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*Research article*

# Ground state solutions for the fractional Schrödinger-Poisson system involving doubly critical exponents

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**Abstract:** In this article, we are dedicated to studying the fractional Schrödinger-Poisson system involving doubly critical exponent. By using the variational method and analytic techniques, we establish the existence of positive ground state solution.

**Keywords:** fractional Schrödinger-Poisson system; doubly critical exponent; variational method; positive solution
**Mathematics Subject Classification:** 35J20, 35J60

## 1. Introduction and main results

In this paper, we study the following fractional Schrödinger-Poisson system with doubly critical exponent

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi|u|^{2_s^*-3}u = f(u) + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $2_s^* = \frac{6}{3-2s}$ is the critical fractional Sobolev exponent, the potential $V(x)$ and the nonlinearity $f$ satisfy the following assumptions:
$(V_1)$ $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$;
$(V_2)$ There exists $h > 0$ such that $\lim_{|y|\to\infty} meas(\{x \in B_h(y) \mid V(x) \leq c\}) = 0$ for any $c > 0$, where $B_h(y)$ is the ball centered at point $y$ with radius $h$;
$(f_1)$ $f \in C(\mathbb{R},\mathbb{R})$ and $f(t) \equiv 0$ for $t \leq 0$;
$(f_2)$ $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
$(f_3)$ $\lim_{t\to+\infty} \frac{f(t)}{t^{2_s^*-1}} = 0$;

$(f_4)$ There exist two constants $\rho \in (2, 2_s^*)$ and $0 < \delta < \frac{(\rho-2)V_0}{2}$ such that $\rho F(t) \leq f(t)t + \delta t^2$, where $F(t) = \int_0^t f(s)ds$;
$(f_5)$ $\lim_{t\to+\infty} \frac{F(t)}{t^{m+1}} = +\infty$, where $m > \max\{1, \frac{6s-3}{3-2s}\}$.

The non-local fractional Laplacian operator $(-\Delta)^s$ in $\mathbb{R}^3$ can be characterized as

$$(-\Delta)^s u(x) = C(s)\text{P.V.} \int_{R^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy,$$

P.V. represents the Cauchy principal value, and $C(s)$ is a positive constant only depending on $s$, see [1]. In the last several years, nonlinear equations involving the fractional Laplacian have been attracted a lot of attention by many scholars. One of the main reasons for this is that the fractional Laplacian operator naturally arises in many different areas, such as such as thin obstacle problem (see [2]), combustion (see [3]), financial mathematics (see [4]), minimal surfaces (see [5]), etc. Another main reason is that the fractional Laplacian $(-\Delta)^s$ is a non-local operator in contrast to the classical Laplacian $\Delta$, and previously developed methods may not be applied directly.

Technically, system (1.1) consists of a fractional Schrödinger equation coupled with a fractional Poisson equation. It can reduce to the following classical Schrödinger-Poisson system for $s = 1$,

$$\begin{cases} (-\Delta)u + V(x)u - \phi|u|^{2^*-3}u = f(u) + |u|^{2^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)\phi = |u|^{2^*-1}, & \text{in } \mathbb{R}^3. \end{cases}$$

Due to the real physical meaning, the classical Schrödinger-Poisson system has been studied extensively by many scholars. Here, we do not try to recall the details on this topic, we refer the interested readers to see [6–9] and the references therein.

From a physical standpoint, the fractional Schrödinger equation was discovered by Laskin as a result of extending Feynman path integral, from the Brownian-like to levy like quantum mechanical path. Since then, fractional Schrödinger-Poisson system has been attracted many scholars′ dinterest, the existence and multiplicity of solutions have been established via applying the variational methods, please see [10–14].

But to the best of our knowledge, there are few papers considering the frational Schrödinger-Poisson system with doubly critical exponent, such as [15, 16]. More precisely, in [15], Feng and Yang used concentration-compactness principle to obtained a ground state solution for the following system

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi|u|^{2_s^*-3}u = K(x)|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^s\phi = |u|^{2_s^*-1}, & \text{in } \mathbb{R}^3, \end{cases}$$

where $s \in (\frac{3}{4}, 1)$, $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$, $K \in L^\infty(\mathbb{R}^3)$. In [16], the author studied system (1.1), by limiting the order $s \in (\frac{3}{4}, 1)$, he obtained a positive solution with (AR) condition. Motivated by the previously mentioned works, in the present paper, we shall investigate the case of $s \in (0, 1)$ for doubly critical problem without (AR) condition, in other words, we will study the existence of positive ground state solution.

Then, our main result can be described as the follows.

**Theorem 1.1.** *Assume that* $V$*,* $f$ *satisfy the assumptions* $(V_1) - (V_2)$ *and* $(f_1) - (f_5)$*, respectively. Then, system* (1.1) *has at least one positive ground state solution.*

**Remark 1.1.** *As far as we know, there have not any works in the present literature for the system* (1.1) *with* $s \in (0,1)$. *The condition* $0 < \mu F(t) = \mu \int_0^t f(\tau)d\tau \leq tf(t)$ *for* $\mu \in (4, 2_s^*)$ *is well knowm as (AR) condition, which was first introduced to obtain a bounded Palais-Smale sequence. In fact, the (AR) condition implies that* $F(t)$ *is a 4-superlinear and subcritical nonlinearity, in order to cover the case where the degree of* $F(t)$ *is between (2, 4), we add a more weaker condition* $(f_4)$.

**Corollary 1.1.** *Assume that* $V$ *satisfy the assumptions* $(V_1) - (V_2)$ *and*

$$f(t) = \begin{cases} t^p, & t > 0 \text{ and } p \in \left(\max\left\{1, \frac{6s-3}{3-2s}\right\}, \frac{3+2s}{3-2s}\right), \\ 0, & t \leq 0, \end{cases}$$

*then system* (1.1) *has at least one positive ground state solution.*

## 2. Variational setting and preliminaries

In this section, firstly we will give the variational framework for the system (1.1). Throughout this paper, we denote by $C$, $C_i > 0$ various positive constants which may vary from line to line and are not essential to the problem.

For any $s \in (0,1)$, $D^{s,2}(\mathbb{R}^3)$ is completion of the set $C_0^\infty(\mathbb{R}^3)$, which consists of infinitely differentiable functions $u : \mathbb{R}^3 \to \mathbb{R}$ with compact support to the following norm

$$[u]_s^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \text{ and } D^{s,2}(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : [u]_s < \infty\}.$$

The fractional Sobolev space is defined by

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : [u]_s < \infty\}$$

is equipped with the norm

$$\|u\|_{H^s}^2 = \|u\|_2^2 + [u]_s^2,$$

we denote by $\|\cdot\|_p$ the usual $L^p$-norm. Due to the appearance of potential function $V(x)$, we will work in the following space:

$$E = \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty\right\},$$

then, $E$ is equipped with the norm $\|u\|^2 = [u]_s^2 + \int_{\mathbb{R}^3} V(x)u^2 dx$. Let $S > 0$ be the best Sobolev constant for the embedding of $D^{s,2}(\mathbb{R}^3)$ in $L^{2_s^*}(\mathbb{R}^3)$, which can be expressed as

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{(\int_{\mathbb{R}^3} |u|^{2_s^*} dx)^{\frac{2}{2_s^*}}}. \tag{2.1}$$

As we all known, for $u \in E$, the Lax-Milgram theorem implies that Poisson equation $(-\Delta)^s \phi = |u|^{2_s^*-1}$ has a unique weak solution

$$\phi_u(x) = C_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dy, x \in \mathbb{R}^3,$$

where $C_s = \frac{\Gamma(\frac{3-2s}{2})}{2^{2s}\pi^{\frac{3}{2}}\Gamma(s)}$. Similar to the usual Schrödinger-Poisson system, we can insert $\phi_u$ into the first equation of the system (1.1). Then system (1.1) can be transformed in a single Schrödinger equation as follows

$$(-\Delta)^s u + V(x)u - \phi_u|u|^{2_s^*-3}u = f(u) + |u|^{2_s^*-2}u, \quad \forall x \in \mathbb{R}^3. \tag{2.2}$$

The Euler functional of Eq (2.2) is defined by $I: E \to \mathbb{R}$, that is,

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx - \int_{\mathbb{R}^3}F(u)dx.$$

Under the assumptions of $f(u)$, we can deduce that functional $I$ is well defined on $E$ and is of class $C^1(E,\mathbb{R})$. For each $u, v \in E$, we have

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3}\left((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(x)uv + \phi_u|u|^{2_s^*-3}uv - |u|^{2_s^*-1}v - f(u)v\right)dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality. It is easy to verify that if $u$ is a critical point of $I$, then the pair $(u, \phi_u)$ is a weak solution of system (1.1).

**Lemma 2.1.** *If $s \in (0,1)$, then for any $u \in E$, the following results hold*
*(1) $\phi_u \geq 0$;*
*(2) $\phi_{tu} = t^{2_s^*-1}\phi_u$ for all $t > 0$;*
*(3) $\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx \leq S^{-1}\|u\|_{2_s^*}^{2(2_s^*-1)}$;*
*(4) If $u_n \rightharpoonup u$ in $E$ as $n \to \infty$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$. Moreover,*

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx \to \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-3}u\varphi dx \text{ for any } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{2.3}$$

*Proof.* The conclusions (1) and (2) are clear from simple calculation.

(3) Since $\phi_u(x)$ is a unique weak solution of $(-\Delta)^s\phi = |u|^{2_s^*-1}$, one has

$$\begin{aligned}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_u|^2dx &\leq \left(\int_{\mathbb{R}^3}|\phi_u|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}\left(\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right)^{\frac{2_s^*-1}{2_s^*}} \\ &\leq S^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_u|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right)^{\frac{2_s^*-1}{2_s^*}}.\end{aligned}$$

Thus, it is easy to check that (3) holds.

(4) Since $u_n \rightharpoonup u$ in $E$ as $n \to \infty$, then $|u_n|^{2_s^*-1} \rightharpoonup |u|^{2_s^*-1}$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$ as $n \to \infty$. Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, the uniqueness of weak solution of Poisson equation implies that

$$\int_{\mathbb{R}^3}\phi_{u_n}\varphi dx = \int_{\mathbb{R}^3}|u_n|^{2_s^*-1}\varphi dx \to \int_{\mathbb{R}^3}|u|^{2_s^*-1}\varphi dx = \int_{\mathbb{R}^3}\phi_u\varphi dx,$$

that is, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$. It is now simple to conclude

$$\int_{\mathbb{R}^3}(\phi_{u_n} - \phi_u)|u|^{2_s^*-3}u\varphi \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Let $q = \frac{2_s^*}{2_s^*-1}$, using Hölder inequality, we have

$$\int_{\mathbb{R}^3} \phi_{u_n}[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]^q dx \leq C\left(\|\phi_{u_n}\|_{2_s^*}^q \|u_n\|_{2_s^*}^{q(2_s^*-2)} + \|\phi_{u_n}\|_{2_s^*}^q \|u_n\|_{2_s^*}^{q(2_s^*-2)}\right) < +\infty,$$

notice that $u_n(x) \to u(x)$ a.e in $\mathbb{R}^3$ can be inferred from weak convergence of $u_n \rightharpoonup u$ in $E$, which implies that

$$\int_{\mathbb{R}^3} \phi_{u_n}[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]\varphi \to 0 \text{ as } n \to \infty. \tag{2.5}$$

Now, we combine (2.4) and (2.5), (2.3) holds. □

**Lemma 2.2.** *Suppose that* $(V_1)$ *and* $(f_1) - (f_3)$ *hold, then*
*(a) there exist* $\alpha > 0, \beta > 0$ *such that* $I(u) \geq \alpha$, *for all* $\|u\| = \beta$,
*(b) there exists* $e \in H$ *such that* $\|e\| > \beta$ *and* $I(e) < 0$.

*Proof.* (a) By $(f_1) - (f_3)$, for all $\varepsilon > 0$ small enough, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|u|^{2_s^*-1} \quad \text{and} \quad |F(t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{2_s^*}|t|^{2_s^*}. \tag{2.6}$$

Thus, by Sobolev inequality and $(V_1)$, there holds

$$\int_{\mathbb{R}^3} F(u)dx \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{C_\varepsilon}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}dx \leq \frac{\varepsilon}{2V_0}\|u\|^2 + \frac{S^{-\frac{2_s^*}{2}}C_\varepsilon}{2_s^*}\|u\|^{2_s^*}. \tag{2.7}$$

It follows from (2.6), Lemma 2.1(3) and Sobolev inequality that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}dx - \int_{\mathbb{R}^3} F(u)dx \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon}{2V_0}\right)\|u\|^2 - \frac{S^{-2_s^*}}{2(2_s^*-1)}\|u\|^{2(2_s^*-1)} - \frac{S^{-\frac{2_s^*}{2}}(C_\varepsilon+1)}{2_s^*}\|u\|^{2_s^*}. \end{aligned}$$

Because $\varepsilon$ is small enough, we can assume $\varepsilon \in (0, V_0)$ and letting $\beta > 0$ small, $\|u\| = \beta$ implies that

$$I(u) \geq \left(\frac{1}{2} - \frac{\varepsilon}{2V_0}\right)\beta^2 - \frac{S^{-2_s^*}}{2(2_s^*-1)}\beta^{2(2_s^*-1)} - \frac{S^{-\frac{2_s^*}{2}}(C_\varepsilon+1)}{2_s^*}\beta^{2_s^*} = \alpha > 0.$$

(b) Fixed $u_0 \in E$ and $u_0 \neq 0$, for any $t > 0$, we have

$$I(tu_0) = \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_{u_0}|u_0|^{2_s^*-1}dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3} |u_0|^{2_s^*}dx - \int_{\mathbb{R}^3} F(tu_0)dx,$$

it is easy to check that $\lim_{t\to+\infty} I(tu_0) = -\infty$. Therefore, there exists $t_0 > 0$ large enough such that $I(t_0u_0) < 0$ and $\|t_0u_0\| > \beta$. Thus, we complete the proof by taking $e = t_0u_0$. □

**Lemma 2.3.** *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_4)$ *hold, then the functional* $I$ *satisfies the* $(PS)_c$ *condition provided* $c \in (0, c^*)$, *where* $c^* = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} S^{\frac{3}{2s}}$.

*Proof.* Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence of $I$, that is,

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Firstly, we claim that the sequence $\{u_n\}$ is bounded in $E$. Indeed, by $(f_4)$ and (2.8), we get that

$$\begin{aligned}
1 + c + o(\|u_n\|) \geq & I(u_n) - \frac{1}{\rho}\langle I'(u_n), u_n\rangle \\
= & \left(\frac{1}{2} - \frac{1}{\rho}\right)\|u_n\|^2 + \left[\frac{1}{\rho} - \frac{1}{2(2_s^* - 1)}\right]\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx \\
& + \left(\frac{1}{\rho} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3} |u_n|^{2_s^*}dx + \int_{\mathbb{R}^3}\left[\frac{1}{\rho}f(u_n)u_n - F(u_n)\right]dx \\
\geq & \left(\frac{1}{2} - \frac{1}{\rho} - \frac{\delta}{V_0\rho}\right)\|u_n\|^2,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $E$. Then we can extract a subsequence, still denoted $\{u_n\} \subset E$, that converges weakly to some $u_* \in E$. Under the conditions $(V_0)$ and $(V_1)$, we know from [10] that, the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is continuous and compact for any $p \in (2, 2_s^*)$, thus we can make sure that

$$\begin{cases}
u_n \rightharpoonup u_* \ \text{ weakly in } E, \\
u_n(x) \to u_*(x) \ \text{ a.e in } \mathbb{R}^3, \\
u_n \to u_* \ \text{ strongly in } L^p(\mathbb{R}^3).
\end{cases}$$

Next, we claim that $u_*$ is a solution of (2.2). It follows from $(f_3)$ and continuity of $f$ that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $f(u_n) \leq C_\varepsilon + \varepsilon u_n^{2_s^*-1}$. Set $0 < \theta < \varepsilon$ small enough, for $\Omega_\theta \subset \mathrm{supp}\varphi$ with $meas(\Omega_\theta) < \theta$ and any $\varphi \in C_0^\infty(\mathbb{R}^3)$, there holds

$$\begin{aligned}
\left|\int_{\Omega_\theta} f(u_n)\varphi dx\right| & \leq C_\varepsilon \int_{\Omega_\theta} |\varphi| dx + \varepsilon\int_{\Omega_\theta} |u_n|^{2_s^*-1}|\varphi| dx \\
& \leq C meas(\Omega_\theta) + \varepsilon\left(\int_{\Omega_\theta} |u_n|^{2_s^*}dx\right)^{\frac{2_s^*-1}{2_s^*}}\left(\int_{\Omega_\theta} |\varphi|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \\
& < C\varepsilon
\end{aligned}$$

due to $\{u_n\}$ is bounded in $L^{2_s^*}(\mathbb{R}^3)$, which implies that $\{f(u_n)\varphi\}$ is equiabsolutely continuous. Making use of Vitali theorem, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)\varphi dx = \int_{\mathbb{R}^3} f(u_*)\varphi dx.$$

It follows from Lemma 2.1(4) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx = \int_{\mathbb{R}^3} \phi_{u_*}|u_*|^{2_s^*-3}u_*\varphi dx.$$

Using $u_n \rightharpoonup u_*$ weakly in $E$ again, we can prove that $\langle I'(u_*), \varphi\rangle = 0$ for $\forall\varphi \in C_0^\infty(\mathbb{R}^3)$. In the end, we claim that $u_n \to u_*$ strongly in $E$. In fact, we can define $v_n = u_n - u_*$, then $v_n \rightharpoonup 0$ in $E$. For any $\varepsilon > 0$, there exists, by assumptions $(f_1),(f_2)$ and $(f_3)$, a $C_\varepsilon>0$ such that

$$|f(t)|\leq\varepsilon(|t|+|t|^{2_s^*-1})+C_\varepsilon|t|^{p-1}\quad\text{and}\quad|F(t)|\leq\frac{\varepsilon}{2}|t|^2+\frac{\varepsilon}{2_s^*}|t|^{2_s^*}+\frac{C_\varepsilon}{p}|t|^p.\tag{2.9}$$

Consequently, by (2.9) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{cases}\lim\limits_{n\to\infty}\int_{\mathbb{R}^3}F(u_n)dx=\int_{\mathbb{R}^3}F(u_*)dx,\\ \lim\limits_{n\to\infty}\int_{\mathbb{R}^3}f(u_n)u_ndx=\int_{\mathbb{R}^3}f(u_*)u_*dx.\end{cases}\tag{2.10}$$

From Brézis-Lieb's lemma (see [17]), it holds that

$$\begin{cases}\|u_n\|^2=\|v_n\|^2+\|u_*\|^2+o_n(1),\\ \int_{\mathbb{R}^3}u_n^{2_s^*}dx=\int_{\mathbb{R}^3}v_n^{2_s^*}dx+\int_{\mathbb{R}^3}u_*^{2_s^*}dx+o_n(1).\end{cases}\tag{2.11}$$

Using Lemma 3.2 in [18], there holds

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx-\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx=\int_{\mathbb{R}^3}\phi_u|u_*|^{2_s^*-1}dx+o_n(1).$$

Summing up, the preceding equalities show that

$$I(u_n)=I(u_*)+\frac{1}{2}\|v_n\|^2-\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx+o_n(1)$$

and

$$\langle I'(u_n),u_n\rangle=\langle I(u_*),u_*\rangle+\|v_n\|^2-\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx-\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx+o_n(1).$$

Therefore, it follows from the hypotheses $I(u_n)\to c$ and $I'(u_n)\to 0$ that

$$\begin{aligned}c=&\lim_{n\to\infty}I(u_n)\\ =&I(u_*)+\lim_{n\to\infty}\left(\frac{1}{2}\|v_n\|^2-\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx\right)\end{aligned}\tag{2.12}$$

and

$$\lim_{n\to\infty}\|v_n\|^2-\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx-\lim_{n\to\infty}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx=0.\tag{2.13}$$

Now, we may assume that

$$\ell_n:=\|v_n\|^2\to\ell,\ a_n:=\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx\to a\ \text{ and }\ b_n:=\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx\to b.\tag{2.14}$$

Since the Lax-Milgram theorem implies that $(-\Delta)^s\phi=|v_n|^{2_s^*-1}$ has a sequence solution $\{\phi_{v_n}\}$, then we have

$$\begin{aligned}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx=&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}\phi_{v_n}(-\Delta)^{\frac{s}{2}}|v_n|dx\\ &\leq\frac{1}{\sqrt{5}-1}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^{2_s^*-1}dx+\frac{\sqrt{5}-1}{4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}v_n|^2dx.\end{aligned}$$

As $n \to \infty$ passing to the limit, it follows that

$$b \leq \frac{1}{\sqrt{5}-1}a + \frac{\sqrt{5}-1}{4}\ell.$$

Using (2.13) and (2.14), we infer that

$$a \geq \frac{3-\sqrt{5}}{2}\ell.$$

On the other hand, we obtain

$$\begin{aligned} I(u_*) =& \left(\frac{1}{2}-\frac{1}{\rho}\right)\|u_*\|^2 + \left[\frac{1}{\rho}-\frac{1}{2(2_s^*-1)}\right]\int_{\mathbb{R}^3}\phi_{u_*}|u_*|^{2_s^*-1}dx \\ &+\left(\frac{1}{\rho}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_*|^{2_s^*}dx + \int_{\mathbb{R}^3}\left(\frac{1}{\rho}f(u_*)u_* - F(u_*)\right)dx \\ \geq& \left(\frac{1}{2}-\frac{1}{\rho}-\frac{\delta}{V_0\rho}\right)\|u_*\|^2 \\ \geq& 0. \end{aligned}$$

It follows from (2.12)–(2.14), one has

$$c \geq \frac{2s}{3+2s}a + \frac{s}{3}b \geq \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}\ell. \tag{2.15}$$

The estimates (2.1) and (2.13) lead to

$$\ell \leq S^{-2_s^*}\ell^{2_s^*-1} + S^{-\frac{2_s^*}{2}}\ell^{\frac{2_s^*}{2}}.$$

Thus, we get either

$$\ell = 0 \text{ or } \ell^{\frac{2_s^*-2}{2}} \geq \frac{\sqrt{5}-1}{2}S^{\frac{2_s^*}{2}}.$$

If $\ell \neq 0$, then from (2.13), we infer that

$$c \geq \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}\ell \geq (\frac{\sqrt{5}-1}{2})^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{3}{2s}} := c^*,$$

which contradicts the fact that $c < c^*$. Hence $\ell = 0$ and we have that $u_n \to u$ in $E$. □

As in [19], the extremal function $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}}u^*(\frac{x}{\varepsilon})$ solves the equation $(-\Delta)^s\phi = |u|^{2_s^*-2}u$ in $\mathbb{R}^3$, where $u^*(x) = \frac{\widetilde{u}(x/S_s^{\frac{1}{2s}})}{\|\widetilde{u}\|_{2_s^*}}$ and $\widetilde{u}(x) = k(\mu_0^2+|x|^2)^{-\frac{3-2s}{2}}$ with $k > 0$ and $\mu_0 > 0$ being fixed constants.

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ is such that $0 \leq \psi(x) \leq 1$ in $\mathbb{R}^3, \psi(x) = 1$ in $B_1$ and $\psi(x) = 0$ in $\mathbb{R}^3\backslash B_2$, we define $v_\varepsilon(x) = \psi(x)U_\varepsilon(x)$. According to Propositions 21 and 22 in [19], we know that

$$\begin{cases} \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|^2 \leq S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \\ \int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx = S^{\frac{3}{2s}} + O(\varepsilon^3), \end{cases} \tag{2.16}$$

$$\int_{\mathbb{R}^3}|v_\varepsilon|^p dx = \begin{cases} O(\varepsilon^{\frac{3(2-p)+2sp}{2}}), & p > \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-p)+2sp}{2}}|log\varepsilon|), & p = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{(3-2s)p}{2}}), & p < \frac{3}{3-2s}. \end{cases} \tag{2.17}$$

**Lemma 2.4.** *Under the assumptions of Theorem 1.1, then* $0 < c < c^*$.

*Proof.* Since the Lax-Milgram theorem implies that $(-\Delta)^s\phi = |v_\varepsilon|^{2_s^*-1}$ has a unique solution $\{\phi_\varepsilon\}$, then we have

$$\begin{aligned}\int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx &= \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}\phi_{v_\varepsilon}(-\Delta)^{\frac{s}{2}}|v_\varepsilon|dx\\ &\le \frac{1}{2}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx + \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|^2dx.\end{aligned}$$

Let $Q(t) = \frac{t^2}{2}[v_\varepsilon]_s^2 - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx, t \ge 0$, it follows from (2.16) that

$$\begin{aligned}Q(t) &\le \left(\frac{t^2}{2} + \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\right)[v_\varepsilon]_s^2 - \left(\frac{t^{2_s^*}}{2_s^*} + \frac{t^{2(2_s^*-1)}}{2_s^*-1}\right)\int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx\\ &\le \left(\frac{t^2}{2} + \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\right)(S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})) - \left(\frac{t^{2_s^*}}{2_s^*} + \frac{t^{2(2_s^*-1)}}{2_s^*-1}\right)(S^{\frac{3}{2s}} + O(\varepsilon^3))\\ &\le \left(\frac{t^2}{2} - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)} - \frac{t^{2_s^*}}{2_s^*}\right)S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})\end{aligned}$$

as $\varepsilon \to 0$. After computation, we have

$$\sup_{t\ge0} Q(t) \le \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}). \tag{2.18}$$

As in Lemma 2.2, we see that $I(tv_\varepsilon) > 0$ for $t > 0$ small, and $I(tv_\varepsilon) \to -\infty$ as $|t| \to \infty$. According to the continuity of $I$, there exist $t_\varepsilon > 0$ such that

$$I(t_\varepsilon v_\varepsilon) = \sup_{t\ge0} I(tv_\varepsilon) > 0.$$

From $(f_5)$, given $M > 0$ large enough, there exists $R_M > 0$ large enough such that

$$|F(u)| \ge Mu^{m+1} \text{ with } |u| > R_M.$$

This together with (2.6) implies that for all $M > 0$, there exists a constant $C_M > 0$ such that

$$F(u) \ge Mu^{m+1} - C_Mu^2 \text{ with } m = \max\left\{1, \frac{2s}{3-2s}\right\}.$$

In addition, we deduce that

$$\int_{\mathbb{R}^3} F(t_\varepsilon v_\varepsilon) \ge M\int_{\mathbb{R}^3}|t_\varepsilon v_\varepsilon|^{m+1}dx - C_M\int_{\mathbb{R}^3}|t_\varepsilon v_\varepsilon|^2dx \ge C_1\|v_\varepsilon\|_{m+1}^{m+1} - C_2\|v_\varepsilon\|_2^2.$$

Using the estimates (2.17) and (2.18), for $\varepsilon > 0$ small enough, we get

$$\begin{aligned}I(t_\varepsilon v_\varepsilon) &\le \sup_{t\ge0} Q(t) + \int_{B_2} V(x)|t_\varepsilon v_\varepsilon|^2dx - C_1\|v_\varepsilon\|_{m+1}^{m+1} + C_2\|v_\varepsilon\|_2^2\\ &\le \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{3}{2s}} - C_1\|v_\varepsilon\|_{m+1}^{m+1} + C_2\|v_\varepsilon\|_2^2.\end{aligned}$$

We distinguish three cases.
**Case 1.** $s \in (0, \frac{3}{4})$, then $\frac{3}{3-2s} < 2$. In this case, as we have seen in (2.17),

$$\|v_\varepsilon\|_{m+1}^{m+1} = O(\varepsilon^{\frac{3(1-m)+2s(m+1)}{2}}), \|v_\varepsilon\|_2^2 = O(\varepsilon^{2s}).$$

Thus, for $\varepsilon$ small enough, $O(\varepsilon^{2s}) - O(\varepsilon^{\frac{3(1-m)+2s(m+1)}{2}}) < 0$ holds because of $3(1-m) + 2s(m+1) - 4s = (3-2s)(1-m) < 0$.
**Case 2.** $s = \frac{3}{4}$, then $\frac{3}{3-2s} = 2$. It is follows from (2.17) that

$$\|v_\varepsilon\|_{m+1}^{m+1} = O(\varepsilon^{\frac{3(1-m)+2s(m+1)}{2}}) = O(\varepsilon^{\frac{9-3m}{4}}), \|v_\varepsilon\|_2^2 = O(\varepsilon^{2s}|log\varepsilon|) = O(\varepsilon^{\frac{3}{2}}|log\varepsilon|).$$

Due to $\lim_{\varepsilon\to 0} \frac{\varepsilon^{\frac{3}{2}}|log\varepsilon|}{\varepsilon^{\frac{9-3m}{4}}} = 0$, as $\varepsilon \to 0$. Thus, we have $O(\varepsilon^{\frac{3}{2}}|log\varepsilon|) - O(\varepsilon^{\frac{9-3m}{4}}) < 0$ for $\varepsilon$ small enough.
**Case 3.** $s \in (\frac{3}{4}, 1)$, then $2 < \frac{3}{3-2s} < 3$. We remark that $m + 1 > \frac{3}{3-2s}$ since $m > \frac{6s-3}{3-2s} > \frac{2s}{3-2s}$, we can obtain

$$\|v_\varepsilon\|_{m+1}^{m+1} = O(\varepsilon^{\frac{3(1-m)+2s(m+1)}{2}}), \|v_\varepsilon\|_2^2 = O(\varepsilon^{3-2s}).$$

Having observed $m > \frac{6s-3}{3-2s}$, it is easy to check that $O(\varepsilon^{3-2s}) - O(\varepsilon^{\frac{3(1-m)+2s(m+1)}{2}}) < 0$ for $\varepsilon$ small enough.

Since $-C_1\|v_\varepsilon\|_{m+1}^{m+1} + C_3\|v_\varepsilon\|_2^2 < 0$ as $\varepsilon \to 0$, the result follows. □

## 3. Proof of Theorem 1.1

*Proof.* From Lemma 2.2, we know that the functional $I$ satisfies the mountain geometry structure. Thus we apply the Mountain-pass lemma, there exists a sequence $\{u_n\} \subset E$ satisfying

$$I(u_n) \to c \geq \alpha > 0, I'(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], E)| : \gamma(0) = 0, I(\gamma(1)) = I(e) < 0\},$$

$\alpha$ and $e$ are defined by Lemma 2.2. By Lemmas 2.3 and 2.4, there exist a convergent subsequence $\{u_n\} \subset E$ (still denoted by itself) and $u_{**} \in E$ such that $u_n \to u_{**}$ in $E$. Thus, we conclude that

$$I(u_{**}) = c \geq \alpha > 0 \text{ and } I'(u_{**}) = 0,$$

which implies that $(u_{**}, \phi_{u_{**}})$ is a nontrivial solution of system (1.1).

In the following, we claim that there exists a positive ground state solution $(v, \phi_v)$ of system (1.1). Define

$$m = \inf_{u\in\mathcal{N}} I(u), \quad \mathcal{N} = \{u \in E\backslash\{0\} \mid I'(u) = 0\},$$

we notice that $\mathcal{N}$ is nonempty because of $u_{**} \in \mathcal{N}$. For any $u \in \mathcal{N}$, It follows from (2.6), Lemma 2.1(3) and Sobolev inequality that

$$\begin{aligned} 0 = \langle I'(u), u\rangle &= \|u\|^2 - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \int_{\mathbb{R}^3} |u|^{2_s^*} dx - \int_{\mathbb{R}^3} f(u)u dx \\ &\geq \left(1 - \frac{\varepsilon}{V_0}\right)\|u\|^2 - S^{-2_s^*}\|u\|^{2(2_s^*-1)} - S^{-\frac{2_s^*}{2}}(C_\varepsilon + 1)\|u\|^{2_s^*}, \end{aligned}$$

which implies that $||u||$ must be larger than some positive constant, thereby 0 is not in $\partial M$. Meanwhile, we decuce that

$$I(u) = I(u) - \frac{1}{\rho}\langle I'(u), u\rangle \geq \left(\frac{1}{2} - \frac{1}{\rho} - \frac{\delta}{V_0\rho}\right)||u||^2$$

for any $u \in \mathcal{N}$. By the fact $u \neq 0$, we have $I(u) > 0$, thus we can get that $m > 0$. Let $v_n \subset \mathcal{N}$ be a minimizing sequence such that

$$I(v_n) \to m, \quad I'(v_n) \to 0 \text{ as } n \to \infty.$$

Since $m \leq c < c^*$, after taking a subsequence, it follows from the proof of Lemma 2.3 that there exists $v \in E$ such that $v_n \to v$ in $E$. Hence, $v$ is a non-trivial critical point of $I$ with $I(v) = m$. Now, we define a new functional as

$$I_+(v) = \frac{1}{2}||v||^2 - \frac{1}{2(2_s^* - 1)}\int_{\mathbb{R}^3} \phi_{v^+}(v^+)^{2_s^*-1}dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}(v^+)^{2_s^*}dx - \int_{\mathbb{R}^3} F(v^+)dx,$$

where $v^+ := \max\{v, 0\}$, $v^- := \min\{v, 0\}$. The condition $\langle I_+'(v), v^-\rangle = 0$ implies that $v \geq 0$ in $\mathbb{R}^3$, which is a non-negation weak solution of system (1.1). By using the strong maximum principle and standard argument, $v$ is a positive ground state solution. This completes the proof of Theorem 1.1. □

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**Conflict of interest**

The authors declare no conflict of interest.

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