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Research article

Trees with the second-minimal ABC energy

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Abstract: The atom-bond connectivity energy (ABC energy) of an undirected graph G, denoted by $\mathcal{E}_{ABC}(G)$, is defined as the sum of the absolute values of the ABC eigenvalues of G. Gao and Shao [The minimum ABC energy of trees, Linear Algebra Appl., 577 (2019), 186–203] proved that the star S_n is the unique tree with minimum ABC energy among all trees on n vertices. In this paper, we characterize the trees with the minimum ABC energy among all trees on n vertices except the star S_n.

Keywords: energy; ABC energy; ABC matrix; ABC eigenvalues; tree
Mathematics Subject Classification: 05C50

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The eigenvalues of adjacency matrix $A(G)$ are called the eigenvalues of G. The energy $\mathcal{E}(G)$ of G is defined as the sum of the absolute values of its eigenvalues of $A(G)$, which is studied in chemistry and used to approximate the total-electron energy of a molecule [3]. The singular values of an $n \times m$ matrix M are the square roots of the eigenvalues of MM^* if $n \geq m$ or M^*M if $n < m$, where M^* is the transpose conjugate of M. Nikiforov [4] extended the concept of energy to all matrices and defined the energy of a matrix M, denoted by $\mathcal{E}(M)$, as the sum of the singular values of M. Clearly, $\mathcal{E}(A(G)) = \mathcal{E}(G)$.

Estrada et al. [12] introduced the atom-bond connectivity index as

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}.$$

Moreover, they introduced the atom-bond connectivity matrix (or ABC matrix for short) ABC_G of G, which is correlated with the ABC index of G. The (i, j)-entry of the matrix ABC_G is equal to $\sqrt{\frac{d_i+d_j-2}{d_id_j}}$ if $v_iv_j \in E(G)$ and 0 otherwise. The eigenvalues of the ABC matrix of G, denoted by $\mu_1, \mu_2, \ldots, \mu_n$, are said to be the ABC eigenvalues of G. The atom-bond connectivity energy (ABC energy) of a connected graph G is defined in [8] as

$$\mathcal{E}_{ABC}(G) = \sum_{i=1}^{n} |\mu_i(G)|.$$

Recently, several theoretical and computational properties of the ABC energy of graphs have been obtained, see e.g., [1, 8, 13]. Estrada [8] and Chen [13] gave an upper bound and a lower bound for the ABC energy in terms of the general Randić index, respectively. Ghorbani et al. [1] established some new bounds for the ABC energy. Gao and Shao [7] determined the unique tree with the minimum ABC energy. In this paper, we determine the trees with the minimum ABC energy among all trees on n vertices except the star S_n.

2. Preliminaries

A matching in a graph is a set of edges without common vertices. A k-matching is a matching consisting of k edges. Let T be a tree, M be a matching of T and $M^k(T)$ be the set of all k-matchings of T. We define $m^*_M(T)$ and $m^*(T, k)$ by

$$m^*_M(T) = \prod_{v_iv_j \in M} (ABC_T)^2_{ij}$$

and

$$m^*(T, k) = \sum_{M \in M^k(T)} m^*_M(T),$$

respectively. By Sachs Theorem [14], the characteristic polynomial $\phi_{ABC}(T, x)$ of the ABC matrix of a tree T can be expressed as

$$\phi_{ABC}(T, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m^*(T, k) x^{n-2k}.$$

Then by Coulson integral formula, we get

$$\mathcal{E}_{ABC}(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^*(T, k) x^{2k} \right] dx. \tag{2.1}$$

Let T_1 and T_2 be two trees on n vertices. If $m^*(T_1, k) \geq m^*(T_2, k)$ for all k, then by (2.1) we get $\mathcal{E}_{ABC}(T_1) \geq \mathcal{E}_{ABC}(T_2)$. Moreover, if there exists some k such that $m^*(T_1, k) > m^*(T_2, k)$, then $\mathcal{E}_{ABC}(T_1) > \mathcal{E}_{ABC}(T_2)$.

Let T be a tree on n vertices, $B = (b_{ij})$ be an $n \times n$ nonnegative real symmetric matrix and $ABC_T \geq B$. Let M be a matching of T, $m^*_M(B) = \prod_{v_iv_j \in M} b^2_{ij}$ and $m^*(B, k) = \sum_{M \in M^k(T)} m^*_M(B)$. Then

$$\mathcal{E}(B) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^*(B, k) x^{2k} \right] dx.$$

Clearly, $m^*(T, k) \geq m^*(B, k)$. Thus $\mathcal{E}_{ABC}(T) \geq \mathcal{E}(B)$. Moreover, if $(ABC_T)_{ij} > b_{ij}$ for some $v_iv_j \in E(T)$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}(B)$. Thus we can get the following lemma.

Lemma 2.1. *Let T be a tree on n vertices and B be an $n \times n$ nonnegative real symmetric matrix. If $ABC_T \geq B$, then $\mathcal{E}_{ABC}(T) \geq \mathcal{E}(B)$. Moreover, if $(ABC_T)_{ij} > b_{ij}$ for some $v_iv_j \in E(T)$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}(B)$.*

Let uv be an edge of a tree T and $T - uv = T_1 \cup T_2$, where $T_1(T_2)$ is the component of $T - uv$ containing $u(v,$ respectively). We denote the sub-matrices of ABC_T spanned by the vertices of T_1 and T_2 by $(ABC_T)_{V(T_1)}$ and $(ABC_T)_{V(T_2)}$, respectively.

By Lemma 2.1, we have the next lemma.

Lemma 2.2. *Let uv be an edge of a tree T and $T - uv = T_1 \cup T_2$, where $T_1(T_2)$ is the component of $T - uv$ containing $u(v,$ respectively). Then*

$$\mathcal{E}_{ABC}(T) > \mathcal{E}((ABC_T)_{V(T_1)}) + \mathcal{E}((ABC_T)_{V(T_2)}).$$

Suppose that uv is not a pendent edge. If $d_T(w) \leq 2$ for any $w \in N_T(u) \setminus \{v\}$, then

$$\mathcal{E}((ABC_T)_{V(T_1)}) \geq \mathcal{E}_{ABC}(T_1).$$

Furthermore, if $d(w) \leq 2$ for any $w \in N_T(u) \cup N_T(v) \setminus \{u, v\}$, then

$$\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(T_1) + \mathcal{E}_{ABC}(T_2).$$

Lemma 2.3. *([7]) Let T be a tree of order $n \geq 3$. Then $\mathcal{E}_{ABC}(T) \geq 2\sqrt{n-2}$, with equality if and only if $T \cong S_n$, where S_n is the star of order n.*

Lemma 2.4. *([7]) Let $t \geq 2, x_i \geq 3$ for $i = 1, \ldots, t$, and $\sum_{i=1}^t x_i \geq 8$. Then $\sum_{i=2}^t \sqrt{x_i - 2} \geq \sqrt{\sum_{i=2}^t x_i + (t-1) - 2}$.*

3. The minimum ABC energy of trees

For two graphs G and H, we define $G \cup H$ to be their disjoint union. In addition, let kG be the disjoint union of k copies of G. Let S^*_n be the tree formed by attaching a vertex to a pendent vertex of the star S_{n-1}. Note that

$$\phi_{ABC}(S^*_n, x) = x^{n-4} \left[x^4 - (1 + \frac{(n-3)^2}{n-2}) x^2 + \frac{(n-3)^2}{2(n-2)} \right].$$

Thus

$$\mathcal{E}_{ABC}(S^*_n) = 2\sqrt{n - 3 + \frac{1}{n-2} + \sqrt{2}\sqrt{n - 4 + \frac{1}{n-2}}}.$$

Lemma 3.1. *Let $x \geq 11$. Then*

$$\sqrt{x-5}+1 > \sqrt{x-3+\frac{1}{x-2}+\sqrt{2}\sqrt{x-4+\frac{1}{x-2}}}. \tag{3.1}$$

Proof. It is equivalent to prove that

$$2\sqrt{x-5}-\sqrt{2}\sqrt{x-4+\frac{1}{x-2}}-\frac{1}{x-2}-1>0. \tag{3.2}$$

Let $f(x) = 2\sqrt{x-5}-\sqrt{2}\sqrt{x-4+\frac{1}{x-2}}-\frac{1}{x-2}-1$ with $x \geq 11$. Then

$$\begin{aligned}
\frac{df}{dx} &= \frac{1}{\sqrt{x-5}}-\frac{\sqrt{2}}{2}\frac{1}{\sqrt{x-4+\frac{1}{x-2}}}(1-\frac{1}{(x-2)^2})+\frac{1}{(x-2)^2}\\
&> \frac{1}{\sqrt{x-5}}-\frac{\sqrt{2}}{2}\frac{1}{\sqrt{x-4+\frac{1}{x-2}}}\\
&= \frac{1}{\sqrt{x-5}}-\frac{1}{\sqrt{2(x-4)+\frac{2}{x-2}}}\\
&> 0.
\end{aligned}$$

Thus $f(x)$ is a strictly monotonously increasing function on x. Noting that $f(11) = 0.0166 > 0$, then the lemma holds. □

From Lemma 3.1, $\mathcal{E}_{ABC}(S_n^*) < 2 + 2\sqrt{n-5}$ for $n \geq 11$.

For $n = 1, 2, 3$, there is only unique tree S_n. For $n = 4$, there are exactly two trees P_4 and S_4. Obviously, P_4 is the tree with the second minimum ABC energy. For $n = 5$, there are exactly three trees P_5, S_5 and S_5^*. By direct calculation, we have $\mathcal{E}_{ABC}(S_5^*) = 3.9831 > \mathcal{E}_{ABC}(P_5) = \sqrt{2}+\sqrt{6} > 2\sqrt{3} = \mathcal{E}_{ABC}(S_5)$. Thus P_5 is the tree with the second minimum ABC energy. Let $P_5 = v_1v_2v_3v_4v_5$, we denote the tree, obtained by attaching a new vertex to v_2 of P_5, by P_6^*. For $n = 6$, there are exactly six trees $T_{2.8}, T_{2.9}, T_{2.10}, T_{2.11}, T_{2.12}, T_{2.13}$ (see tables of graph spectra in [14]), where $T_{2.8} \cong S_6, T_{2.9} \cong S_6^*$, $T_{2.11} \cong P_6^*$ and $T_{2.13} \cong P_6$. By direct calculation, $\mathcal{E}_{ABC}(T_{2.12}) = 5.0590 > \mathcal{E}_{ABC}(P_6) = 4.9412 > \mathcal{E}_{ABC}(T_{2.10}) = 4.8074 > \mathcal{E}_{ABC}(S_6^*) = 4.6352 > \mathcal{E}_{ABC}(P_6^*) = 4.6260 > \mathcal{E}_{ABC}(S_6) = 4$.

By simple calculations, we obtain the following lemma.

Lemma 3.2. *Let T be an n-vertex tree not isomorphic to S_n, where $7 \leq n \leq 10$. Then $\mathcal{E}_{ABC}(T) \geq \mathcal{E}_{ABC}(S_n^*)$ with equality if and only if $T \cong S_n^*$.*

Lemma 3.3. *Let T be a tree on $n \geq 11$ vertices.*

(i) Let $u_1v_1 \in E(G)$ and $T - u_1v_1 = T_1 \cup T_2$, where $T_1(T_2)$ is the component of $T - u_1v_1$ containing $u_1(v_1$, respectively). If $d(w) \leq 2$ for any $w \in N(u_1) \cup N(v_1) \setminus \{u_1, v_1\}$ and $|V(T_1)| = n_1 \geq |V(T_2)| = n_2 \geq 3$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(S_n^*)$.

(ii) Let $u_2v_2, u_3v_3 \in E(G)$, $T - u_2v_2 \cong P_2 \cup T_3$ and $T_3 - u_3v_3 \cong P_2 \cup T_4$, where T_3 is one of the component of $T - u_2v_2$ and T_4 is one of the component of $T_3 - u_3v_3$. If $d_T(w_1) \leq 2$ for any $w_1 \in N_T(u_2) \cup N_T(v_2) \setminus \{u_2, v_2\}$ and $d_{T_3}(w_2) \leq 2$ for any $w_2 \in N_{T_3}(u_3) \cup N_{T_3}(v_3) \setminus \{u_3, v_3\}$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(S_n^*)$.

Proof. (i) By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &> \mathcal{E}_{ABC}(T_1)+\mathcal{E}_{ABC}(T_2)\\ &\geq 2\sqrt{n_1-2}+2\sqrt{n_2-2}\\ &\geq 2\sqrt{n-5}+2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

(ii) Similarly, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &> \mathcal{E}_{ABC}(T_3)+\mathcal{E}\left((ABC_T)_{V(P_2)}\right)\\ &> \mathcal{E}_{ABC}(T_4)+\sqrt{2}+\sqrt{2}\\ &\geq 2\sqrt{n-6}+2\sqrt{2}\\ &\geq 2\sqrt{n-5}+2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

We complete the proof. □

Lemma 3.4. *Let $n \geq 11$. Then $\mathcal{E}_{ABC}(P_n) > \mathcal{E}_{ABC}(S_n^*)$.*

Proof. By Lemma 2.2, we have $\mathcal{E}_{ABC}(P_n) > \mathcal{E}_{ABC}(P_3)+\mathcal{E}_{ABC}(P_{n-3}) \geq 2+2\sqrt{n-5} > \mathcal{E}_{ABC}(S_n^*)$. □

A tree is called starlike if it has exactly one vertex of degree greater than two.

Lemma 3.5. *Let $T \ncong S_n$ be a starlike tree with order $n \geq 11$ and v be the unique vertex with degree at least three. Let $T-v = n_1P_1 \cup n_2P_2 \cup \cdots \cup n_mP_m$ and $\sum_{i=1}^{m} in_i + 1 = n$. Then $\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(S_n^*)$.*

Proof. If $n_i \geq 1$ for some $i \geq 3$, then there exists an edge uv such that $T-uv = T_1 \cup T_2$, where $T_1(T_2)$ is the component of $T-uv$ containing $u(v,$ respectively), $|V(T_1)|, |V(T_2)| \geq 3$ and $d(w) \leq 2$ for any $w \in N(u) \cup N(v) \setminus \{u, v\}$. Then by (i) of Lemma 3.3 we can get the result. Suppose that $n_i = 0$ for all $i \geq 3$. If $n_2 = 0$, then $T \cong S_n$. If $n_2 = 1$, then $T \cong S_n^*$. If $n_2 \geq 2$, then by (ii) of Lemma 3.3 we can get the result. □

Let T be a tree and $R(T)$ be set of vertices of degree greater than two in T.

Lemma 3.6. *Let T be a tree with $n \geq 11$ vertices and $|R(T)| \geq 2$. If there are no adjacent vertices in $R(T)$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(S_n^*)$.*

Proof. Let $d(u) \geq 3$ and $d(v) \geq 3$ and $P_l = uv_1 \ldots v_{l-1}v$ be the single path connecting u and v with $d(v_1) = \cdots = d(v_{l-1}) = 2$. Clearly, $l \geq 2$. Without loss of generality, we suppose that $T-uv_1 = T_1 \cup T_2$ such that T_1 is a starlike-tree or a path, where $u \in V(T_1)$.

If $l \geq 3$, then by (i) of Lemma 3.3 we can get the result.

Suppose now that $l = 2$. Let $T_1'(T_2')$ be the component of $T-uv_1-v_1v$ containing $u(v$, respectively), $s = |V(T_1')|$ and $t = |V(T_2')|$. Obviously, $s+t+1 = n$.

If $s = 3$, then the ABC matrix of T can be written as

$$ABC_T = \begin{bmatrix} B & C \\ C^{\top} & D \end{bmatrix},$$

where

$$B=\begin{bmatrix} 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & \sqrt{\frac{1}{2}} & 0 \end{bmatrix}, \quad C=\begin{bmatrix} 0_{3\times 1} & 0_{3\times(t-1)} \\ \sqrt{\frac{1}{2}} & 0_{1\times(t-1)} \end{bmatrix},$$

and $D=(ABC_T)_{V(T_2')}$. Let

$$A=\begin{bmatrix} B & 0 \\ 0 & ABC_{T_2'} \end{bmatrix}.$$

Obviously, $D \geq ABC_{T_2'}$. Thus $ABC_T > A$. By Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \mathcal{E}_{ABC}(T) &> \mathcal{E}(A)=\mathcal{E}(B)+\mathcal{E}_{ABC}(T_2') \\ &\geq 2\sqrt{1+\frac{1}{3}+\frac{1}{2}}+2\sqrt{n-6} \\ &\geq 2\sqrt{n-5}+2 \\ &> \mathcal{E}_{ABC}(S_n^*). \end{aligned}$$

Suppose that $s=4$. Then $T_1' \cong S_4$ or P_4. If $T_1' \cong S_4$, then the ABC matrix of T can be written as

$$ABC_T=\begin{bmatrix} F & H \\ H^\top & K \end{bmatrix},$$

where

$$F=\begin{bmatrix} 0 & 0 & 0 & \sqrt{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{4}} & 0 \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{3}{4}} & \sqrt{\frac{3}{4}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \end{bmatrix}, \quad H=\begin{bmatrix} 0_{4\times 1} & 0_{4\times(t-1)} \\ \sqrt{\frac{1}{2}} & 0_{1\times(t-1)} \end{bmatrix},$$

and $K=(ABC_T)_{V(T_2')}$. Let

$$M=\begin{bmatrix} F & 0 \\ 0 & ABC_{T_2'} \end{bmatrix}.$$

Obviously, $K \geq ABC_{T_2'}$. Thus $ABC_T > M$. By Lemmas 2.1 and 2.3 we have

$$\begin{aligned} \mathcal{E}_{ABC}(T) &\geq \mathcal{E}(M)=\mathcal{E}(F)+\mathcal{E}_{ABC}(T_2') \\ &\geq 2\sqrt{2+\frac{1}{4}+\frac{1}{2}}+2\sqrt{n-7} \\ &\geq 2\sqrt{n-5}+2 \\ &> \mathcal{E}_{ABC}(S_n^*). \end{aligned}$$

Suppose now that $T_1' \cong P_4$. Let $T_1' = u_1uu_2u_3$. Then by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &\geq \sqrt{2} + \mathcal{E}_{ABC}(T - u_2 - u_3)\\ &\geq \sqrt{2} + \mathcal{E}_{ABC}(P_3) + \mathcal{E}_{ABC}(T_2')\\ &\geq \sqrt{2} + 2 + 2\sqrt{n-7}\\ &\geq 2\sqrt{n-5} + 2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

By symmetry, we now suppose that $5 \leq s, t \leq n-6$, then by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &> \mathcal{E}_{ABC}(T_1') + \mathcal{E}_{ABC}(T_2')\\ &\geq 2\sqrt{s-2} + 2\sqrt{t-2}\\ &\geq 2\sqrt{3} + 2\sqrt{n-8}\\ &> 2 + 2\sqrt{n-5}\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

□

Lemma 3.7. *Let T be a tree with $n \geq 11$ vertices and $|R(T)| \geq 2$. If there exist adjacent vertices in $R(T)$, then $\mathcal{E}_{ABC}(T) > \mathcal{E}_{ABC}(S_n^*)$.*

Proof. Let $E^0 = \{uv \in E(T) | d(u), d(v) \geq 3\}$, and $T - E^0 = xP_1 \cup yP_2 \cup T_1 \cup \cdots \cup T_z$, where $T_1, \ldots, T_z$ are components of $T - E^0$ with at least three vertices. Let $xP_1 = \{v_1, \ldots, v_x\}$ and $yP_2 = \{v_{x+1}v_{x+2}, \ldots, v_{x+2y-1}v_{x+2y}\}$. Then $d_T(v_i) \geq 3$ with $1 \leq i \leq x$, $d_T(v_{x+2j-1}) \geq 3$ and $d_T(v_{x+2j}) = 1$ with $1 \leq j \leq y$, and for each component T_i with $1 \leq i \leq z$, there exists a vertex $v_i \in V(T_i)$ such that $d_T(v_i) \geq d_{T_i}(v_i) + 1$. Let $|V(T_i)| = s_i$ for $1 \leq i \leq z$. Thus we have

$$\begin{aligned}2(n-1) &= \sum_{v \in V(T)} d_T(v) \geq 3x + 4y + \sum_{i=1}^{z}\Big(\sum_{v \in V(T_i)} d_{T_i}(v) + 1\Big)\\ &= 3x + 4y + \sum_{i=1}^{z} 2(s_i - 1) + z\\ &= x + 2n - z.\end{aligned}$$

Thus we get that $z \geq x + 2$. We discuss the following four cases.

Case 1. $y = 0$ and $z = 2$.

Then $x = 0$ and $s_1 + s_2 = n$. By Lemmas 2.2 and 2.3, we get

$$\begin{aligned}\mathcal{E}_{ABC}(T) &> \mathcal{E}_{ABC}(T_1) + \mathcal{E}_{ABC}(T_2)\\ &\geq 2\sqrt{s_1 - 2} + 2\sqrt{s_2 - 2}\\ &\geq 2\sqrt{n-5} + 2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

Case 2. $y = 0$ and $z \geq 3$.

Then $x + \sum_{i=1}^{z} s_i = n$. Without loss of generation, we suppose that $3 \le s_z \le s_{z-1} \le \cdots \le s_2 \le s_1$.

If $\sum_{i=1}^{z-1} s_i = 6$, then $z = 3$, $s_z = 3$ and $x \le 1$. Thus $n \le 10$, a contradiction.

If $\sum_{i=1}^{z-1} s_i = 7$, then $z = 3$, $s_1 = 4$, $s_2 = s_3 = 3$. Thus $x = 1$ and $n = 11$. Obviously, $T_2 \cong T_3 \cong S_3$ and $T_1 \cong S_4$ or P_4. By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &\ge \mathcal{E}_{ABC}(T_1) + \mathcal{E}((ABC_T)_{V(T_2)}) + \mathcal{E}((ABC_T)_{V(T_3)})\\ &\ge 2\sqrt{4-2} + 4 \times \frac{2}{\sqrt{3}}\\ &= 7.448 > 6.8742 = \mathcal{E}_{ABC}(S_{11}^*).\end{aligned}$$

Suppose that $\sum_{i=1}^{z-1} s_i \ge 8$. By Lemmas 2.2–2.4, we have that

$$\begin{aligned}\mathcal{E}_{ABC}(T) &> \sum_{i=1}^{z} \mathcal{E}_{ABC}(T_i) \ge 2\sum_{i=1}^{z} \sqrt{s_i - 2}\\ &\ge 2\sqrt{\sum_{i=1}^{z-1} s_i + (z-1) - 3} + 2\sqrt{s_z - 2}\\ &\ge 2\sqrt{n - x - s_z + x + 2 - 4} + 2\sqrt{s_z - 2}\\ &= 2\sqrt{n - s_z - 2} + 2\sqrt{s_z - 2}\\ &\ge 2\sqrt{n-5} + 2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned} \tag{3.3}$$

Case 3. $y \ge 1$ and $z \ge 3$.

Then $\sum_{i=1}^{z-1} s_i + s_z + x + 2y = n$. By Lemmas 2.2 and 2.3, we have

$$\mathcal{E}_{ABC}(T) \ge 2y\sqrt{\frac{2}{3}} + \sum_{i=1}^{z} \mathcal{E}_{ABC}(T_i) \ge 2y\sqrt{\frac{2}{3}} + 2\sum_{i=1}^{z} \sqrt{s_i - 2}.$$

If $\sum_{i=1}^{z-1} s_i \ge 8$, then by Lemma 2.4, we have

$$\begin{aligned}\mathcal{E}_{ABC}(T) &\ge 2y\sqrt{\frac{2}{3}} + 2\sqrt{\sum_{i=1}^{z-1} s_i + (z-1-1) - 2} + 2\sqrt{s_z - 2}\\ &\ge 2y\sqrt{\frac{2}{3}} + 2\sqrt{n - s_z - 2y - x + x + 2 - 4} + 2\sqrt{s_z - 2}\\ &= 2y\sqrt{\frac{2}{3}} + 2\sqrt{n - s_z - 2y - 2} + 2\sqrt{s_z - 2}\\ &\ge 2y\sqrt{\frac{2}{3}} + 2\sqrt{n - 3 - 2y - 2} + 2\sqrt{3-2}\\ &\ge 2\sqrt{n-5} + 2\\ &> \mathcal{E}_{ABC}(S_n^*).\end{aligned}$$

Here, the last but one inequality holds because $f(y) = 2y\sqrt{\frac{2}{3}} + 2\sqrt{n - 5 - 2y} + 2$ is increasing for $0 \le y \le \frac{2n-13}{4}$.

Suppose that $\sum_{i=1}^{z-1} s_i \leq 7$. Then $z = 3$. Thus $s_1 + s_2 = 6$ or 7, $s_3 = 3$ and $x \leq 1$.
Suppose first that $s_1 + s_2 = 7$ and $s_3 = 3$. If $x = 0$, then $n = 2y + 10$. Hence

$$\begin{aligned}
\mathcal{E}_{ABC}(T) &\geq 2y\sqrt{\frac{2}{3}} + \mathcal{E}_{ABC}(T_1) + \mathcal{E}_{ABC}(T_2) + \mathcal{E}_{ABC}(T_3) \\
&\geq 2y\sqrt{\frac{2}{3}} + 2\sqrt{2} + 2 + 2 \\
&= (n-10)\sqrt{\frac{2}{3}} + 2\sqrt{2} + 4 \\
&\geq 2\sqrt{n-5} + 2 \\
&> \mathcal{E}_{ABC}(S_n^*).
\end{aligned}$$

Suppose now that $x = 1$. Then $n = 2y + 11$. Hence

$$\begin{aligned}
\mathcal{E}_{ABC}(T) &> 2y\sqrt{\frac{2}{3}} + 2\sqrt{2} + 2 + 2 \\
&= (n-11)\sqrt{\frac{2}{3}} + 2\sqrt{2} + 4.
\end{aligned}$$

Let $f(x) = (x-11)\sqrt{\frac{2}{3}} + 4 + 2\sqrt{2} - 2\sqrt{x - 3 + \frac{1}{x-2} + \sqrt{2}\cdot\sqrt{x - 4 + \frac{1}{x-2}}}$. It is easy to get that $f'(x) > 0$ for $x \geq 11$. Then $f(x)$ is an increasing function on x and $f(x) \geq f(11) > 0$. Thus

$$\begin{aligned}
\mathcal{E}_{ABC}(T) &> (n-11)\sqrt{\frac{2}{3}} + 4 + 2\sqrt{2} \\
&> 2\sqrt{n - 3 + \frac{1}{n-2} + \sqrt{2}\cdot\sqrt{n - 4 + \frac{1}{n-2}}}.
\end{aligned}$$

By a similar discussion as above, we can get the result for the case $s_1 + s_2 = 6$ and $s_3 = 3$.
Case 4. $y \geq 1$ and $z = 2$.
Then $x = 0, n = 2y + s_1 + s_2$. If $n - 2y \geq 11$, then by Lemmas 2.1–2.3, we have

$$\begin{aligned}
\mathcal{E}_{ABC}(T) &\geq 2y\sqrt{\frac{2}{3}} + \mathcal{E}_{ABC}(T_1) + \mathcal{E}_{ABC}(T_2) \\
&\geq 2y\sqrt{\frac{2}{3}} + 2\sqrt{s_1 - 2} + 2\sqrt{s_2 - 2} \\
&\geq 2y\sqrt{\frac{2}{3}} + 2\sqrt{n - 2y - 3 - 2} + 2\sqrt{3 - 2} \\
&\geq 2\sqrt{n-5} + 2 \\
&> \mathcal{E}_{ABC}(S_n^*).
\end{aligned}$$

Suppose that $n - 2y \leq 10$. Then $6 \leq s_1 + s_2 \leq 10$. If $s_1 + s_2 = 10$, then by Lemmas 2.1–2.3, we have

$$\mathcal{E}_{ABC}(T) > 2y\sqrt{\frac{2}{3}} + 2\sqrt{3-2} + 2\sqrt{7-2}$$

$$
\begin{aligned}
&= (n-10)\sqrt{\frac{2}{3}} + 2 + 2\sqrt{5} \\
&\geq 2\sqrt{n-5} + 2 \\
&> \mathcal{E}_{ABC}(S_n^*).
\end{aligned}
$$

Similarly, for each $6 \leq s_1 + s_2 \leq 9$, we may also get the result. □

Combining Lemmas 3.2 and 3.4–3.7, we get our main result.

Theorem 3.1. *Among all trees (except the star) on $n \geq 5$ vertices, P_5 is the unique tree with the minimum ABC energy for $n = 5$, P_6^* is the unique tree with the minimum ABC energy for $n = 6$ and S_n^* is the unique tree with the minimum ABC energy for $n \geq 7$.*

4. Conclusions

In this paper, motivated by the unique tree with the minimum ABC energy, we determine the trees with the minimum ABC energy among all trees on n vertices except the star S_n.

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Conflict of interest

The authors declare that they have no competing interests.

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