta(\varrho) + \mathscr{K}(\varrho, \vartheta(\varrho)) = 0, \quad \varrho \in \mathcal{J} := [a, b], \\ \quad \vartheta(a) + \vartheta(b) = 0, \quad \vartheta'(a) + \vartheta'(b) = 0, \end{cases}$$

where ${}^{c}\mathcal{D}^{\nu,\varphi}_{0^+}$ are the φ-Caputo FD of order $\nu \in (0, 1)$, and $\varphi \in C^1(\mathcal{J}, \mathcal{X})$ is an increasing function such that $\varphi'(\varrho) \neq 0$ for all $\varrho \in \mathcal{J}$.

Here, it is remarkable that S-L problems have many applications in the solutions of other applied problems, like heat, wave, Laplace, and Poisson equations. Also, the said problems have some applications in dynamical systems like the mathematical model of lifting phenomenon. Keeping these important applications, the said problems have been investigated very well for qualitative results. The mentioned problems have not yet been investigated under the ϑ-Caputo derivative of fractional order involving the p-Laplacian operator. To fill this gap, and motivated from the above-mentioned work, we establish sufficient conditions for the existence of solutions to the following S-L BVPs involving the p-Laplacian operator by using ϑ-Caputo derivative of non-integer order as

$$\begin{cases} {}^{c}\mathcal{D}^{\nu,\varphi}_{0^+}\left(\phi_p\left[{}^{c}\mathcal{D}^{\mu,\varphi}_{0^+}(\vartheta(\varrho)\right]\right) + \mathscr{K}(\varrho, \vartheta(\varrho)) = 0, \quad \varrho \in \mathcal{J} := (0, 1]\,, \\ \xi\vartheta(0) + \eta\vartheta'(0) = 0, \quad \gamma\vartheta(1) + \delta\vartheta'(1) = 0, \quad {}^{c}\mathcal{D}^{\mu,\varphi}_{0^+}\vartheta(0) = 0, \end{cases} \tag{1.1}$$

where $1 < \mu \leq 2$, $0 < \nu \leq 1$, ${}^{c}\mathcal{D}^{\mu,\varphi}_{0^+}, {}^{c}\mathcal{D}^{\nu,\varphi}_{0^+}$ are the φ-Caputo FD, $\phi_p(\varsigma)$ is a p-Laplacian operator, (i.e., $\phi_p(\varsigma) = |\varsigma|^{p-2}\varsigma$) for $p > 1, \phi_p^{-1} = \phi_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi \in C^1(\mathcal{J}, \mathcal{X})$ is an increasing function such that $\varphi'(\varrho) \neq 0$ for all $\varrho \in \mathcal{J}$ and $\mathscr{K} : \mathcal{J} \times \mathcal{X} \longrightarrow \mathcal{X}$ is a given function. Some assumptions satisfied by functions are given later in this paper. Here, $\mathcal{X}$ is the Banach space, and θ refers to the null vector in the space $\mathcal{X}$, where $\xi, \eta, \gamma, \delta$ are constants.

Using the usual fixed point theory often needs strong compact conditions, which produce restriction in the using of such tools. Therefore, to relax the criteria from strong to weak contraction, measure of non-compactness plays a main role. In this regard, some results which utilize a measure of non-compactness have been established, among which Mönch's fixed point result is very important. This theorem uses slightly relaxed criteria compared to usual fixed point results. Therefore, based on the said theorem, we investigate our considered problems for qualitative results. Further, we remark that the proposed problem is more general than those studied earlier, like by Yang and Zhao [30, 31]. Our problem generalizes many results studied earlier in the literature. Also, by using various values of function φ in the proposed problem (1.1) some new problems can be predicted which have not been investigated yet. Here, it is interesting that if we put the following values for the function φ, the concerned problem (1.1) gives the given problem like follows.

(1) If $\varphi(\varrho) = \log\varrho$, then (1.1) reduces to fractional order p-Laplacian S-L BVPs with Caputo-Hadamard derivative.

(2) If $\varphi(\varrho) = \varrho^{\rho}$, then (1.1) reduces to fractional order p-Laplacian S-L BVPs with Caputo-Katugampola derivative.
(3) If some one takes $\varphi(\varrho) = \varrho$, $\mu = 2$ and $\nu = 1$, then (1.1) reduces to fractional order p-Laplacian S-L BVPs studied in [30, 31].
(4) Also if $\varphi(\varrho) = \varrho$, $\mu = 2, \nu = 1$, $p = 2$, we get traditional S-L BVP with order three.

Further, we stress to use use the measures of non-compactness and Darbo's or Mönch's fixed point results. With the help of the aforesaid theorems, various important results have been investigated already in literature like [32–38] for FDEs. Using Reiamman-Liouville and Caputo type operators, sufficient conditions have been established by constructing a suitable Banach space (see detail in Banaś et al. [39]).

Our work is designed as follows up: Section 1 is devoted to some literature overview about the area and problem we have proposed. Some fundamental notions and results are recollected in Section 2. The main results related to qualitative theory are established in Section 3. Section 4 is enriched by constructing a pertinent application. The last portion is concluded with some future directions and remarks.

2. Fundamental notions and results

Some necessary results needed onward for our analysis are given bellow.

Let $\mathcal{X}$ be a Banach space, and $\mathcal{Q} := C(\mathcal{J}, \mathcal{X})$ is the Banach space of all continuous functions $\vartheta : \mathcal{J} \to \mathcal{X}$, with the usual supremum norm defined by

$$\|\vartheta\|_{\infty} = \sup\{\|\vartheta(\varrho)\|, \varrho \in \mathcal{J}\}.$$

Further, $L^1(\mathcal{J})$ is the space of Bochner-integrable functions equipped with norm

$$\|\vartheta\|_{L^1} = \max_{t\in[0,1]} \int_0^1 |\vartheta(t)| dt.$$

The measure of non-compactness due to Kuratowski on metric space is recollected as follows:

Definition 2.1. *[40] Let $\Omega_{\mathcal{X}}$ the bounded subsets of Banach space, $\mathcal{X}$. Then, the map $\kappa \in C\Big[\Omega_{\mathcal{X}}, [0, \infty)\Big]$ is defined by*

$$\kappa(\mathcal{B}) = \inf\Big\{\varepsilon > 0 : \mathcal{B} \subset \cup_{j=1}^{m} \mathcal{B}_j,\ \mathcal{B} \in \Omega_{\mathcal{X}} \text{ and } \varepsilon \geq \operatorname{diam}(\mathcal{B}_j)\Big\}.$$

Definition 2.2. *[41]The function $\mathcal{K} : [0, 1] \times \mathcal{X} \longrightarrow \mathcal{X}$ $\mathcal{K}$ is said to be Carathéodory if for $\vartheta \in \mathcal{X}$, the function $\mathcal{K}(\varrho, \vartheta)$ is measurable with respect to ϱ and continuous corresponding to $\vartheta \in \mathcal{X}$ a.e. $\varrho \in \mathcal{J}$.*

For n-fold integrals, we recall the well-known Cauchy formula given in [4] as

$$\int_0^{\varrho} dt_1 \int_0^{\varrho_1} dt_2 \int_0^{\varrho_2} dt_3 \cdots \int_0^{\varrho_{n-1}} \mathcal{K}(\varrho_n) dt_n = \frac{1}{(n-1)!} \int_0^{\varrho} (\varrho - \varsigma)^{\mu-1} \mathcal{K}(\varsigma) d\varsigma.$$

Here, we give some generalization in $\varphi-$ notation for the R-L integral.

Definition 2.3. *[1] Let $\mu > 0$. Then the φ-R-L (RL) fractional integral of order μ of a function $\vartheta \in L^1([0,1])$ is defined by*

$$\mathcal{I}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = \int_0^{\varrho} \frac{\varphi'(\varsigma)(\varphi(\varrho) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\vartheta(\varsigma)d\varsigma, \quad \mu > 0,$$

provided that the integral on the right side converges over $(0,\infty)$.

Definition 2.4. *[17] Let $n-1 < \mu \le n$. Then the Caputo derivative of order μ of a function $\vartheta \in AC^n([0,1])$ is represented by*

$$\begin{aligned} {}^c\mathcal{D}^{\mu,\varphi}\vartheta(\varrho) &= \left(\frac{1}{\varphi'(\varrho)}\frac{d}{d\varrho}\right)^n \mathcal{I}_{0^+}^{n-\mu,\varphi}\vartheta(\varrho) \\ &= \left(\frac{1}{\varphi'(\varrho)}\frac{d}{d\varrho}\right)^n \int_0^{\varrho} \frac{\varphi'(\varsigma)(\varphi(\varrho) - \varphi(\varsigma))^{n-\mu-1}}{\Gamma(n-\mu)}\vartheta(\varsigma), \end{aligned}$$

provided that the integral on the right side converges over $(0,\infty)$.

Lemma 2.1. *[1, 17] Let $\mu > \nu > 0$, and $\vartheta \in L^1([0,1])$. Then, we have*

- $\mathcal{I}_{0^+}^{\mu,\varphi}\mathcal{I}_{0^+}^{\nu,\varphi}\vartheta(\varrho) = \mathcal{I}_{0^+}^{\mu+\nu,\varphi}\vartheta(\varrho)$,
- ${}^c\mathcal{D}_{0^+}^{\mu,\varphi}\mathcal{I}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = \vartheta(\varrho)$,
- ${}^c\mathcal{D}_{0^+}^{\nu,\varphi}\mathcal{I}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = \mathcal{I}_{0^+}^{\mu-\nu,\varphi}\vartheta(\varrho)$.

Lemma 2.2. *[17] The FDE, with $n-1 < \mu \le n$,*

$$({}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta)(\varrho) = 0,$$

has the following solution

$$\vartheta(\varrho) = \sum_{j=0}^{n-1} c_j(\varphi(\varrho) - \varphi(0))^j, \quad c_j \in \mathbb{R}, j = 0 \dots n-1.$$

Lemma 2.3. *[17] If $y \in C[0,1] \cap L[0,1]$, the FDE*

$${}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = y(\varrho),$$

has a solution given by

$$\vartheta(\varrho) = \mathcal{I}_{0^+}^{\mu,\varphi}[y(\varrho)] + \sum_{j=0}^{n-1} c_j(\varphi(\varrho) - \varphi(0))^j,$$

for some $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, n-1$.

The given result of [42] plays an important role in our analysis.

Theorem 2.1. *Let $0 \in \mathcal{D} \subset \mathcal{X}$ be a bounded, closed and convex set and let $\mathcal{N} : \mathcal{D} \to \mathcal{D}$ be a continuous mapping with $V = \overline{conv}\mathcal{N}(V)$ or $V = \mathcal{N}(V) \cup \{0\}$, such that $\kappa(V) = 0$ at each set $V \subset \mathcal{D}$. Then $\mathcal{N}$ has a fixed point.*

Lemma 2.4. *[43] Let $\mathcal{H}$ be a bounded and an equicontinuous subset of $C(\mathcal{J}, \mathcal{X})$. Then, the function $\varrho \to \kappa(\mathcal{H}(\varrho))$ is continuous on $\mathcal{J}$, and*

$$\kappa_C(\mathcal{H}) = \max_{\varrho\in\mathcal{J}} \kappa(\mathcal{H}(\varrho)),$$

and

$$\kappa\left(\int_{\mathcal{J}} \vartheta(\varsigma)d\varsigma\right) \le \int_{\mathcal{J}} \kappa(\mathcal{H}(\varsigma))d\varsigma,$$

where $\mathcal{H}(\varsigma) = \{\vartheta(\varsigma) : \vartheta \in \mathcal{H}, \varsigma \in \mathcal{J}\}$, and κ_C is the Kuratowski *measure of non-compactness defined on the bounded sets of $C(\mathcal{J}, \mathcal{X})$.*

3. Qualitative analysis

Here we establish our main results for the considered problem (1.1) by using Mönch's theorem [42]. Here we derive our main results.

Lemma 3.1. *The solution of the following linear problem with $\mathscr{Y} \in C[0, 1] \cup L[0, 1]$,*

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\nu,\varphi}\left(\phi_p\left[{}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(\varrho)\right]\right) + \mathscr{Y}(\varrho) = 0, \quad \varrho \in \mathcal{J} := [0, 1], \\ \xi\vartheta(0) + \eta\vartheta'(0) = 0, \quad \gamma\vartheta(1) + \delta\vartheta'(1) = 0, \quad {}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(0) = 0, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} \vartheta(\varrho) &= \frac{-\eta\vartheta'(0)}{\rho}\Big[\gamma\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(1)] + \delta\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(1)]\Big] + \frac{\xi(\vartheta(\varrho) - \vartheta(0))}{\rho}\Big[\gamma\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(1)] + \delta\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(1)]\Big] - \mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(\varrho)] \\ &= \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{Y}(u)du\Big)d\varsigma \\ &+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu}}{\Gamma(\nu)}\mathscr{Y}(u)du\Big)d\varsigma\Big] \\ &- \frac{\xi(\vartheta(\varrho) - \vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu}}{\Gamma(\nu)}\mathscr{Y}(u)du\Big)d\varsigma \\ &+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-2}}{\Gamma(\nu-1)}\mathscr{Y}(u)du\Big)d\varsigma\Big] \\ &+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{Y}(u)du\right)d\varsigma. \end{aligned} \tag{3.2}$$

where $\rho = \gamma\xi(\varphi(1) - \varphi(0)) + \delta\xi\varphi'(1) - \eta\gamma\varphi'(0) \neq 0$.

Proof. Assume that $\vartheta(\varrho)$ is a solution of Eq (3.1). Applying the operator $\mathcal{I}_{0^+}^{\nu,\varphi}$ on both sides of Eq (3.1) and using Definition 2.3, we obtain

$$\phi_p\left[{}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(\varrho)\right] = -\int_0^\varrho \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{Y}(\varsigma)d\varsigma + a_0.$$

By the condition ${}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(0) = 0$, we get $a_0 = 0$, and hence

$${}^c\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = -\phi_q\left(\int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho) - \varphi(\varsigma))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(\varsigma, \vartheta(\varsigma))d\varsigma\right) = \mathrm{g}(\varrho). \tag{3.3}$$

Thus, from (3.3), we write

$$ {}^{c}\mathcal{D}_{0^+}^{\mu,\varphi}\vartheta(\varrho) = -\mathrm{g}(\varrho) \text{ (say)}. \tag{3.4}$$

Applying the operator $\mathcal{I}_{0^+}^{\mu,\varphi}$ on both sides of Eq (3.4) and using Definition 2.3, we obtain

$$\vartheta(\varrho) = -\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(\varrho)] + a_1 + a_2(\varphi(\varrho) - \varphi(0)) \tag{3.5}$$

and

$$\vartheta'(\varrho) = -\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(\varrho)] + a_2\varphi'(\varrho). \tag{3.6}$$

By using the condition $\xi\vartheta(0) + \eta\vartheta'(0) = 0$, we obtain

$$\xi a_1 + \eta a_2\vartheta'(0) = 0, \tag{3.7}$$

and $\gamma\vartheta(1) + \delta\vartheta'(1) = 0$, gives

$$\gamma a_1 + a_2\gamma(\varphi(1) - \varphi(0)) + \delta a_2\varphi'(1) - \gamma\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(1)] - \delta\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(1)]. \tag{3.8}$$

From (3.7), we see that

$$a_1 = \frac{\eta a_2\vartheta'(0)}{\xi}. \tag{3.9}$$

Putting (3.9) in (3.8) gives

$$a_2 = \frac{\xi}{\rho}\Big[\gamma\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(1)] + \delta\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(1)]\Big]. \tag{3.10}$$

Hence, using (3.10) in (3.9), we get

$$a_1 = \frac{-\eta\vartheta'(0)}{\rho}\Big[\gamma\mathcal{I}_{0^+}^{\mu,\varphi}[\mathrm{g}(1)] + \delta\mathcal{I}_{0^+}^{\mu-1,\varphi}[\mathrm{g}(1)]\Big]. \tag{3.11}$$

Substituting the values of $a_1,\ a_2$ into Eq (3.6), we obtain the required solution given in (3.2). □

Theorem 3.1. *Let* $1 < \mu \leq 2, 0 < \nu \leq 1$*, and then in view of Lemma 3.1, the solution of BVP* (1.1) *is given by*

$$\begin{aligned}
\vartheta(\varrho) &= \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u, \vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-2}}{\Gamma(\mu - 1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u, \vartheta(u))du\Big)d\varsigma\Big] \\
&- \frac{\xi(\vartheta(\varrho) - \vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u, \vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1) - \varphi(\varsigma))^{\mu-2}}{\Gamma(\mu - 1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u, \vartheta(u))du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho) - \varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma) - \varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u, \vartheta(u))du\right)d\varsigma.
\end{aligned} \tag{3.12}$$

Theorem 3.2. *Assume the following hypotheses hold:*

(A_1) *The function* $\mathscr{K} : [0,1] \times \mathcal{X} \to \mathcal{X}$ *satisfies Carathéodory conditions.*

(A_2) *There exist* $p_{\mathscr{K}} \in L^\infty(\mathcal{J}, \mathbb{R}_+)$ *and a continuous nondecreasing function* $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$\|\mathscr{K}(\varrho, \vartheta)\| \leq \phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|)) \text{ for a.e. } \varrho \in \mathcal{J} \text{ and each } \vartheta \in C(\mathcal{J}, \mathcal{X}).$$

(A_3) *For any bounded set* $\mathcal{D} \subset \mathcal{X}$, *and each* $\varrho \in \mathcal{J}$, *the following inequality holds*

$$\kappa(\mathscr{K}(\varrho, \mathcal{D})) \leq \phi_p(p_{\mathscr{K}}(\varrho)\kappa(\mathcal{D})).$$

If

$$\mathcal{M}_{\mathscr{K}} = \frac{\Gamma(\nu(q-1)+1)\|p_{\mathscr{K}}\|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}(|\gamma|+|\delta|)}{\Gamma(\mu+\nu(q-1)+1)\Gamma(\nu+1)^{q-1}}\left[\frac{|\eta|\vartheta'(0)}{|\rho|} + \frac{|\xi|(\vartheta(\varrho)-\vartheta(0))}{|\rho|} + 1\right] < 1, \tag{3.13}$$

then the problem (1.1) *has at least one solution defined on* $\mathcal{J}$.

Proof. Define the operator $\mathcal{N} : C(\mathcal{J}, \mathcal{X}) \longrightarrow C(\mathcal{J}, \mathcal{X})$ by:

$$\begin{aligned}
\mathcal{N}\vartheta(\varrho) &= \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&- \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\right)d\varsigma.
\end{aligned}$$

Clearly, the fixed points of the operator $\mathcal{N}$ are a solution of the problem (1.1).

Let $\varpi > 0$, such that

$$\varpi \geq \mathcal{M}_{\mathscr{K}}\psi(\varphi).$$

Now, let us consider the ball

$$\mathcal{B}_\varpi = B(\varpi, \theta) = \Big\{\vartheta \in C(\mathcal{J}, \mathcal{X}) : \|\vartheta\|_\infty \leq \varpi\Big\}.$$

Our goal is to show that the operator $\mathcal{N}$ meets all assumptions of Theorem 2.1. Let $\vartheta \in \mathcal{B}_\varpi, \varrho \in \mathcal{J}$. Then, we have

$$\begin{aligned}
\|\mathcal{N}\vartheta(\varrho)\| &\leq \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-2}}{\Gamma(\nu-1)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\right)d\varsigma.
\end{aligned}
$$

Using hypothesis (A_2), we have

$$
\begin{aligned}
\|\mathcal{N}\vartheta(\varrho)\| &\leq \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu-1)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&+ \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-2}}{\Gamma(\nu-1)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\right)d\varsigma \\
&\leq \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\Big(\phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|))\Big)du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu}}{\Gamma(\nu)}\Big(\phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|))\Big)du\Big)d\varsigma\Big] \\
&+ \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\Big(\phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|))\Big)du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\Big(\phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|))\Big)du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\Big(\phi_p(p_{\mathscr{K}}(\varrho)\psi(\|\vartheta\|))\Big)du\right)d\varsigma \\
&\leq p_{\mathscr{K}}\psi(\|\vartheta\|)\frac{|\eta|\vartheta'(0)}{|\rho|}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma \\
&+ |\delta|\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma\Big] \\
&+ \frac{\xi p_{\mathscr{K}}\psi(\|\vartheta\|)(\vartheta(\varrho)-\vartheta(0))}{|\rho|}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma\Big] \\
&+ p_{\mathscr{K}}\psi(\|\vartheta\|)\int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\Big)du\right)d\varsigma. \qquad (3.14)
\end{aligned}
$$

Upon simplification of (3.14), we get

$$
\begin{aligned}
\|\mathcal{N}\vartheta(\varrho)\| &\leq \|p_{\mathscr{K}}\|\psi(\|\vartheta\|)\frac{|\eta|\vartheta'(0)}{|\rho|}\Big[\frac{|\gamma|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}}{\Gamma(\mu)\Gamma(\nu+1)^{q-1}}B(\mu,\nu(q-1)+1)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\\
&+ \frac{|\delta|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}}{\Gamma(\mu)\Gamma(\nu+1)^{q-1}}B(\mu-1,\nu(q-1)+1)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\Big]\\
&+ \|p_{\mathscr{K}}\|\psi(\|\vartheta\|)\frac{|\xi|(\vartheta(\varrho)-\vartheta(0))}{|\rho|}\Big[\frac{|\gamma|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}}{\Gamma(\mu)\Gamma(\nu+1)^{q-1}}B(\mu,\nu(q-1)+1)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\\
&+ \frac{|\delta|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}}{\Gamma(\mu)\Gamma(\nu+1)^{q-1}}B(\mu-1,\nu(q-1)+1)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\Big]\\
&+ \|p_{\mathscr{K}}\|\psi(\|\vartheta\|)\frac{|\eta|}{|\rho|}\frac{|\gamma|(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}}{\Gamma(\mu)\Gamma(\nu+1)^{q-1}}B(\mu,\nu(q-1)+1)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\\
&\leq \frac{\Gamma(\nu(q-1)+1)\|p_{\mathscr{K}}\|\psi(\|\vartheta\|)}{\Gamma(\mu+\nu(q-1)+1)\Gamma(\nu+1)^{q-1}}(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}\\
&\leq \frac{\Gamma(\nu(q-1)+1)\|p_{\mathscr{K}}\|\psi(\|\vartheta\|)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}(|\gamma|+|\delta|)}{\Gamma(\mu+\nu(q-1)+1)\Gamma(\nu+1)^{q-1}}\Big[\frac{|\eta|\vartheta'(0)}{|\rho|}+\frac{|\xi|(\vartheta(\varrho)-\vartheta(0))}{|\rho|}+1\Big]\\
&\leq \mathcal{M}_{\mathscr{K}}\psi(\varpi)\\
&\leq \varpi.
\end{aligned}
$$

Thus,

$$\|\mathcal{N}\vartheta\| \leq \varpi.$$

This means that $\mathcal{N}$ transforms $\mathcal{B}_\varpi$ into itself. Further, for any $\vartheta \in \mathcal{B}_\varpi$ and $\varrho \in \mathcal{J}$, we have

$$\|(\mathcal{N}\vartheta)(\varrho)\| \leq \frac{\Gamma(\nu(q-1)+1)\|p_{\mathscr{K}}\|\psi(\|\vartheta\|)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}(|\gamma|+|\delta|)}{\Gamma(\mu+\nu(q-1)+1)\Gamma(\nu+1)^{q-1}}\Big[\frac{|\eta|\vartheta'(0)}{|\rho|}+\frac{|\xi|(\vartheta(\varrho)-\vartheta(0))}{|\rho|}+1\Big].$$

Thus, the operator $\mathcal{N}$ is bounded. In order to derive conditions of Theorem 2.1, we perform the given steps.

Step 1. The operator $\mathcal{N}:\mathcal{B}_\varpi \longrightarrow \mathcal{B}_\varpi$ is continuous. For sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}_\varpi$, with $\vartheta_n \to \vartheta$ in $\mathcal{B}_\varpi$, at each $\varrho \in \mathcal{J}$, one has

$$
\begin{aligned}
\Big\|(\mathcal{N}\vartheta_n)(\varrho) &- (\mathcal{N}\vartheta)(\varrho)\Big\| \leq \frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\\
&\times |\mathscr{K}(u,\vartheta_n(u)-\mathscr{K}(u,\vartheta(u))|\,du\Big)d\varsigma\\
&+ \delta\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta_n(u)-\mathscr{K}(u,\vartheta(u))|\,du\Big)d\varsigma\Big]\\
&+ \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\\
&\times |\mathscr{K}(u,\vartheta_n(u)-\mathscr{K}(u,\vartheta(u))|\,du\Big)d\varsigma \qquad (3.15)\\
&+ \delta\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-2}}{\Gamma(\nu-1)}|\mathscr{K}(u,\vartheta_n(u)-\mathscr{K}(u,\vartheta(u))|\,du\Big)d\varsigma\Big]
\end{aligned}
$$

$$+\int_0^{\varrho}\frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\,|\mathscr{K}(u,\vartheta_n(u)-\mathscr{K}(u,\vartheta(u))|\,du\right)d\varsigma.$$

Since $\vartheta_n\to\vartheta$ as $n\to\infty$, $\mathscr{K}$, $\phi_q(\cdot)$ are continuous. Also, $\mathcal{N}$ is bounded and so is uniformly continuous. Therefore, by the dominated convergence theorem due to Lebesgue, one has

$$\|\mathcal{N}\vartheta_n-\mathcal{N}\vartheta\|\to 0,\quad \text{as}\quad n\to\infty.$$

Step 2. To show boundedness and equicontinuity of $\mathcal{N}(\mathcal{B}_{\varpi})$ in $\mathcal{B}_{\varpi}$, we have $\mathcal{N}(\mathcal{B}_{\varpi})\subset\mathcal{B}_{\varpi}$ and also that $\mathcal{B}_{\varpi}$ is bounded, which means $\mathcal{N}(\mathcal{B}_{\varpi})$ is bounded. If $0<\varrho_1<\varrho_2<1,\vartheta\in\mathcal{B}_{\varpi}$, then one has

$$\begin{aligned}
\Big|\mathcal{N}(\vartheta)(\varrho_2) &- \mathcal{N}(\vartheta)(\varrho_1)\Big|\le\frac{|\xi||\vartheta(\varrho_2)-\vartheta(\varrho_1)|}{|\rho|}\\
&\times\Big[\gamma\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma\\
&+\delta\int_0^1\frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-2}}{\Gamma(\nu-1)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma\Big]\\
&+\int_0^{\varrho_1}\Big[\frac{\varphi'(\varsigma)(\varphi(\varrho_2)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}-\frac{\varphi'(\varsigma)(\varphi(\varrho_1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\Big]\\
&\times\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma+\int_{\varrho_1}^{\varrho_2}\frac{\varphi'(\varsigma)(\varphi(\varrho_2)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\\
&\times\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}|\mathscr{K}(u,\vartheta(u))|du\Big)d\varsigma.
\end{aligned}$$

After some simple calculation and using (A_2), we obtain

$$\begin{aligned}
|\mathcal{N}(\vartheta)(\varrho_2) &- \mathcal{N}(\vartheta)(\varrho_1)|\le\frac{\Gamma(\nu(q-1)+1)\|p_{\mathscr{K}}\|\psi(\|\vartheta\|)(\varphi(1)-\varphi(0))^{\mu+\nu(q-1)}(|\gamma|+|\delta|)}{\Gamma(\mu+\nu(q-1)+1)\Gamma(\nu+1)^{q-1}}\\
&\times\Big[\frac{|\xi||\vartheta(\varrho_2)-\vartheta(\varrho_1)|}{|\rho|}+\int_0^{\varrho_1}\Big(\frac{\varphi'(\varsigma)(\varphi(\varrho_2)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}-\frac{\varphi'(\varsigma)(\varphi(\varrho_1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\Big)\\
&\times\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma\\
&+\int_{\varrho_1}^{\varrho_2}\frac{\varphi'(\varsigma)(\varphi(\varrho_2)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^{\varsigma}\frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}du\Big)d\varsigma\Big].
\end{aligned}\tag{3.16}$$

We see that the right side of (3.16) goes to zero when $\varrho_2\to\varrho_1$. Hence, we have that for $\varrho_2\to\varrho_1$, the left side $|\mathcal{N}(\vartheta)(\varrho_2)-\mathcal{N}(\vartheta)(\varrho_1)|\to 0$. As already proved, $\mathcal{N}$ is bounded and continuous. Therefore, it will be uniformly continuous, and therefore we have

$$\|\mathcal{N}(\vartheta)(\varrho_2)-\mathcal{N}(\vartheta)(\varrho_1)\|\to 0,\ \text{for}\ \varrho_2\to\varrho_1.$$

Therefore, $\mathcal{N}$ is equi-continuous.

Step 3. In this step, we need to prove that the condition of Theorem 2.1 holds. For that, let $V\subset\mathcal{B}_{\varpi}$ such that the bounded and equicontinuous $V\subset\overline{\mathcal{N}(V)}\cup\{0\}$. Then, $\varrho\to\vartheta(\varrho)=\kappa(V(\varrho))$ is continuous on $\mathcal{J}$. According to condition (A_3) and using κ, one has at each $\varrho\in\mathcal{J}$ that

$$\vartheta(\varrho)\quad\le\kappa(\overline{\mathcal{N}(V)}\cup\{0\})\le\kappa(\mathcal{N}(V)(\varrho)).$$

To derive required result, we proceed as

$$\begin{aligned}
|\vartheta(\varrho)| &\leq \kappa\Big|\frac{\eta\vartheta'(0)}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&- \frac{\xi(\vartheta(\varrho)-\vartheta(0))}{\rho}\Big[\gamma\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma \\
&+ \delta\int_0^1 \frac{\varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\mu-2}}{\Gamma(\mu-1)}\phi_q\Big(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\Big)d\varsigma\Big] \\
&+ \int_0^\varrho \frac{\varphi'(\varsigma)(\varphi(\varrho)-\varphi(\varsigma))^{\mu-1}}{\Gamma(\mu)}\phi_q\left(\int_0^\varsigma \frac{\varphi'(u)(\varphi(\varsigma)-\varphi(u))^{\nu-1}}{\Gamma(\nu)}\mathscr{K}(u,\vartheta(u))du\right)d\varsigma\Big| \\
&\leq \mathcal{M}_{\mathscr{K}}\|\vartheta\|_\infty,
\end{aligned}$$

which gives

$$\|\vartheta\|_\infty \leq \mathcal{M}_{\mathscr{K}}\|\vartheta\|_\infty.$$

This means that

$$\|\vartheta\|_\infty(1-\mathcal{M}_{\mathscr{K}}) \leq 0. \tag{3.17}$$

Since $\mathcal{M}_{\mathscr{K}} < 1$, we have $\|\vartheta\|_\infty \leq 0$, but $\|\vartheta\|_\infty \geq 0$ always. Therefore, one has $\|\vartheta\|_\infty = 0$. Hence, $\vartheta(\varrho) = \kappa(V(\varrho)) = 0$, for each $\varrho \in \mathcal{J}$. Thus, $V(\varrho)$ is relatively compact in $\mathcal{X}$. Via the Ascoli-Arzelá theorem, the operator V is relatively compact in $\mathcal{B}_\varpi$. Hence, according to Theorem 2.1, we conclude that $\mathcal{N}$ has a fixed point, which is a solution of the problem (1.1). □

4. Pertinent application

We enrich this portion of our work by pertinent application to demonstrate the applicability of our main results, where

$$\mathcal{X} = c_0 = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ldots) : \vartheta_n \to 0\},\ \text{with } n \to \infty,$$

is the complete normed space under the given norm

$$\|\vartheta\|_\infty = \sup_{n\geq 1}|\vartheta_n|.$$

Example 1. *Consider the given problem*

$$\begin{cases} {}^c\mathscr{D}_{0^+}^{\frac{9}{10},t}\left(\phi_5\left[{}^c\mathscr{D}_{0^+}^{\frac{3}{2},t}\vartheta(\varrho)\right]\right)(\varrho) + \phi_p\left(\mathscr{K}(\varrho,\vartheta(\varrho)\right) = 0, \quad \varrho \in \mathcal{J} := [0,1], \\ 5\vartheta(0) - \frac{1}{100}\vartheta'(0) = 0, \quad 10\vartheta(1) + \frac{1}{100}\vartheta'(1) = 0, \quad {}^c\mathscr{D}_{0^+}^{\frac{3}{2},t}\vartheta(0) = 0. \end{cases} \tag{4.1}$$

Taking the following particular values in (1.1)*, we see from the problem* (4.1)

$$\mu = \frac{3}{2}, \nu = \frac{1}{2} = 1, p = 5, q = \frac{5}{4}, \xi = 5, \eta = \frac{1}{100}, \gamma = 10, \delta = \frac{1}{100}, \varphi(\varrho) = \varrho,$$

and $\mathscr{K} : \mathcal{J} \times c_0 \longrightarrow c_0$ *given by*

$$\mathscr{K}(\varrho, \vartheta) = \left\{ \frac{1}{(\varrho^2+2)^2} \left(\frac{\varrho}{n^2} + \ln(1+|\vartheta_n|) \right) \right\}_{n\geq 1}, \quad \text{for } \varrho \in \mathcal{J}, \vartheta = \{\vartheta_n\}_{n\geq 1} \in c_0.$$

Obviously the hypothesis (A_1) *holds. Also,*

$$\begin{aligned} |\mathscr{K}(\varrho, \vartheta)| &= \left| \phi_p \left(\frac{1}{(\varrho^2+2)^2} \left(\frac{1}{n^2} + \ln(1+|\vartheta_n|) \right) \right) \right| \\ &\leq \phi_p \left(\frac{1}{(\varrho^2+2)^2} (1 + \|\vartheta\|) \right) \\ &= \phi_p \left(p_{\mathscr{K}}(\varrho) \psi(\|\vartheta\|) \right), \end{aligned}$$

which yields that hypothesis (A_2) *is satisfied with* $p_{\mathscr{K}}(\varrho) = \frac{1}{(\varrho^2+2)^2}, \varrho \in \mathcal{J}$ *and* $\psi(x) = 1 + x, \ x \in [0, \infty)$. *Let for bounded set* $\mathcal{D} \subset c_0$*, one has*

$$\kappa(\mathscr{K}(\varrho, \mathcal{D})) \leq \phi_p \left(\frac{1}{(\varrho^2+2)^2} \kappa(\mathcal{D}) \right), \text{ for each } \varrho \in \mathcal{J}.$$

Thus, (A_3) *is satisfied. Further on calculation, we have*

$$\mathcal{M}_{\mathscr{K}} = 0.4555 < 1,$$

and

$$\mathcal{M}_{\mathscr{K}}(\varpi + 1) < \varpi,$$

thus

$$\varpi > \frac{\mathcal{M}_{\mathscr{K}}}{1 - \mathcal{M}_{\mathscr{K}}} = 0.836547.$$

Then, ϖ *can be chosen as* $\varpi = 1.5$. *Consequently, Theorem 3.2 implies that problem* (4.1) *has at least one solution* $\vartheta \in C(\mathcal{J}, c_0)$.

5. Conclusions

In this work, we have developed some adequate results for qualitative analysis of a solution to a fractional order S-L BVP containing a p-Laplacian operator. Further, the considered problem has been investigated under the φ-Caputo fractional derivative, which is a more general operator than the traditional one. Upon the applications of measure of non-compactness combined with Mönch's fixed point approach, we have established the required results. The measure of non-compactness is a much more flexible condition than the usual strong compact condition. The considered problem has various interesting properties. We have mentioned earlier that by fixing the values for the function φ, we get various problems as special cases of (1.1). Hence, by adopting the following values for the function φ, we predict for the new problem as:

- If we take $\varphi(\varrho) = \varrho$, $\mu = 2$, and $\nu = 1$, then the problem (1.1) reduces to the the fractional S-L BVP with p-Laplacian investigated in [30, 31].

- In (1.1), if we choose $\varphi(\varrho) = \varrho$, $\mu = 2, \nu = 1$, and $p = 2$, then we get classical S-L BVP of order three.
- Also we remark that we have taken here abstract space instead of scaler space as adopted by Yang and Zhao's [30, 31] in their work.
- Our current problem is more general than that addressed in the literature and involves a generalized fractional derivative which results in many special cases being considered according to our results, including Caputo-Hadamard fractional S-L BVP with p-Laplacian and Caputo-Katugampola fractional S-L BVP with p-Laplacian, etc.
- In this way, it is possible to solve S-L BVP of FDEs with (φ, p)-Laplacian and boundary conditions that contain fractional derivatives as

$$\begin{cases} {}^{c}\mathscr{D}_{0^+}^{\nu;\varphi}\left(\phi_p\left[{}^{c}\mathscr{D}_{0^+}^{\mu;\varphi}(\vartheta(\varrho))\right]\right) + \mathscr{K}(\varrho, \vartheta(\varrho)) = 0, \ \varrho \in \mathcal{J} := [0, 1], \\ \xi x(0) - \eta {}^{c}\mathscr{D}_{0^+}^{\sigma;\varphi}(0) = 0, \quad \gamma\vartheta(1) + \delta {}^{c}\mathscr{D}_{0^+}^{\sigma;\varphi}(1) = 0, \quad {}^{c}\mathscr{D}_{0^+}^{\mu;\varphi}\vartheta(0) = 0. \end{cases}$$

Briefly, our considered problem is the generalization of various problems already studied in the literature by using the usual strong compact conditions. Here, we have studied a more general problem by using slightly relaxed criteria of measure of non-compactness. By pertinent example, we have demonstrated our results.

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Conflict of interest

There is no competing interest regarding this work.

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